Available online at www.sciencedirect.com

ScienceDirect

Procedia Computer Science 195 (2021) 427–436

Procedia Computer Science

www.elsevier.com/locate/procedia

XI Latin and American Algorithms, Graphs and Optimization Symposium

Cliques in exact distance powers of graphs of given maximum degree☆

Florent Foucaud[a,1], Suchismita Mishra[c,b], Narayanan Narayanan[b], Reza Naserasr[d], Petru Valicov[e]

[a]LIMOS, CNRS UMR 6158, Université Clermont Auvergne, Aubière, France
Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, Talence, France.
Univ. Orléans, INSA Centre Val de Loire, LIFO EA 4022, F-45067 Orléans Cedex 2, France.
[b]Department of Mathematics, IIT Madras, Chennai 600036, India.
[c]Indian Statistical Institute, Kolkata, India.
[d]Université de Paris, CNRS, IRIF, F-75006, Paris, France.
[e]LIRMM, CNRS, Université de Montpellier, France.

Abstract

The exact distance p-power of a graph G, denoted $G^{[\sharp p]}$, is a graph on vertex set $V(G)$ in which two vertices are adjacent if they are at distance exactly p in G. Given integers k and p, we define $f(k, p)$ to be the maximum possible order of a clique in the exact distance p-powers of graphs with maximum degree $k+1$. It is easily observed that $f(k, 2) \leq k^2 + k + 1$. We prove that equality may only hold if a connected component of G is isomorphic to a member of the class $\mathcal{P}_k$ of incidence graphs of finite projective k-geometries. (These famous combinatorial structures are known to exist when k is a prime power, and are conjectured not to exist for other values of k.) We then study the case of graphs of maximum degree $k+1$ with clique number $k^2 + k$. One way to obtain such a graph is to remove a vertex from a graph in $\mathcal{P}_k$; we call $\mathcal{P}'_k$ the class of all such resulting graphs. We prove that for any graph G of maximum degree $k+1$ whose exact square has a (k^2+k)-clique, either G has a subgraph isomorphic to a graph in $\mathcal{P}'_k$, or a connected component of G is a $(k+1)$-regular bipartite graph of order $2(k^2+k)$. We call O_k the class of such bipartite graphs, and study their structural properties. These properties imply that (if they exist) the graphs in O_k must be highly symmetric. Using this structural information, we show that O_2 contains only one graph, known as the Franklin graph. We then show that O_3 also consists of a single graph, which we build. Furthermore, we show that O_4 and O_5 are empty.

For general values of p, we prove that $f(k, p) \leq (k+1)k^{\lfloor p/2 \rfloor} + 1$, and that the bound is tight for every odd integer $p \geq 3$. This implies that $f(k, 2) = f(k, 3)$ whenever there exists a finite projective k-geometry, however, in such a case, the bound of $f(k, 3)$ could also be reached by highly symmetric graphs built from a finite k-geometry, which is not the case for other values of k.

Keywords: Exact distance powers; Powers of graphs; Clique number; Maximum degree

☆ This research was supported by the IFCAM project "Applications of graph homomorphisms" (MA/IFCAM/18/39), by SERB OVDF program ("*ODF*_2018_001449"), by the ANR project HOSIGRA (ANR-17-CE40-0022) and by the ANR project DISTANCIA (ANR-17-CE40-0015).

[1] Email: florent.foucaud@uca.fr

1877-0509 © 2021 The Authors. Published by Elsevier B.V.
This is an open access article under the CC BY-NC-ND license (https://creativecommons.org/licenses/by-nc-nd/4.0)
Peer-review under responsibility of the scientific committee of the XI Latin and American Algorithms, Graphs and Optimization Symposium
10.1016/j.procs.2021.11.052

1. Introduction

In this paper, we study exact distance powers of graphs, and more precisely, we focus on the structure and possible sizes of their maximum cliques. Graphs considered here are finite, without loops and multiple edges. Given a graph G, we denote $G^{[\sharp p]}$ the *exact distance p-power* of G, that is, the graph obtained by taking the vertices of G and adding an edge between any two distinct vertices at distance exactly p. For $p = 2$, we call $G^{[\sharp 2]}$ the *exact square* of G. The graph $G^{[\sharp p]}$ is a subgraph of the usual distance p-power G^p of G (in which any two distinct vertices at distance *at most* p in G are joined by an edge), which is a classic and well-studied notion. Thus, exact distance powers can be seen as refinements of usual graph powers. Although exact distance powers have not been studied much, as we will see, they give rise to many interesting problems, with deep connections to other parts of mathematics.

The concept of exact distance powers of graphs is a special case of exact distance graphs arising from metric spaces [13], which appears in many contexts. In particular, it is the object of the famous Hadwiger-Nelson problem of "coloring the plane" [8, 9]. One of the first explicit definitions of the concept of exact distance powers of graphs is due to Simić [16], who studied graphs which had an exact distance power isomorphic to their line graph. Among recent studies of the subject, one may refer to [2], which is about the search for graphs that are isomorphic to their exact square, or to [4], which is about the structure of exact distance powers of products of graphs. The notion of exact distance powers also appears in the context of the theory of sparse graphs, see the book [14, Section 11.9]. Following this line of work, the chromatic number of exact distance powers of graphs in various graph classes has been studied in several papers, see [3, 6, 11, 13, 15].

The main focus of this work is on the maximum possible clique number of the exact distance power of graphs of given maximum degree. We denote by $\omega(G)$ the clique number of a graph G. Bounds on the clique number of usual powers of graphs have been the subject of many studies. In particular, it is easily observed that the usual square of a graph of maximum degree d has clique number at most $d^2 + 1$. It is a well-known result of Hoffman and Singleton [12] that the bound can be achieved only for $d \in \{2, 3, 7, 57\}$ and while there are unique examples for $d = 2, 3, 7$, it remains an open question whether such a graph exists for $d = 57$. Our work, however, suggests that the notion of exact distance power is perhaps a better fit to the study of cliques. It leads to the discovery of highly symmetric graphs and provides strong connections to other areas of mathematics, such as finite geometries, geometrical configurations, Latin squares and orthogonal arrays. More precisely we ask the following.

What is the maximum possible clique number $f(k, p)$ among exact p-distance powers of graphs with maximum degree $k + 1$?

Noticing that the maximum degree of $G^{[\sharp p]}$ is at most $d(d-1)^{p-1}$ (where $d = k + 1$ is the maximum degree of G), it is clear that $f(k, p) \leq (k + 1)k^{p-1} + 1$.

Before answering the above question we give the following two definitions. A *finite projective k-geometry* is a combinatorial structure that consists of a finite set of points and a finite set of lines where each line contains exactly $k+1$ points, any pair of lines intersect in exactly one point and any pair of points belong to a unique line. The incidence graph of a finite projective k-geometry is a bipartite graph where one part consists of all the points and the other part consists of all the lines. A point is adjacent to a line if it belongs to it. The class of incidence graphs of all finite projective k-geometries is denoted by $\mathcal{P}_k$.

We first focus on the case $p = 2$ and show in Section 2 that $f(k, 2) = k^2 + k + 1$ if and only if there exists a finite projective k-geometry. More precisely, $\omega(G^{[\sharp 2]}) = k^2+k+1$ if and only if a connected component of G is isomorphic to some graph in $\mathcal{P}_k$. In that case, the exact square of this component results in the disjoint union of two complete graphs, each of order $k^2 + k + 1$. Note that finite projective k-geometries are highly regular structures, and it is a prominent open problem to determine for which integers k they exist. There are a number of classical constructions based on finite fields, thus for those values of k which are prime powers. It is conjectured that there does not exist one for other values of k. Currently, $k = 12$ is the smallest open case of this conjecture [5].

In Section 3, we turn our attention to cliques in exact squares having size $k^2 + k$ (that is, one less than the trivial upper bound). The removal of a vertex from a finite projective k-geometry gives a graph with a clique of size $k^2 + k$. We define $\mathcal{P}'_k$ as the class of all such graphs (graphs formed by deleting a vertex from the incidence graph of a finite projective k-geometry). We provide two other interesting constructions, that do not come from finite projective k-

geometries, for the cases where $k = 2, 3$ with clique sizes 6 and 12, respectively. For $k = 2$ this is the Franklin graph, and for $k = 3$ the construction is obtained by doubling the Franklin graph in a certain way. Moreover, we show that for the cases $k = 2, 3$, these are the only examples. This is done by exploring the (highly constrained) structure of potential examples for general values of k. Using these structural properties, we further prove that for $k = 4, 5$, no such alternative constructions exist (besides those arising from finite projective k-geometries). This could indicate that the two constructions for $k = 2, 3$ are exceptional small cases, and that $f(k, 2) < k^2 + k$ whenever no finite projective k-geometry exists.

In Section 4, we turn our attention to powers for larger distances. We show that $f(k, p) \leq (k + 1)k^{\lfloor p/2 \rfloor} + 1$, which is tight for every odd $p \geq 3$. Interestingly, this implies that $f(k, 2) = f(k, 3)$ whenever there exists a finite projective k-geometry.

We conclude in Section 5.

Notation. The open neighbourhood of a vertex v is denoted $N(v)$ and the k^{th} neighbourhood, that is the set of vertices at distance exactly k from v, is denoted $N_k(v)$. For a set S of vertices, $N(S) := \{u \notin S | u \text{ is a neighbour of some } v \in S\}$. The degree of v is denoted $d(v)$. The S-degree of v, denoted $d_S(v)$, is the number of neighbours of v in set S.

2. Cliques of order $k^2 + k + 1$ in exact squares of graphs of maximum degree $k + 1$, and finite projective k-geometries

It follows by a simple counting argument that a finite projective k-geometry must have exactly $k^2 + k + 1$ points and the same number of lines. Moreover there is a symmetry between points and lines, indeed any two points must belong to a unique common line. Examples of finite projective k-geometries when k is a prime power are known. Perhaps the most important conjecture of finite geometry states that for other values of k (i.e., when k is not a prime power), there is no finite projective k-geometry [5].

We will now see that there is a strong connection between cliques in exact squares and finite projective k-geometries. The idea of the proof is similar to the ones in [10, Theorem 16] and [1, Section 3] which deal with related topics of injective chromatic number and usual graph powers, respectively.

Theorem 2.1. *If G is a graph of maximum degree $k + 1$, then $\omega(G^{[\sharp 2]}) \leq k^2 + k + 1$. Moreover, equality holds only if a connected component of G is isomorphic to a graph in the class $\mathcal{P}_k$.*

Proof. Let G be a graph of maximum degree at most $k + 1$. Let K be a clique of maximum order in $G^{[\sharp 2]}$, and let u be a vertex of K. Then, all other vertices of K are in $N_2(u)$, however $|N_2(u)| \leq k^2 + k$. This proves the first part of the statement.

Suppose equality holds for G, and observe that if $|N_2(u)| = k^2 + k$, then: (i) u and all its neighbours are of degree exactly $k + 1$, (ii) any two distinct neighbours of u in G have exactly one common neighbour, which is u (each must contribute k distinct vertices to $N_2(u)$). By the definition of an exact square, K must induce an independent set in G. Applying properties (i) and (ii) to every other vertex of K, we conclude that G is a bipartite $(k + 1)$-regular graph.

It remains to show that this graph is isomorphic to the incidence graph of a finite projective k-geometry. Consider the finite geometry built as follows: the points of the geometry are the vertices of K, and for each neighbour x of a vertex u of K, we define the line L_x as the set of all $k + 1$ neighbours of x. Note that if any two lines intersect, they can only intersect once by Property (ii). Thus, to show that our points and lines form a finite projective geometry, it remains only to prove that any two lines intersect.

To see this, consider two lines L_x and L_y. Let u be a neighbour of x and v be a neighbour of y (in G). Recall that the $k + 1$ neighbours $y, y_1, y_2, \ldots, y_k$ of v form $k + 1$ lines, any pair of which has only v as a common point. If L_x does not intersect L_y, then, by the pigeonhole principle, it must intersect one of $L_{y_1}, L_{y_2}, \ldots, L_{y_k}$ at least in two points, but this contradicts Property (ii).

Conversely, the clique number of the exact square of any incidence graph of a projective k-geometry is $k^2 + k + 1$, in fact the exact square of any such graph consists of two cliques of size $k^2 + k + 1$, since any two lines intersect and any two points belong to a common line. □

By Brook's theorem, Theorem 2.1 implies that the chromatic number of $G^{[\sharp 2]}$ is at most $k^2 + k$ (where G has maximum degree $k + 1$), unless when a component of G is isomorphic to the incidence graph of a finite projective k-geometry (then it is exactly $k^2 + k + 1$).

3. Cliques of order $k^2 + k$ in exact squares of graphs of maximum degree $k + 1$

In this section, we show that there are two possible types of graphs of maximum degree $k + 1$ whose exact square has a clique of order $k^2 + k$. The first type is essentially obtained from a projective k-geometry by removing a vertex. Those of the second type are graphs that are essentially $(k + 1)$-regular bipartite graphs. The existence of this second type of graphs for a given k is the main subject of study in this work.

Lemma 3.1. *Let G be a graph of maximum degree $k+1$ and assume a set K of k^2+k vertices induces a clique in $G^{[\sharp 2]}$. Then the bipartite graph with vertex set $K \cup N(K)$ and whose edge set is the set of all edges incident with a vertex of K is either:*

1. *isomorphic to a graph in $\mathcal{P}'_k$, or*
2. *a $(k + 1)$-regular bipartite subgraph of G (and thus a component). Morover, in this case, for each vertex x of K there is a unique vertex $c(x)$ in K (called its* cousin*), such that x and $c(x)$ have exactly two common neighbours.*

Proof. By analogy with finite projective geometries, the vertices of K and $L = N(K)$ will be called points and lines, respectively.

Let x be any point of K and let $\ell_1, \ell_2, \ldots, \ell_{d(x)}$ be the neighbours of x. As each vertex ℓ_i is adjacent to at most k other points of K, and as there are $k^2 + k - 1$ such other points in K, we must have that $d(x) = k + 1$. For the same reason, only one of the ℓ_i's can be of K-degree k.

Now, let $b_1, b_2, \ldots, b_r$ $(r \geq 0)$ be the lines in L which are of K-degree k (thus by the previous paragraph, the other lines are of K-degree $k + 1$). Observe that no pair of b_i's has a common neighbour in K, since by the previous paragraph, for each neighbour x of b_i, x has b_i as its unique neighbour of K-degree k. Thus, the points of K can be partitioned into r sets, each of size k (the sets $N(b_i) \cap K$), and a last set of size $k^2 + k - rk$. This implies that r is at most $k + 1$.

Case a: $r = k + 1$. We show that this case corresponds to the first case of the statement. We restrict G to K and L and remove edges inside L (note that K remains a clique in the exact square of the new graph). Then, we add a new vertex u joined by an edge to all the lines of K-degree k. Then, u is at distance 2 from all the points of K, thus $K \cup \{u\}$ forms a clique of order $k^2 + k + 1$ in the new graph, whose maximum degree remains $k + 1$. By Theorem 2.1, this new graph must be the incidence graph of a finite projective k-geometry, and we are indeed in the first case.

Case b: $r < k + 1$. We will show that the second case of the statement holds.

Note that (by the first two paragraphs of the proof), there are exactly $k^2 + k - rk > 0$ points in K that have no neighbour of L with K-degree k. Consider such a point, and name it u. Since all neighbours of u are of K-degree $k+1$, and there are only $k^2 + k - 1$ points in K other than u, there has to be a unique pair ℓ_1 and ℓ_2 of neighbours of u which have another common neighbour in K: call it u', the *cousin* of u. Observe that u uniquely determines u' and that this relation is symmetric: u' must also have $k+1$ neighbours, each of K-degree $k+1$, so u is the cousin of u'. Thus, overall there are exactly $\frac{k^2+k-rk}{2}$ pairs of cousins.

We now count the total number of pairs of points of K in two ways:

- On one hand, since K has order $k^2 + k$, we have a total of $\binom{k^2+k}{2}$ pairs.
- On the other hand, each vertex ℓ of L contributes to counting $\binom{d_K(\ell)}{2}$ pairs of points of K, where overall there will be exactly $\frac{k^2+k-rk}{2}$ repetitions.

Let ℓ be the number of lines. Counting the pairs in K as above, we have

$$r\binom{k}{2} + (\ell - r)\binom{k+1}{2} - \frac{k^2 + k - rk}{2} = \binom{k^2 + k}{2}.$$

Simplifying this identity, we have $(k + 1)(\ell - (k^2 + k)) = r$. As r is an integer between 1 and $k + 1$, and as from this equation it must be divisible by $k + 1$, we conclude that $r = 0$. This implies that $\ell = k^2 + k$, G is bipartite and $(k + 1)$-regular, and that K can be partitioned into pairs of cousins. Thus, the second point of the statement holds. □

The first case of Lemma 3.1 is about the classic problem of the existence of finite projective k-geometries (but then, we know by Theorem 2.1 that we actually have $f(k, 2) = k^2 + k + 1$). The class of all graphs coming from the second case is denoted $\mathcal{O}_k$. Note that any graph G in $\mathcal{O}_k$ is a bipartite graph, with partite sets, say, (K, L). By analogy with finite geometries, we call vertices of K, *points* and vertices of L, *lines*. This will lead to the introduction of highly symmetric structures, perhaps so structured that they do not exist unless $k = 2$ or 3. The study of this case is the main focus of this work.

Thus, for the next lemmas, we will work with a graph G of maximum degree $k + 1$ having a clique K of maximum size $k^2 + k$ in its exact square, and we assume that the second point of Lemma 3.1 holds for G. Recall that then, by Lemma 3.1, each point x of K has a unique *cousin* x' in K.

Lemma 3.2. *Any two lines of a graph G in $\mathcal{O}_k$ have a common neighbour.*

Proof. Consider any line l. We first show that l sees both vertices of a unique pair of cousin points. If l is not adjacent to any pair of cousins, every neighbour of l is adjacent to k other lines, and all these lines are distinct. This gives a total of $(k + 1)k + 1$ lines, contradicting Lemma 3.1 (there are only $k^2 + k$ lines). So, l sees a pair of cousins. There are $\frac{k^2+k}{2}$ pairs of cousins in K, and by Lemma 3.1 the vertices of each cousin pair have exactly two lines as common neighbours. Since by Lemma 3.1 there are only $k^2 + k$ lines, every line can only see one pair of cousins. Thus l sees exactly one pair of cousins (say $\{x, x'\}$), as claimed.

Any line seeing a vertex of $R = N(l) \setminus \{x, x'\}$ does not see any other vertex of $N(l)$. So, l is at distance 2 of $(k - 1)k$ lines through the vertices of R, and $2(k - 1)$ lines through x and x'. This makes a total of $k^2 + k - 1$ distinct lines, that is, all the lines, and the claim is proved. □

Lemma 3.2 implies that the points and lines are in fact interchangeable: indeed L also is a clique of size $k^2 + k$ in the exact square of G, and so $G^{[\sharp 2]}$ is isomorphic to two copies of the complete graph K_{k^2+k}. (This is reminiscent of finite projective k-geometries, which also enjoy this symmetry.) Hence, all the properties described until now for the points, are valid for the lines; in particular, by Lemma 3.1, every line l has a unique *cousin* l'. This implies that the two common neighbours of any pair of cousins are themselves cousins. We can now prove the following (which thus holds when x, x' are cousin points or cousin lines, thus we do not make the distinction, although we use the notation for points).

Lemma 3.3. *Let $\{x, x'\}$ be a pair of cousins in a graph G in $\mathcal{O}_k$. For every neighbour l of x, the cousin l' of l is a neighbour of x'.*

Proof. Let a, a' be the two common neighbours of x and x', and let l be some neighbour of x. We wish to prove that the cousin l' of l is adjacent to x'. If $l \in \{a, a'\}$ (say $l = a$), we are done, since then $l' = a'$ is the second common neighbour of x and x'. Thus, assume that l is not adjacent to x'. Let $A = N(l) \setminus \{x\}$ and let $B = N(x') \setminus \{a, a'\}$. Since x' and all vertices of A are in K, x' must have a common neighbour with each of them. This common neighbour cannot be a or a', since otherwise x would have two cousins, contradicting Lemma 3.1. Hence, every vertex of A has a neighbour in B. But since $|B| < |A|$, by the pigeonhole principle, there is a vertex in B that has two neighbours in A. Thus, this vertex is the cousin l' of l and is a neighbour of x', as claimed. □

By the structural properties of the graphs in $\mathcal{O}_k$ proved so far, we can see that for any graph G in $\mathcal{O}_k$, $V(G)$ can be partitioned into vertex-disjoint 4-cycles (each composed of one pair of cousin points and one pair of cousin lines). There is no other 4-cycle in G, since any two cousins have just two common neighbours. By Lemma 3.3, the adjacencies of cousins are in a way parallel, we can thus reduce the graph G by introducing the following construction, without losing much information. We denote by $C(K)$ and $C(L)$ the sets of cousin pairs in K and L, respectively.

Definition 3.4. *For any graph G in $\mathcal{O}_k$, let G' be the bipartite graph with partite sets $C(K)$, $C(L)$ obtained from G by identifying every pair of cousins as a single vertex. We define $\mathcal{O}'_k := \{G' \mid G \in \mathcal{O}_k\}$.*

We now prove some properties of $\mathcal{O}'_k$.

Lemma 3.5. *Let G be a graph in $\mathcal{O}_k$. Then G' is k-regular and has a perfect matching $\varphi : K \to L$ whose edges $x\varphi(x)$ correspond to the 4-cycles of G. Moreover, any two vertices in the same part of G either have exactly one common neighbour (this common neighbour is matched by φ to one of the two vertices), or exactly two common neighbours* *(and none of these common neighbours is matched by φ to any of these two vertices). In particular, there is no $K_{2,3}$ in G'.*

Proof. The first part of the statement follows directly from the definition of G' and from Lemma 3.3.

Then, consider two vertices, say x and y, in the same part of G', say $C(K)$, and let $P(x)$ and $P(y)$ be the pairs of cousins of K corresponding to x and y. Clearly, x and y must have a common neighbour, for otherwise the vertices in $P(x)$ would not have a common neighbour with those of $P(y)$ in G. Suppose first that $\varphi(x)$ is their common neighbour. If x and y had another common neighbour, this would imply that the vertices of $P(x)$ have two common neighbours with the vertices of $P(y)$, a contradiction to Lemma 3.1. Assume thus that x and y have a common neighbour z and $z \notin \{\varphi(x), \varphi(y)\}$. Then, one can see that only two of the pairs in $P(x) \times P(y)$ have a common neighbour in G, a contradiction. Moreover, if x and y had more than two common neighbours, this would imply the existence of a pair in $P(x) \times P(y)$ having two common neighbours in G, a contradiction too. □

Note that given $G' \in \mathcal{O}'_k$, the original graph $G \in \mathcal{O}_k$ is not uniquely determined. Indeed, for an edge xl of G' that is not in the matching φ, it is not clear which were the two edges in G connecting the two pairs of cousins corresponding to x and l. However, the construction will still be enough for the purposes of our proofs.

3.1. Characterizations for $k = 2, 3, 4, 5$

We now characterize graphs having maximum degree $k + 1$ with their exact square having clique number $k^2 + k$, when $k \leq 5$. The first case involves the Franklin graph [7] (Fig. 1a) and the subgraph of the Heawood graph (Fig. 1b).

Theorem 3.6. *Let G be a subcubic graph with $\omega(G^{[\sharp 2]}) = 6$. Then either G is isomorphic to the Franklin graph, (that is $\mathcal{O}_2$ has only one graph that is the Franklin graph), or G has a subgraph isomorphic to the Heawood graph with one vertex deleted (that is, in $\mathcal{P}'_2$).*

Proof. By Lemma 3.1, either G is in $\mathcal{P}'_2$, or G is a cubic graph with specific properties. In the former case, it is known that G must be the Heawood graph with a vertex deleted (there is only one finite projective 2-geometry, the Fano plane, whose incidence graph is the Heawood graph [5]).

In the latter case, by the results in Section 3, we can consider the corresponding graph $G' \in \mathcal{O}'_2$ defined in Definition 3.4. We know that any graph in $\mathcal{O}'_2$ is 2-regular, bipartite, and has order 6; thus, it must be a 6-cycle. One can now reconstruct the original graph G; even though the three edges that are not in the matching φ give room to several graphs, they are all isomorphic to the Franklin graph. □

(a) The Franklin graph, in $\mathcal{O}_2$ (b) The Heawood graph with one vertex deleted, in $\mathcal{P}'_2$

Figure 1: Two subcubic graphs whose exact squares have clique number 6.

We can obtain next graph of interest by "doubling" the Franklin graph in the following way: we make two copies H_1 and H_2 of the Franklin graph, and for each edge xy of the Franklin graph that is not in a 4-cycle, we connect the copy of x in H_1 either to the copy of y in H_2, or to the copy of its cousin y' in H_2, in a certain way (see Fig. 2). Using the structural properties from Section 3, we can also prove the following theorem.

Theorem 3.7. *Let G be a graph with maximum degree 4 and $\omega(G^{[\sharp 2]}) = 12$. Then either G is isomorphic to the graph from Fig. 2, or G has a subgraph isomorphic to the incidence graph of a finite projective 3-geometry with one vertex deleted, (that is, in $\mathcal{P}'_3$). That is the graph mentioned in Fig. 2 is the only graph in O_3.*

Proof. By Lemma 3.1, if G has no subgraph isomorphic to the incidence graph of a finite projective 3-geometry with one vertex deleted, we can use the results of Section 3. However, we will not use Definition 3.4, as we lose some information when reducing the graph, and here we need to be more precise.

By the results in Section 3, G is a connected 4-regular bipartite graph, with bipartition (K, L), where $K = \{x_i, x'_i | 1 \leq i \leq 6\}$, $L = \{l_i, l'_i | 1 \leq i \leq 6\}$. We assume that the pairs $\{x_i, x'_i\}$ and $\{l_i, l'_i\}$ are the cousin pairs, and that for each $i \in \{1, \dots, 6\}$, $x_i l_i x'_i l'_i$ is a 4-cycle.

Without loss of generality, we assume that the two neighbours of x_1 besides l_1 and l'_1 are l_2 and l_3. Lemma 3.3 implies that l'_2, l'_3 are neighbours of x'_1. Moreover, l_1 must have two neighbours other than x_1 and x'_1. None of them is x_2, as otherwise x_1 and x_2 would be cousins. Similarly, we can show that x'_2, x_3, x'_3 are non-neighbours of l_1. Without loss of generality, we may assume that x_4 and x_5 are the other neighbours of l_1. By Lemma 3.3, x'_4 and x'_5 are neighbours of l'_1.

Now, since L is a clique in $G^{[\sharp 2]}$, then l'_2, l_3 have a common neighbour. This common neighbour is not x_2, as otherwise x_1 and x_2 would be cousins. Similarly, we can show that this common neighbour is different from x'_2, x_3, x'_3, x_4, x'_4, x_5, x'_5. So, one of x_6 and x'_6 is a common neighbour of l_2 and l'_3. Similarly, we can show that either l_6 or l'_6 is a common neighbour of x_4 and x'_5. Without loss of generality, we may assume that x_6 is the common neighbour of l'_2 and l_3; l_6 is a common neighbour of x_4, x'_5. By Lemma 3.3, l_2, l'_3 are neighbours of x'_6 and x'_4, x_5 are neighbours of l'_6.

Again, since L is a clique in $G^{[\sharp 2]}$, vertices l_2 and l_4 must have a common neighbour. The neighbours of l_2 are x_1, x_2, x'_2 and x_6. Because of the degree of x_1 and x_6, either x_2 or x'_2 is a neighbour of l_4. Again both of them are not neighbours of l_4, otherwise l_2 is a cousin of l_4. Hence, $|N(l_4) \cap \{x_2, x'_2\}| = 1$. Similarly, we can show that $|N(l_4) \cap \{x_3, x'_3\}| = 1 = |N(l'_4) \cap \{x_2, x'_2\}| = |N(l'_4) \cap \{x_3, x'_3\}| = |N(l_5) \cap \{x_2, x'_2\}| = |N(l_5) \cap \{x_3, x'_3\}| = |N(l'_5) \cap \{x_2, x'_2\}| = |N(l'_5) \cap \{x_3, x'_3\}|$. Again, x_2 and x_3 have a common neighbour, and x_2 and x'_3 also have a common neighbour. Without loss of generality, we may assume that $x_2 l_4$, $x_3 l_4$, $x_2 l'_5$, $x_3 l_5$ are edges of G. Thus, by Lemma 3.3, $x'_2 l'_4$, $x'_3 l'_4$, $x'_2 l_5$, $x'_3 l'_5$ are also edges of G. Since L is a clique in $G^{[\sharp 2]}$, l_1 and l_6 have a common neighbour. Again l_1 and l'_6 have a common neighbour. Hence one of the vertices x_4 or x_5 is a neighbour of l_6 and other is a neighbour of l'_6. Without loss of generality, we assume that x_5 is a neighbour of l_6 and x_4 is a neighbour of l'_6. Hence $l'_6 x'_5$ and $l_6 x'_4$ are two edges. We have obtained the graph of Fig. 2. □

It seems unlikely that the doubling construction used to create the graph of Fig. 2 from the Franklin graph H can be extended. Indeed the cases of $k = 2, 3$ are special since for $k = 2$ we have $2(k^2 + k) = (k + 1)^2 + (k + 1)$. We now show that for $k = 4, 5$ the only constructions are the ones arising from finite projective k-geometries.

Theorem 3.8. *Let G be a graph of maximum degree 5 with $\omega(G^{[\sharp 2]}) = 20$. Then, G has a subgraph isomorphic to the incidence graph of a finite projective 4-geometry with one vertex deleted (that is, in $\mathcal{P}'_4$).*

Proof. Assume by contradiction that G has no subgraph isomorphic to a graph in $\mathcal{P}'_4$. Then, by Lemma 3.1 and the results in Section 3, we may consider the 4-regular graph G' of order 20 from Definition 3.4. Let x be any vertex of G'. Let 1, 2 and 3 be the three neighbours of x other than $\varphi(x)$. Since none of the edges $1x$, $2x$ and $3x$ are matched by φ, by Lemma 3.5, any two of these three vertices must have exactly two common neighbours, so each pair needs a common neighbour besides x. For two of these vertices i and j, we denote by ij their second common neighbour. Any two of these three new vertices, say 12 and 13, have a common neighbour (here, 1) and $1 \notin \{\varphi(12), \varphi(13)\}$. Thus, again by Lemma 3.5, 12 and 13 have a second common neighbour. It follows that 12, 13 and 23 (since each of them has degree 3 when not considering the edges matched by φ), have a single common neighbour, let us call it 123. Consider the connected component containing x in the cubic graph G'' obtained from G' by deleting all edges matched by φ. This component is thus isomorphic to the hypercube of dimension 3. Repeating the same arguments, we obtain that all connected components of G'' must have eight vertices, thus the order of G' must be divisible by 8, a contradiction since it is 20. This completes the proof. □

Similar, but more involved arguments lead to the following.

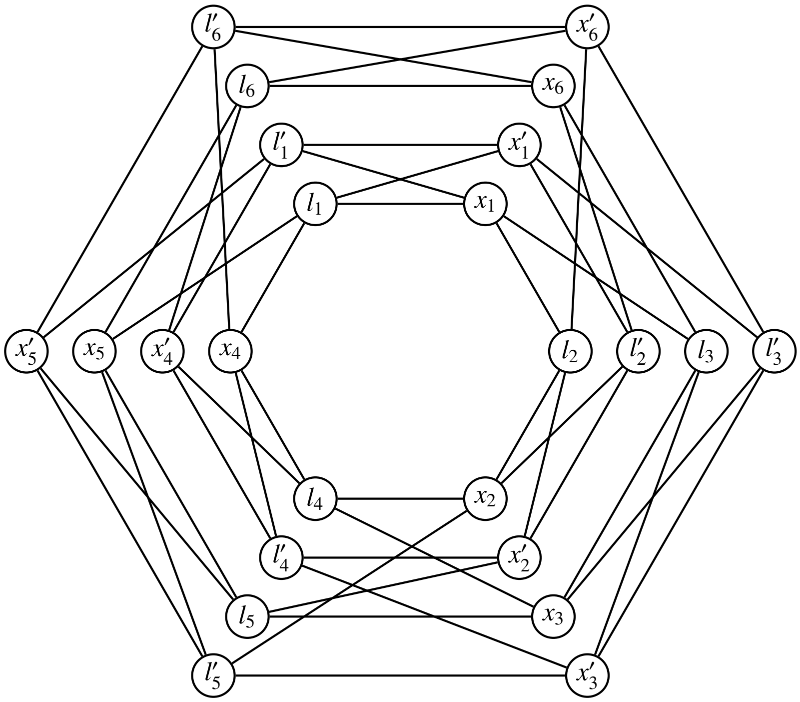

Figure 2: Doubling the Franklin graph: a 4-regular graph G with $\omega(G^{[\sharp 2]}) = 12$.

Theorem 3.9. *Let G be a graph of maximum degree 6 with $\omega(G^{[\sharp 2]}) = 30$. Then, G has a subgraph isomorphic to the incidence graph of a finite projective 5-geometry with one vertex deleted (that is, in $\mathcal{P}'_5$).*

Proof. Assume by contradiction that G has no subgraph isomorphic to the incidence graph of a finite projective 5-geometry with one vertex deleted. Then, using Section 3, we consider the 5-regular graph G' of order 30 from Definition 3.4. Let x be any vertex of G'. Let us call $1, 2, 3, 4$ the four neighbours of x other than $\varphi(x)$. By Lemma 3.5, any two of these four vertices must have exactly two common neighbours, so each pair needs a common neighbour besides x. For two of these vertices i and j, we denote by ij their second common neighbour. By similar arguments, we also deduce that 12 and 13 need a common neighbour besides 1, let us name it u. Similarly, 12 and 14 also have a common neighbour besides 1, say v. Note that $u \neq v$ since otherwise there would be a $K_{2,3}$ in G', contradicting Lemma 3.5. We distinguish two cases following Lemma 3.5.

Case 1: 12 and 34 have only one common neighbour. (By Lemma 3.5, this means that this common neighbour is matched to one of them by φ.) Note that 12 and 23 have another common neighbour besides 2, which must be u or v; without loss of generality, we can assume that it is u. Similarly, we can assume that v is adjacent to 24. Since 13 and 14 have 1 as a common neighbour through, they have another common neighbour, say w. Now, 14 and 34 also need another common neighbour, which must be w. By similar arguments, 23, 24 and 34 have a common neighbour, which must be a single vertex, say t. Finally, u, v, w, t need a common neighbour, which is also a single vertex. The connected component containing x obtained from G' by removing the edges matched by φ is now 4-regular, so no other non-matched edge exists in it. But then, the unique common neighbour of 12 and 34 must be u, v, t or w, which in each case creates a $K_{2,3}$, contradicting Lemma 3.5.

Case 2: 12 and 34 have two common neighbours. Note that u and v must have a second common neighbour with 3 and 4, respectively, and at least a common neighbour with both of them. A rapid case analysis implies (considering the symmetries of the construction so far) that without loss of generality, we can assume that u has two neighbours among 23, 24, 34 and that v has one neighbour among them: say, u is adjacent to 24 and 34, and v, to 23. But then, we also need v to be adjacent to 34 so that v and 4 have a second common neighbour. Now, any two of 13, 14, 23, 24 need a second common neighbour, and so far, three of their four non-φ-matched neighbours are fixed. So, they must all have a single common neighbour, say w. The connected component containing x in the 4-regular graph G'' obtained from G' by removing the edges of the φ-matching is now fixed by the previous discussion, and is order is 14. By the same arguments, this holds for all connected components of G''. But since 14 does not divide 30, this is impossible and gives a contradiction. □

4. The case of exact powers for larger distances

We now turn our attention to exact distance powers for larger distances. As we will show, for odd values of the distance p, $f(k, p)$ seems much easier to settle than for even values. Moreover, interestingly, $f(k, 3) = f(k, 2)$ (whenever there exists a finite projective k-geometry).

Theorem 4.1. *Let $k, p \geq 2$ be two integers. If G is a graph of maximum degree $k+1$, then $\omega(G^{[\sharp p]}) \leq (k+1)k^{\lfloor p/2 \rfloor} + 1$, and moreover this bound is tight for every odd integer p. In particular, $f(k, 3) = k^2 + k + 1$.*

Proof. Let x be a vertex of a clique K of $G^{[\sharp p]}$. There can be at most $(k + 1)k^{\lfloor p/2 \rfloor}$ vertices at distance $\lfloor p/2 \rfloor + 1$ of x. Let N be the set of these vertices. Each vertex in $K \setminus \{x\}$ is at distance $p - (\lfloor p/2 \rfloor + 1) = \lceil p/2 \rceil - 1$ of some vertex in N. However, there cannot be two vertices of K both at distance $\lceil p/2 \rceil - 1$ of a same vertex in N, as otherwise they would be at distance less than p from each other. Thus, there are at most $(k + 1)k^{\lfloor p/2 \rfloor}$ vertices in $K \setminus \{x\}$, and the bound is proved.

We now show that the bound is tight for every $k \geq 2$ and odd $p \geq 3$ using the following construction. Let $t = (k + 1)k^{\lfloor p/2 \rfloor} + 1$ be the size of the clique we wish to construct, and consider a complete graph K_t of order t. Let $s = (k + 1)k^{\lfloor p/2 \rfloor - 1}$. We iteratively perform the following process on every vertex v of K_t:

- delete the edges incident to v and replace v with a set S_v of s vertices
- make each of the s vertices of S_v adjacent to $(t - 1)/s = k$ vertices of the original neighbours of v such that the s sets of neighbours are pairwise disjoint.

Once all vertices of K_t are processed, we have the property that any two sets S_u and S_v are connected by an edge, and the resulting graph is k-regular. Now, we create a copy K of $V(K_t)$ (K is an independent set), and for each copy of a vertex x of K_t, we create a complete $(k + 1)$-ary tree T_x of height $\lfloor p/2 \rfloor$ (with the root x of degree $k + 1$). This tree has s leaves, and we identify each leaf of the tree with a distinct vertex of S_x, resulting in a $(k + 1)$-regular graph. The vertices of K are all mutually at distance p, and so they form a clique of size $(k + 1)k^{\lfloor p/2 \rfloor} + 1$ in $G^{[\sharp p]}$. □

5. Concluding remarks

We have seen that the question whether $f(k, 2) = k^2 + k + 1$ is equivalent to the one whether there exists a finite projective k-geometry. On the other hand, we have shown that cliques of size $k^2 + k$ that do not come from such a structure exist for $k = 2, 3$ and do not exist for $k = 4, 5$. Perhaps they also do not exist for greater values of k. In that case, there would be a "gap" in the values taken by $f(k, 2)$: either $k^2 + k + 1$, or less than $k^2 + k$. If they do exist, however, they would have to arise from highly symmetric graphs. Both these outcomes would be very interesting. If it is true that $f(k, 2) < k^2 + k$ whenever no finite projective k-geometry exists, then this behaviour would be reminiscent of a similar fact in the theory of Latin squares. Indeed, it is a well-known fact that the maximum number of mutual orthogonal Latin squares of order n is $n - 1$, which is reached if and only if there exists a finite projective n-geometry. Moreover, when this is not the case, it is known that this number is less than $n - 2$ (indeed it is known that $n - 2$ such Latin squares imply the existence of an $(n - 1)$-th one), see [5].

The best general upper bound for $f(k, p)$ for both $p = 2, 3$ is $k^2 + k + 1$. But for $f(k, 2)$ to reach the upper bound, there must exist a finite projective k-geometry, while $f(k, 3)$ always reaches it. So at first glance, there seems to be a contrast between $f(k, 2)$ and $f(k, 3)$. However, there exists a stronger connection. If in the proof of the second part of Theorem 4.1, after processing K_{k^2+k+1}, we want the vertices of the newly constructed graph to be partitionable into $k^2 + k + 1$ cliques, then this is only possible if a finite projective k-geometry exists. In other words, for $p = 2$ all cliques of high order must be highly symmetric and for $p = 3$, while cliques of large order always exist, to have ones in highly symmetric graphs, we are still dependent on the existence of a finite projective k-geometry.

Acknowledgements.
We thank Hervé Hocquard and Éric Sopena for initial discussions on the topic of this paper.

References

[1] Alon, N., Mohar, B., 2002. The chromatic number of graph powers. Combinatorics, Probability and Computing 11(1):1–10.
[2] Azimi, A., Farrokhi, M., 2017. Derakhshandeh Ghouchan. Self 2-distance graphs. Canadian Mathematical Bulletin 60:26–42.
[3] Bousquet, N., Esperet, L., Harutyunyan, A., de Joannis de Verclos, R., 2019. Exact distance colouring in trees. Combinatorics, Probability and Computing 28(2):177–186.
[4] Brešar, B., Gastineau, N., Klavžar, S., Togni, O., 2019. Exact distance graphs of product graphs. Graphs and Combinatorics 35:1555–1569.
[5] Laywine, C. F., Mullen, G. L., 1998. Discrete Mathematics using Latin squares. Wiley.
[6] Foucaud, F., Hocquard, H., Mishra, S., Narayanan, N., Naserasr, R., Sopena, É., Valicov, P., 2021. Exact square coloring of subcubic planar graphs. Discrete Applied Mathematics 293:74-89.
[7] Franklin, P., 1934. A six color problem. *Journal of Mathematics and Physics* 13:363–379.
[8] de Grey, A. D. N. J., 2018. The chromatic number of the plane is at least 5. Geombinatorics 28(1):5–18.
[9] Hadwiger, H., Debrunner, H., Klee, V., 1964. Combinatorial geometry in the plane. Holt, Rinehart and Winston, New York.
[10] Hahn, G., Kratochvíl, J., Sotteau, D., Širáň, J., 2002. On the injective chromatic number of graphs. Discrete Mathematics 256:179–192.
[11] van den Heuvel, J., Kierstead, H. A., Quiroz, D. A., 2018. Chromatic numbers of exact distance graphs. Journal of Combinatorial Theory, Series B 134:143–163.
[12] Hoffman, A. J., Singleton, R. R., 1960. Moore graphs with diameter 2 and 3. IBM Journal of Research and Development 5(4):497–504.
[13] Kloeckner, B. R., 2015. Coloring distance graphs: a few answers and many questions. Geombinatorics 24(3):117–134.
[14] Nešetřil, J., Ossona de Mendez, P., 2012. *Sparsity - Graphs, Structures, and Algorithms*. Springer.
[15] Quiroz, D. A., 2020. Colouring exact distance graphs of chordal graphs. Dicrete Mathematics 343(5):111769.
[16] Simić, S. K., 1983. Graph equations for line graphs and n-distance graphs. Publications de l'Institut de Mathématiques de Beograd 33(47):203–216.