On Three Domination-Based Identification Problems in Block Graphs

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Abstract. The problems of determining the minimum-sized *identifying*, *locating-dominating* and *open locating-dominating codes* of an input graph are special search problems that are challenging from both theoretical and computational viewpoints. In these problems, one selects a dominating set C of a graph G such that the vertices of a chosen subset of $V(G)$ (i.e. either $V(G) \setminus C$ or $V(G)$ itself) are uniquely determined by their neighborhoods in C. A typical line of attack for these problems is to determine tight bounds for the minimum codes in various graph classes. In this work, we present tight lower and upper bounds for all three types of codes for *block graphs* (i.e. diamond-free chordal graphs). Our bounds are in terms of the number of maximal cliques (or *blocks*) of a block graph and the order of the graph. Two of our upper bounds verify conjectures from the literature - with one of them being now proven for block graphs in this article. As for the lower bounds, we prove them to be linear in terms of both the number of blocks and the order of the block graph. We provide examples of families of block graphs whose minimum codes attain these bounds, thus showing each bound to be tight.

Keywords: identifying code · locating-dominating · domination number · block graph · maximal clique · order of a graph · articulation

1 Introduction

For a graph (or network) G that models a facility or a multiprocessor network, detection devices can be placed at its vertices to locate an intruder (like a faulty processor, a fire or a thief). Depending on the features of the detection devices,

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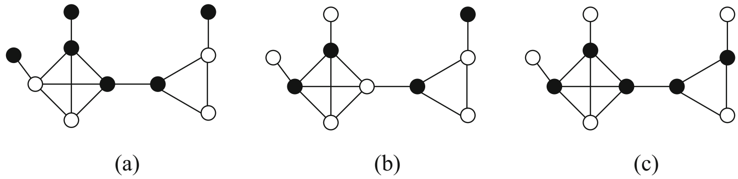


Fig. 1. Examples of (a) ID-code, (b) LD-code and (c) OLD-code. The set of black vertices in each of the three graphs constitute the respective code of the graph.

different types of dominating sets can be used to determine the optimum distributions of these devices across the vertices of G. In this article, we study three problems arising in this context, namely three types of dominating sets - the *identifying codes*, *locating-dominating codes* and *open locating-dominating codes* - of a given graph. Each of these problems has been extensively studied during the last decades. These three types of codes are among the most prominent notions within the larger research area of identification problems in discrete structures pioneered by Rényi [24], with numerous applications, for example in fault-diagnosis [23], biological testing [21] or machine learning [8].

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the set of vertices (also called the *vertex set*) and the set of edges (also called the *edge set*), respectively, of G. The *(open) neighborhood* of a vertex $u \in V(G)$ is the set $N_G(u)$ of all vertices of G adjacent to u; and the set $N_G[u] = \{u\} \cup N_G(u)$ is called the *closed neighborhood* of u.

A vertex subset $C \subseteq V(G)$ is called an *identifying code* [20] (or an *ID-code* for short) of G if

(1) $N_G[u] \cap C \neq \emptyset$ for each vertex u (the property of *domination*); and
(2) $N_G[u] \cap C \neq N_G[v] \cap C$ for all distinct vertices $u, v \in V(G)$ (the property of *closed-separation* in G).

See Fig. 1(a) for an example of an ID-code. A graph G admits an ID-code if and only if G has no *closed-twins* (i.e. a pair of distinct vertices $u, v \in V(G)$ with $N_G[u] = N_G[v]$).

A subset $C \subseteq V(G)$ is called a *locating-dominating code* [26] (or an *LD-code* for short) of G if

(1) $N_G[u] \cap C \neq \emptyset$ for each vertex u (the property of domination); and
(2) $N_G(u) \cap C \neq N_G(v) \cap C$ for all distinct vertices $u, v \in V(G) \setminus C$ (the property of *location* in G).

See Fig. 1(b) for an example of an LD-code. Every graph has an LD-code.

Finally, a subset $C \subseteq V(G)$ is called an *open locating-dominating code* [25] (or an *OLD-code* for short) of G if

(1) $N_G(u) \cap C \neq \emptyset$ for each vertex u (the property of *open-domination*); and
(2) $N_G(u) \cap C \neq N_G(v) \cap C$ for all distinct vertices $u, v \in V(G)$ (the property of *open-separation* in G).

See Fig. 1(c) for an example of an OLD-code. A graph G admits an OLD-code if and only if G has neither isolated vertices nor *open-twins* (i.e. a pair of distinct vertices $u, v \in V(G)$ with $N_G(u) = N_G(v)$).

A graph with no open twins, no closed twins or neither open- nor closed-twins is also called *open-twin-free*, *closed-twin-free* and *twin-free*, respectively. For the rest of this article, we often simply use the word *code* to mean any of the above three ID-, LD- or OLD-codes without distinction. Given a graph G, the *identifying code number* $\gamma^{ID}(G)$ (or *ID-number* for short), the *locating-dominating number* $\gamma^{LD}(G)$ (or *LD-number* for short) and the *open locating-dominating number* $\gamma^{OLD}(G)$ (or *OLD-number* for short) of G are the minimum cardinalities among all ID-codes, LD-codes and OLD-codes, respectively, of G. In other words, for simplicity, for any symbol X $\in$ {ID, LD, OLD}, we have the X-number: $\gamma^X(G) = \min\{|C| : C \text{ is an X-code of } G\}$. In the case that all three codes are addressed together as one unit anywhere in the text, i.e. any specific symbol for X $\in$ {ID, LD, OLD} is irrelevant to the context, we then simply refer to the X-numbers as the *code numbers* of G.

For any two sets A and B, let $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denote the *symmetric difference* between A and B. Then, for a vertex subset $C \subset V(G)$ and distinct vertices $u, v \in V(G)$, if there exists a vertex $w \in (N_G(u) \cap C) \triangle (N_G(v) \cap C)$ (resp. $(N_G[u] \cap C) \triangle (N_G[v] \cap C)$), then both C and the vertex w are said to *open-separate* (resp. *closed-separate*) the vertices u and v (in C).

Known Results. Given a graph G, determining $\gamma^{ID}(G)$, $\gamma^{LD}(G)$ or $\gamma^{OLD}(G)$ is, in general, NP-hard [7,25] and remains so for several graph classes like bipartite graphs [7], split and interval graphs [15] where other hard problems become easy to solve. The problems are also hard to approximate within a factor of $\log |V(G)|$ [10]. As these problems are computationally hard, a typical line of attack is to determine bounds on the code numbers for specific graph classes. Lower bounds for all three code numbers for several graph classes like interval graphs, permutation graphs, cographs [14] and lower bounds for ID-numbers for trees [5], line graphs [12], planar graphs [22] and many others of bounded VC-dimension [6] have been determined. Upper bounds for ID-codes (See e.g. [4,9]), LD-codes (see e.g. [4,13,16]) and OLD-codes (see [18]) for certain graph classes have also been obtained.

Our Work. In this paper, we consider the family of block graphs, a subclass of chordal graphs defined by Harary in [17] (see also [19] for equivalent characterizations). A *block graph* is a graph in which every maximal 2-connected subgraph (or *block*) is complete. Linear-time algorithms to compute all three code numbers in block graphs have been presented in [2]. In this paper, we complement these results by determining tight lower and upper bounds on all three code numbers for block graphs. We give bounds using (i) the number of vertices, i.e. the order of a graph, as has been done for several other classes of graphs; and (ii) the number of blocks of a block graph, a quantity equally relevant to block graphs. In doing so, we also prove the following conjectures.

Conjecture 1 ([1], Conjecture 1). The ID-number of a closed-twin-free block graph is bounded above by the number of blocks in the graph.

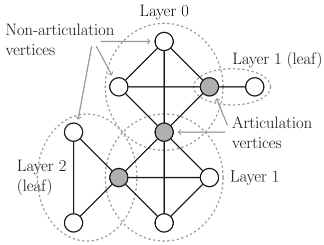


Fig. 2. Example of different layer numbers, articulation vertices (grey) and non-articulation vertices (white) of a connected block graph.

Conjecture 2 ([13,16]*, Conjecture 2).* Every twin-free graph G with no isolated vertices satisfies $\gamma^{LD}(G) \leq \frac{|V(G)|}{2}$.

Terminologies. For a block graph G, let $\mathcal{K}(G)$ denote the set of all blocks of G, i.e. the set of all complete subgraphs of G of maximal order. Note that the vertex sets of any two distinct blocks of G can intersect at a single vertex at most; and any such vertex at the intersection of the vertex sets of two distinct blocks is called an *articulation vertex* of both the blocks. Any vertex that is *not* an articulation vertex, is called a *non-articulation vertex* of G. For a connected block graph G, we fix a *root block* K_0 of G and define a system of assigning numbers to every block of G depending on "how far" the latter is from K_0. So, define a *layer function* $f : \mathcal{K}(G) \to \mathbb{Z}$ *on* G by: $f(K_0) = 0$ and, for any other (*non-root*) block K, define inductively $f(K) = i$ if $V(K) \cap V(K') \neq \emptyset$ for some block K' other than K such that $f(K') = i - 1$. Any block K with $f(K) = i$ is said to be in *Layer* i. See Fig. 2 for a demonstration of the layers.

For a pair of distinct blocks K, K' of G such that their vertex sets intersect and that $f(K) = f(K') + 1$, we call the (only) vertex in the intersection $V(K) \cap V(K')$ *the negative articulation vertex* of K and *a positive articulation vertex* of K'. Note that the root block does not have any negative articulation vertex and every other block has exactly *one* negative articulation vertex. Finally, any block of G that has exactly one articulation vertex is called a *leaf block*, and whereas any block that is *not* a leaf block is called a *non-leaf block* of G.

Structure of the Paper. Sections 2 and 3 of this paper are dedicated to our results on the upper bounds and lower bounds, respectively, on the code numbers of block graphs. We conclude the paper in Sect. 4. In this extended abstract, Theorems 4 and 6 are presented with their proof sketches only, whereas Theorem 9 and all lemmas are presented with their statements only. Theorems 3 and 8, however, are presented with their proofs in full. For the purposes of this abstract, all results marked with ($\star$) are either presented with only their statements or with only sketches of their proofs.

2 Upper Bounds

In this section, we establish upper bounds on the ID-, LD- and OLD-numbers for block graphs. Two of these upper bounds are proving Conjectures 1 and 2.

2.1 Identifying Codes

The number of blocks is as relevant a quantity for block graphs as is the number of vertices for trees. Next, we prove Conjecture 1 to provide an upper bound on $\gamma^{ID}(G)$ for a block graph G in terms of its number of blocks.

Theorem 3. *Let G be a closed-twin-free block graph and let $\mathcal{K}(G)$ be the set of all blocks of G. Then $\gamma^{ID}(G) \leq |\mathcal{K}(G)|$.*

Proof. As the ID-number of a graph is the sum of the ID-numbers of all its components, it is enough to assume that the block graph G is connected. Now, assume by contradiction that there is a closed-twin-free block graph G of minimum order such that $\gamma^{ID}(G) > |\mathcal{K}(G)|$. We also assume that G has at least four vertices since it can be easily checked that the theorem is true for block graphs with at most three vertices. Suppose that $K \in \mathcal{K}(G)$ is a leaf-block of G. Due to the closed-twin-free property of G, one can assume that $V(K) = \{x, y\}$. Without loss of generality, suppose that x is the non-articulation and y the negative articulation vertex of K. Let $G' = G - x$ be the graph obtained by deleting the vertex $x \in V(G)$ (and the edge incident on x) from G. Then G' is a block graph with $|\mathcal{K}(G')| = |\mathcal{K}(G)| - 1$. We now consider the following two cases.

Case 1 (G' is closed-twin-free). By the minimality of the order of G, there is an ID-code C' of G' such that $|C'| \leq |\mathcal{K}(G')| = |\mathcal{K}(G)| - 1$. First, assume that $y \notin C'$. Then by the property of domination of C', there exists a vertex $z \in V(G')$ such that $z \in N_{G'}(y) \cap C'$. We claim that $C = C' \cup \{x\}$ is an ID-code of G. First of all, that C is a dominating set of G is clear from the fact that C' is a dominating set of G'. To prove that C is a closed-separating set of G, we see that x is closed-separated in C from all vertices in $V(G') \setminus \{y\}$ by itself and is closed-separated in C from y by the vertex $z \in C'$. Moreover, all other pairs of distinct vertices closed-separated by C' and are also closed-separated by C. Thus, C, indeed, is an ID-code of G. This implies that $\gamma^{ID}(G) \leq |C| \leq |\mathcal{K}(G)|$, contrary to our assumption.

We therefore assume that $y \in C'$. If again, there exists a vertex $z \in N_{G'}(y) \cap C'$, then by the same reasoning as above, $C = C' \cup \{x\}$ is an ID-code of G. Otherwise, we have $N_{G'}[y] \cap C' = \{y\}$. Now, since G is connected, we have $deg_G(y) > 1$ and therefore, there exists a vertex $w \in N_G(y) \setminus \{x\}$. Then $C = C' \cup \{w\}$ is an ID-code of G. To prove so, we only need to check that C closed-separates x from every vertex in $V(G')$. Now, y closed-separates x from every vertex in $V(G') \setminus \{y, w\}$ in C; w closed-separates x from y in C; and w closed-separates itself from x in C. Moreover, C is clearly also a dominating set of G. Hence, this leads to the same contradiction as before.

Case 2 (G' has closed-twins). Assume that vertices u, $v \in V(G')$ are a pair of closed-twins of G'. Since u and v were not closed-twins in G, it means that x is adjacent to u, say, without loss of generality. This implies that $u = y$. Note that v is then unique with respect to being a closed-twin with y in G'. This is because, if y and some vertex $v'(\neq v) \in V(G')$ were also closed-twins in G', then it would mean that v and v' were closed-twins in G, contrary to our assumption. Now, let $G'' = G' - v$. We claim the following.

Claim 2A. G" is closed-twin-free.

Proof of Claim 2A. Toward a contradiction, if vertices $z, w \in V(G'')$ were a pair of closed-twins in G'', it would mean that $z \in N_{G'}(v)$, without loss of generality, and $w \notin N_{G'}(v)$. This would, in turn, imply that $z \in N_{G'}(y)$ (since y and v are closed-twins in G'). Or, in other words, $y \in N_{G''}(z)$. Now, since z and w are closed-twins in G'', we have $w \in N_{G'}(y)$. Again, by virtue of y and v being closed-twins in G', we have $w \in N_{G'}(v)$, contrary to our assumption. ■

We also note here that the vertices y and v must be from the same block for them to be closed-twins in G'. Thus, G'' is a connected closed-twin-free block graph. Therefore, by the minimality of the order of G, there is an ID-code C'' of G'' such that $|C''| \leq |\mathcal{K}(G'')| < |\mathcal{K}(G)|$. If $y \notin C''$, then we claim that $C = C'' \cup \{x\}$ is an identifying code of G. This is true because, firstly, C is a dominating set of G (note that, by the property of domination of C'' in G'', there exists a vertex $z \in N_{G''}(y) \cap C''$; and since y and v are closed-twins in G', we have $z \in N_G(v) \cap C$). Moreover, x is closed-separated in C from every other vertex in $V(G) \setminus \{y\}$ by x itself; and x and y are closed-separated in C by some vertex in $N_{G''}(y) \cap C''$ that dominates y. The vertices y and v are closed-separated in C by x; y is closed-separated in C'' from all vertices in $V(G'') \setminus \{y\}$ and so is v, since y and v have the same closed neighborhood in G'. Finally, every pair of distinct vertices closed-separated by C'' still remain so by C. Thus, C, indeed, is an ID-code of G. This implies that $\gamma^{ID}(G) \leq |C| \leq |\mathcal{K}(G)|$; again a contradiction.

Let us, therefore, assume that $y \in C''$. This time, we claim that $C = (C' \setminus \{y\}) \cup \{x, v\}$ is an ID-code of G. That C is a dominating set of G is clear. As for the closed-separating property of C, as before, x is closed-separated in C from every vertex in $V(G) \setminus \{y\}$ by x itself; and x and y are closed-separated in C by v. Vertices y and v are closed-separated in C by x; and v and x are closed-separated in C by v. Since y and v have the same closed neighbourhood in G' and since y is closed-separated in C'' from every other vertex in $V(G'')$, both v and y are each closed-separated in C from every vertex in $V(G'') \setminus \{v, y\}$. Finally, every pair of distinct vertices of G'' closed-separated by C'' still remain so by C. This proves that C is an ID-code of G and hence, again, we are led to the contradiction that $\gamma^{ID}(G) \leq |C| \leq |\mathcal{K}(G)|$. This proves the theorem. □

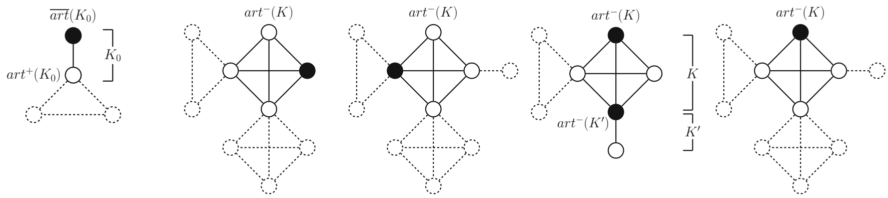
Besides for stars, the upper bound in Theorem 3 is attained by the ID-numbers of thin headless spiders [3]. These graphs, therefore, serve as examples to show that the bound in Theorem 3 is tight.

2.2 Locating-Dominating Codes

In our next result, we prove Conjecture 2 for block graphs.

Theorem 4 ($\star$). *Let G be a twin-free block graph with no isolated vertices. Then we have $\gamma^{LD}(G) \leq \frac{|V(G)|}{2}$.*

Proof (sketch). It is enough to prove the theorem for a connected twin-free block graph G. The proof follows from partitioning the vertex set of G into two *parts*



(a) Rule 1 (b) Rule 2(i) (c) Rule 2(ii) (d) Rule 3(i) (e) Rule 3(ii)

Fig. 3. The rules in the proof of Theorem 4. The symbols $art^+(K)$, $art^-(K)$ and $\overline{art}(K)$ represent the set of all positive articulation, negative articulation and non-articulation vertices, respectively of K. The black and white vertices represent those picked in the sets C^* and D^*, respectively. The blocks with dashed edges represent those that are yet to be analysed for their choices of vertices in C^* and D^*.

C^* and D^* and showing that both the parts are LD-codes of G. So, assign a leaf block of G to be the root block and define a layer function f on G with the root block in Layer 0. Then construct the sets C^* and D^* by the following rules.

(1) The root block is of size 2, as G is twin-free. So, pick the positive articulation vertex of the root block in D^* and the other vertex in C^*. See Fig. 3(a). Next, assume that K is non-root block of G.
(2) Let the negative articulation vertex of K be in D^*. (i) If K has one non-articulation vertex, pick it in C^*. Moreover, pick all positive articulation vertices of K in D^*. (ii) If K has no non-articulation vertices, pick *one* of its positive articulation vertices in C^*, and the rest in D^*. See Figs. 3(b) and 3(c).
(3) Let the negative articulation vertex of K be in C^*. (i) If K has one non-articulation vertex, pick it in D^*. Pick *one* positive articulation vertex (if available) of K in C^*, and the rest in D^*. (ii) If K has no non-articulation vertices, pick all its positive articulation vertices in D^*. See Figs. 3(d) and 3(e).

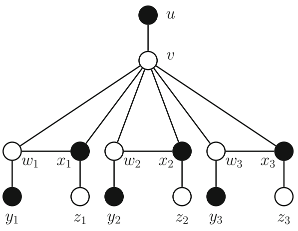
Clearly, the sets C^* and D^* are complements of each other in $V(G)$; and every block of G has at least one vertex in each of them. Thus, both are dominating sets of G. Next, we show that both C^* and D^* are locating sets of G each. We start with C^* and show that any two distinct vertices $u, v \in D^*$ are open-separated in C^*. As G is twin-free, there exist distinct blocks $K, K' \in \mathcal{K}(G)$ such that $u \in V(K)$ and $v \in V(K')$. Then, it is enough to show the following claim.

Claim: Either u or v is an articulation vertex of K or K', respectively.

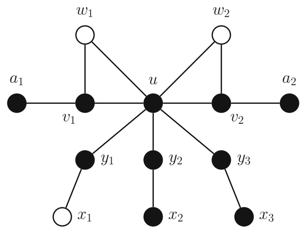
Proof of Claim. Toward a contradiction, let us assume that both u and v are non-articulation vertices of K and K', respectively. Since both $V(K)$ and $V(K')$ have non-empty intersection with C^*, the only non-trivial case to investigate is

$$V(K) \cap C^* = V(K') \cap C^* = V(K) \cap V(K'). \tag{1}$$

Case 1 ($f(K') = f(K) + 1$): Here, K must be a non-root block (by Rule 1) and has its negative articulation vertex in D^*. Since u is a non-articulation vertex of K, by Rule 2(i), u must belong to C^*, a contradiction to our assumption $u \in D^*$.



(a) Block graph H_3 whose LD-number attains the upper bound in Theorem 4.



(b) Block graph $G_{2,3}$ whose OLD-number attains the upper bound in Theorem 6.

Fig. 4. The black vertices constitute a minimum respective code of each graph.

Case 2 ($f(K) = f(K')$). Here, the negative articulation vertices of both K and K' are the same and is in C^*. Assume K to be a non-leaf block (as one of K, K' must be, for G to be twin-free). Since v is a non-articulation vertex of K', by Rule 3(i), K has a positive articulation vertex in C^*, which contradicts (1). ■

This proves the above claim and that C^* is a locating set of G. The proof for D^* being a locating set of G is carried out in a very similar manner. □

The trees whose LD-codes attain the bound in Theorem 4 were characterized in [13]. There are also arbitrarily large connected twin-free block graphs - that are not trees - and whose LD-numbers attain the bound in Theorem 4. Examples of such graphs are, for instance, those of the type in Fig. 4(a). We therefore have the following proposition.

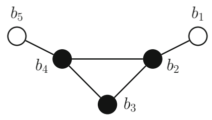
Proposition 5 ($\star$). *There exist arbitrarily large connected twin-free block graphs whose LD-numbers are equal to half the number of vertices.*

2.3 Open Locating-Dominating Codes

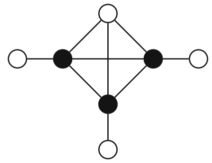
We now focus our attention on upper bounds on OLD-numbers of block graphs.

Theorem 6 ($\star$). *Let G be a connected open-twin-free block graph such that G is neither a copy of P_2 nor of P_4. Let $m_Q(G)$ be the number of non-leaf blocks of G with at least one non-articulation vertex. Then $\gamma^{OLD}(G) \leq |V(G)| - m_Q(G) - 1$.*

Proof (sketch). It is easy to check that the result holds when G is iomorphic to a bull graph (a K_3 with two leaves each adjacent to a distinct vertex of the K_3; see Fig. 5(a)); So, we assume that G is not a bull graph. We define a particular type of "join" of two graphs: Assume G' to be any graph and X to be either a 4-path or a bull graph. For a fixed vertex $q \in V(G')$, we define a new graph $G' \rhd_q X$ to be the graph obtained by identifying a vertex $q \in V(G')$ with an articulation vertex of X. Next, we choose a root block of G according to whether $G \cong G' \rhd_q X$ or $G \ncong G' \rhd_q X$, for some block graph G'. Thereafter, we construct a particular vertex subset $C \subset V(G)$ and, through various case analyses, show that C indeed is an OLD-code of G and is of size $|V(G)| - m_Q(G) - 1$. □



(a) The Bull graph.



(b) Z: Graph of largest size whose minimum OLD-code is a 3-clique.

Fig. 5. The black vertices constitute a minimum respective code of each graph.

Foucaud et al. [11] have shown that, for any open-twin-free graph G with no isolated vertices, $\gamma^{OLD}(G) \leq |V(G)| - 1$ unless G is a special kind of bipartite graph called a *half-graph* (a *half-graph* is a bipartite graph with both parts of the same size, where each part can be ordered so that the open neighbourhoods of consecutive vertices differ by exactly one vertex). Since P_2 and P_4 are the only block graphs that are half-graphs, Theorem 6 is a refinement of their result for block graphs.

We now show that the upper bound on the OLD-numbers for block graphs in Theorem 6 is tight and is attained by arbitrarily large connected block graphs of the type in Fig. 4(b).

Proposition 7 ($\star$). *There exist arbitrarily large connected open-twin free block graphs whose OLD-numbers equal the upper bound in Theorem 6.*

3 Lower Bounds

The general lower bound for the size of an identifying code using the number of vertices is $\gamma^{ID}(G) \geq \lceil \log_2(|V(G)| + 1) \rceil$ [20]. However, to reach this bound, a graph needs to have a large VC-dimension [6] (the *VC-dimension* of a graph G is the size of a largest *shattered set*, that is, a set S of vertices such that for every subset S' of S, some closed neighbourood in G intersects S exactly at S'). Indeed, if a graph has VC-dimension c, then any identifying code has size at least $O(|V(G)|^{1/c})$ [6]. The value $1/c$ is not always tight, see for example the case of line graphs which have VC-dimension at most 4 but for which the tight order for the lower bound is $\Omega(|V(G)|^{1/2})$ [12]. Similar results hold for LD- and OLD-codes (using the same techniques as in [6]). Block graphs have VC-dimension at most 2 (one can check that a shattered set of size 3 would imply the existence of an induced 4-cycle or diamond), and thus, using results from [6], their ID-number is lower bounded by $\Omega(|V(G)|^{1/2})$. In this section, we improve this lower bound to a linear one which is also tight. Our first result of this section is the following.

Theorem 8. *Let G be a connected block graph. Then we have*

$\gamma^{ID}(G) \geq \frac{|V(G)|}{3} + 1$, $\gamma^{LD}(G) \geq \frac{|V(G)|+1}{3}$ *and, for G not isomorphic to Z,* $\gamma^{OLD}(G) \geq \frac{|V(G)|}{3} + 1$*; where Z is the graph K_4 with three leaves each adjacent to a distinct vertex of the K_4.*

See Fig. 5(b) for the graph Z. Extremal cases where these bounds are attained can be constructed as follows (see Fig. 6). Consider the graph with one path on vertices $u_1, ..., u_k$ (the vertices in the code) and attach further vertices as follows.

(1) for an ID-code C: attach a single vertex to each u_i and vertices to the pairs u_i, u_{i+1} for $1 < i < k - 1$,
(2) for an OLD-code C: attach a single vertex to u_1, u_k and each u_i for $2 < i < k - 1$ and vertices to all the pairs u_i, u_{i+1},
(3) for an LD-code C: attach a single vertex to each u_i and vertices to all the pairs u_i, u_{i+1}.

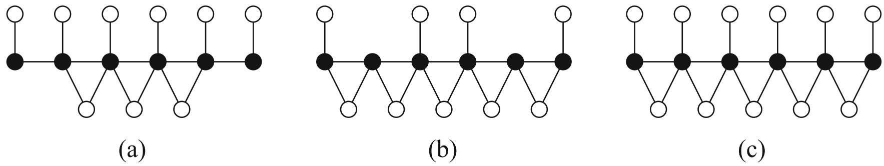


Fig. 6. Extremal cases where the lower bounds are attained, black vertices form a minimum (a) ID-code, (b) OLD-code, (c) LD-code.

Note that the graphs presented here are all the possible extremal cases for ID-codes, whereas further extremal graphs for OLD-codes and for LD-codes exist. If we now consider the parameter $|\mathcal{K}(G)|$, we can use the relation $|V(G)| \geq |\mathcal{K}(G)| + 1$ to obtain a similar lower bound. However, this lower bound can be improved as our next theorem shows.

Theorem 9 ($\star$). *Let G be a connected block graph and $\mathcal{K}(G)$ be the set of all blocks of G. Then we have*

$\gamma^{ID}(G) \geq \frac{3(|\mathcal{K}(G)|+2)}{7}$, $\gamma^{LD}(G) \geq \frac{|\mathcal{K}(G)|+2}{3}$ *and* $\gamma^{OLD}(G) \geq \frac{|\mathcal{K}(G)|+3}{2}$.

To prove Theorems 8 and 9, we introduce the following notations and terminologies. By $n_i(G)$ we shall mean the number of vertices of degree i in a graph G. For a given code C of a connected block graph G, let the subgraph $G[C]$ of G have k components and that $C_1, C_2, ..., C_k$ are all of its components. Note that each C_i is a block graph and so is $G[C]$, therefore. Then, $V(G)$ is partitioned into the four following parts. Starting with $V_1 = C$, we define the other parts.

(1) $V_2 = \{v \in V(G) \setminus V_1 : |N(v) \cap C| = 1\}$,
(2) $V_3 = \{v \in V(G) \setminus V_1 :$ there exist distinct $i, j \leq k$ such that $N(v) \cap C_i \neq \emptyset$ and $N(v) \cap C_j \neq \emptyset\}$, and
(3) $V_4 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Note that, for all $v \in V_4$, $N(v) \cap C \subset V(C_i)$ for some i and that $|N(v) \cap V(C_i)| \geq 2$.

Our next lemmas establish upper bounds on the sizes of V_1, V_2, V_3 and V_4.

Lemma 10 (⋆). *Let G be a connected block graph and C be a code of G. Then following are upper bounds on the size of the vertex subset V_2 of G.*

(1) $|V_2| \leq |C| - n_0(G[C])$ *if C is an ID-code.*
(2) $|V_2| \leq |C|$ *if C is an LD-code.*
(3) $|V_2| \leq |C| - n_1(G[C])$ *if C is an OLD-code.*

Lemma 11 (⋆). *Let G be a connected block graph and C be a code of G such that $G[C]$ has k components. Then, we have $|V_3| \leq k - 1$.*

Lemma 12 (⋆). *Let G be a connected block graph and C be a code of G such that $G[C]$ has k components. Then, we have $|V_4| \leq |C| - k$. In particular,*

(1) $|V_4| \leq |C| - 3k + 2n_0(G[C])$ *if C is an ID-code;*
(2) $|V_4| \leq |\mathcal{K}(G[C])| \leq |C| - 2k_1 - 3k_2 + n_1(G[C])$ *if C is an OLD-code; where $k_1 = |\{C_i : C_i$ is a component of $G[C]$ and $C_i \cong K_3\}|$ and $k_2 = k - k_1$.*

Proof of Theorem 8. Let C be a code of G and that $G[C]$ have k components. We prove the theorem using the relation $|V(G)| = |C| + |V_2| + |V_3| + |V_4|$ and the upper bounds for $|V_2|$ $|V_3|$ and $|V_4|$ in Lemmas 10, 11 and 12, respectively.

If C is an ID-code, then we have

$$\begin{aligned}|V(G)| =& C| + |V_2| + |V_3| + |V_4| \\ \leq& |C| + |C| - n_0(G[C]) + k - 1 + |C| - 3k + 2n_0(G[C]) \\ =& 3\,C| - 2k - 1 + n_0(G[C]).\end{aligned}$$

Now, there must be at least as many components of $G[C]$ as there are isolated vertices in $G[C]$, i.e. we have $k \geq n_0(G[C])$. This implies that $|V(G)| \leq 3|C| - k - 1$. Thus, for $k \geq 2$, the result holds. Moreover, when $k = 1$, we must have $n_0(G[C]) = 0$ and so, again, the result holds.

If C is an LD-code, then the result holds because we have

$$|V(G)| = |C| + |V_2| + |V_3| + |V_4| \leq |C| + |C| + k - 1 + |C| - k = 3|C| - 1.$$

Finally, if C is an OLD-code, then we have

$$\begin{aligned}|V(G)| =& |C| + |V_2| + |V_3| + |V_4| \\ \leq& |C| + |C| - n_1(G[C]) + k_1 + k_2 - 1 + |C| - 2k_1 - 3k_2 + n_1(G[C]) \\ =& 3\,C| - k_1 - 2k_2 - 1.\end{aligned}$$

This implies that the result holds when either $k_1 \geq 2$ or when $k_2 \geq 1$.

If however, $k_1 = 1$ and $k_2 = 0$, then $G[C]$ is isomorphic to K_3. If $n \leq 6$, the result holds since $|V(G)| \leq 3|C| - 3$. Thus, let $|V(G)| = 7$. Since no vertex $v \in V(G) \setminus C$ can be adjacent to exactly two vertices of C (or else, the last vertex of C would not be open-separated from v), each vertex in $V(G) \setminus C$ must be adjacent to either exactly one or all three vertices of C. Therefore, $G \cong Z$ in Fig. 5(b). Hence, the result holds for all connected block graphs $G \ncong Z$. □

The proof of Theorem 9 is by using similar bounding techniques as in the proof of Theorem 8, but on $|\mathcal{K}(G)|$ instead of $|V(G)|$. Using $|\mathcal{K}(G)| = |E(G)| = |V(G)|-1$ for any tree G, the bounds in Theorem 9 are equivalent to the known lower bounds for trees in terms of number of vertices (see [5] for ID-codes, [26] for LD-codes and [25] for OLD-codes). In fact, the code numbers of infinite families of trees attain the three bounds in Theorem 9.

4 Conclusion

Block graphs form a subclass of chordal graphs for which all three considered identification problems can be solved in linear time [2]. In this paper, we complemented this result by presenting tight lower and upper bounds for the optimum sizes of all the three types of codes. We gave bounds in terms of both the number of vertices - as it has been done for several other classes of graphs - and also the number of blocks of G - a parameter more fitting for block graphs. In particular, we verified Conjecture 1 on an upper bound on the ID-number for block graphs from [1] and Conjecture 2 on the LD-numbers from [16] for the special case of block graphs. Moreover, we addressed the questions to find block graphs where the provided lower and upper bounds are attained.

The structural properties of block graphs have enabled us to prove interesting bounds for the three considered problems. It would be further interesting to study other structured classes in a similar way. It would also be interesting to prove Conjecture 2 for a larger class of graphs, like chordal graphs, for example.

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