Relation Between Broadcast Domination and Multipacking Numbers on Chordal Graphs

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Abstract. For a graph $G = (V, E)$ with a vertex set V and an edge set E, a function $f : V \rightarrow \{0, 1, 2, ..., diam(G)\}$ is called a *broadcast* on G. For each vertex $u \in V$, if there exists a vertex v in G (possibly, $u = v$) such that $f(v) > 0$ and $d(u, v) \leq f(v)$, then f is called a dominating broadcast on G. The cost of the dominating broadcast f is the quantity $\sum_{v \in V} f(v)$. The minimum cost of a dominating broadcast is the broadcast domination number of G, denoted by $\gamma_b(G)$.

A multipacking is a set $S \subseteq V$ in a graph $G = (V, E)$ such that for every vertex $v \in V$ and for every integer $r \geq 1$, the ball of radius r around v contains at most r vertices of S, that is, there are at most r vertices in S at a distance at most r from v in G. The multipacking number of G is the maximum cardinality of a multipacking of G and is denoted by $\operatorname{mp}(G)$.

It is known that $\operatorname{mp}(G) \leq \gamma_b(G)$ and that $\gamma_b(G) \leq 2\operatorname{mp}(G) + 3$ for any graph G, and it was shown that $\gamma_b(G) - \operatorname{mp}(G)$ can be arbitrarily large for connected graphs (as there exist infinitely many connected graphs G where $\gamma_b(G)/\operatorname{mp}(G) = 4/3$ with $\operatorname{mp}(G)$ arbitrarily large). For strongly chordal graphs, it is known that $\operatorname{mp}(G) = \gamma_b(G)$ always holds.

We show that, for any connected chordal graph G, $\gamma_b(G) \leq \left\lceil \frac{3}{2} \operatorname{mp}(G) \right\rceil$. We also show that $\gamma_b(G) - \operatorname{mp}(G)$ can be arbitrarily large for connected chordal graphs by constructing an infinite family of connected chordal graphs such that the ratio $\gamma_b(G)/\operatorname{mp}(G) = 10/9$, with $\operatorname{mp}(G)$ arbitrarily large. This result shows that, for chordal graphs, we cannot improve the bound $\gamma_b(G) \leq \left\lceil \frac{3}{2} \operatorname{mp}(G) \right\rceil$ to a bound in the form $\gamma_b(G) \leq c_1 \cdot \operatorname{mp}(G) + c_2$, for any constant $c_1 < 10/9$ and c_2.

Keywords: Chordal graph · Multipacking · Dominating broadcast

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1 Introduction

Covering and packing problems are fundamental in graph theory and algorithms [6]. In this paper, we study two dual covering and packing problems called *broadcast domination* and *multipacking*. The broadcast domination problem has a natural motivation in telecommunication networks: imagine a network with radio emission towers, where each tower can broadcast information at any radius r for a cost of r. The goal is to cover the whole network by minimizing the total cost. The multipacking problem is its natural packing counterpart and generalizes various other standard packing problems. Unlike many standard packing and covering problems, these two problems involve arbitrary distances in graphs, which makes them challenging. The goal of this paper is to study the relation between these two parameters in the class of chordal graphs, which are those graphs that do not contain any induced cycle of a length at least 4.

For a graph $G = (V, E)$ with a vertex set V, an edge set E and the diameter $diam(G)$, a function $f : V \to \{0, 1, 2, ..., diam(G)\}$ is called a *broadcast* on G. Suppose G be a graph with a broadcast f. Let $d(u, v) =$ the length of a shortest path joining the vertices u and v in G. We say $v \in V$ is a *tower* of G if $f(v) > 0$. Suppose $u, v \in V$ (possibly, $u = v$) such that $f(v) > 0$ and $d(u, v) \leq f(v)$, then we say v *broadcasts* (or *dominates*) u and u *hears* the broadcast from v.

For each vertex $u \in V$, if there exists a vertex v in G (possibly, $u = v$) such that $f(v) > 0$ and $d(u, v) \leq f(v)$, then f is called a *dominating broadcast* on G. The *cost* of the broadcast f is the quantity $\sigma(f)$, which is the sum of the weights of the broadcasts over all vertices in G. So, $\sigma(f) = \sum_{v \in V} f(v)$. The minimum cost of a dominating broadcast in G (taken over all dominating broadcasts) is the *broadcast domination number* of G, denoted by $\gamma_b(G)$. So, $\gamma_b(G) = \min_{f \in D(G)} \sigma(f) = \min_{f \in D(G)} \sum_{v \in V} f(v)$, where $D(G) =$ set of all dominating broadcasts on G.

Suppose f is a dominating broadcast with $f(v) \in \{0, 1\}$ $\forall v \in V(G)$, then $\{v \in V(G) : f(v) = 1\}$ is a *dominating set* on G. The minimum cardinality of a dominating set is the *domination number* which is denoted by $\gamma(G)$.

An *optimal broadcast* or *optimal dominating broadcast* on a graph G is a dominating broadcast with a cost equal to $\gamma_b(G)$. A dominating broadcast is *efficient* if no vertex hears a broadcast from two different vertices. So, no tower can hear a broadcast from another tower in an efficient broadcast. There is a theorem that says, for every graph there is an optimal efficient dominating broadcast [7]. Define a ball of radius r around v by $N_r[v] = \{u \in V(G) : d(v, u) \leq r\}$. Suppose $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let c and x be the vectors indexed by (i, k) where $v_i \in V(G)$ and $1 \leq k \leq diam(G)$, with the entries $c_{i,k} = k$ and $x_{i,k} = 1$ when $f(v_i) = k$ and $x_{i,k} = 0$ when $f(v_i) \neq k$. Let $A = [a_{j,(i,k)}]$ be a matrix with the entries

$$a_{j,(i,k)} = \begin{cases} 1 & \text{if } v_j \in N_k[v_i] \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the broadcast domination number can be expressed as an integer linear program:

$$\gamma_b(G) = \min\{c.x : Ax \geq \mathbf{1}, x_{i,k} \in \{0,1\}\}.$$

The *maximum multipacking problem* is the dual integer program of the above problem. Moreover, multipacking is a generalization of packing problems. A *multipacking* is a set $M \subseteq V$ in a graph $G = (V, E)$ such that $|N_r[v] \cap M| \leq r$ for each vertex $v \in V(G)$ and for every integer $r \geq 1$. The *multipacking number* of G is the maximum cardinality of a multipacking of G and it is denoted by $\mathrm{mp}(G)$. A *maximum multipacking* is a multipacking M of a graph G such that $|M| = \mathrm{mp}(G)$. If M is a multipacking, we define a vector y with the entries $y_j = 1$ when $v_j \in M$ and $y_j = 0$ when $v_j \notin M$. So,

$$\mathrm{mp}(G) = \max\{y.\mathbf{1} : yA \leq c, y_j \in \{0,1\}\}.$$

Broadcast domination is a generalization of domination problems and multipacking is a generalization of packing problems. Erwin [8,9] introduced broadcast domination in his doctoral thesis in 2001. Multipacking was introduced in Teshima's Master's Thesis [15] in 2012 (also see [3,6,7,14]). For general graphs, an optimal dominating broadcast can be found in polynomial-time $O(n^6)$ [12]. The same problem can be solved in linear time for trees [4]. However, until now, there is no known polynomial-time algorithm to find a maximum multipacking of general graphs (the problem is also not known to be NP-hard). However, polynomial-time algorithms are known for trees and more generally, strongly chordal graphs [4]. See [10] for other references concerning algorithmic results on the two problems.

It is known that $\mathrm{mp}(G) \leq \gamma_b(G)$, since broadcast domination and multipacking are dual problems [5]. It is known that $\gamma_b(G) \leq 2\,\mathrm{mp}(G) + 3$ [1] and it is a conjecture that $\gamma_b(G) \leq 2\,\mathrm{mp}(G)$ for every graph G [1]. Hartnell and Mynhardt [11] constructed a family of connected graphs such that the difference $\gamma_b(G) - \mathrm{mp}(G)$ can be arbitrarily large and in fact, for which the ratio $\gamma_b(G)/\mathrm{mp}(G) = 4/3$. Therefore, for general connected graphs,

$$\frac{4}{3} \leq \lim_{\mathrm{mp}(G)\to\infty} \sup\left\{\frac{\gamma_b(G)}{\mathrm{mp}(G)}\right\} \leq 2.$$

A natural question comes to mind: What is the optimal bound on this ratio for other graph classes? It is known that $\gamma_b(G) = \mathrm{mp}(G)$ holds for strongly chordal graphs [4]. Thus, a natural class to study is the class of chordal graphs.

In this paper, we establish an improved relation between $\gamma_b(G)$ and $\mathrm{mp}(G)$ for connected chordal graphs by showing that $\gamma_b(G) \leq \left\lceil \frac{3}{2}\,\mathrm{mp}(G) \right\rceil$. We then construct a family of connected chordal graphs such that the difference $\gamma_b(G) - \mathrm{mp}(G)$ can be arbitrarily large and the ratio $\gamma_b(G)/\mathrm{mp}(G) = 10/9$ for every member G of that family. Thus, for chordal connected graphs G, we have:

$$\frac{10}{9} \leq \lim_{\mathrm{mp}(G)\to\infty} \sup\left\{\frac{\gamma_b(G)}{\mathrm{mp}(G)}\right\} \leq \frac{3}{2}.$$

We also make a connection with the *fractional* versions of the two concepts, as introduced in [2].

In Sect. 2, we show that for any connected chordal graph G, $\gamma_b(G) \leq \left\lceil \frac{3}{2}\operatorname{mp}(G) \right\rceil$ and there is a polynomial-time algorithm to construct a multipacking of G of size at least $\left\lceil \frac{2\operatorname{mp}(G)-1}{3} \right\rceil$. In Sect. 3, we prove our main result which says that the difference $\gamma_b(G) - \operatorname{mp}(G)$ can be arbitrarily large for connected chordal graphs, and we conclude in Sect. 4.

2 An Inequality Linking Broadcast Domination and Multipacking Numbers of Chordal Graphs

In this section, we use results from the literature to show that the general bound connecting multipacking number and broadcast domination number can be improved for chordal graphs.

Theorem 1 **([11]).** *If G is a connected graph of order at least 2 having diameter d and multipacking number $\operatorname{mp}(G)$, where $P = v_0, \ldots, v_d$ is a diametral path of G, then the set $M = \{v_i : i \equiv 0 \ (mod\ 3), i = 0, 1, \ldots, d\}$ is a multipacking of G of size $\left\lceil \frac{d+1}{3} \right\rceil$ and $\left\lceil \frac{d+1}{3} \right\rceil \leq \operatorname{mp}(G)$.*

Theorem 2 **([9,15]).** *If G is a connected graph of order at least 2 having radius r, diameter d, multipacking number $\operatorname{mp}(G)$, broadcast domination number $\gamma_b(G)$ and domination number $\gamma(G)$, then $\operatorname{mp}(G) \leq \gamma_b(G) \leq min\{\gamma(G), r\}$.*

Theorem 3 **([13]).** *If G is a connected chordal graph with radius r and diameter d, then $2r \leq d + 2$.*

Proposition 1. *If G is a connected chordal graph, then $\gamma_b(G) \leq \left\lceil \frac{3}{2}\operatorname{mp}(G) \right\rceil$.*

Proof. From Theorem 1, $\left\lceil \frac{d+1}{3} \right\rceil \leq \operatorname{mp}(G)$ which implies that $d \leq 3\operatorname{mp}(G) - 1$. Moreover, from Theorem 2 and Theorem 3, $\gamma_b(G) \leq r \leq \left\lfloor \frac{d+2}{2} \right\rfloor \leq \left\lfloor \frac{(3\operatorname{mp}(G)-1)+2}{2} \right\rfloor = \left\lfloor \frac{3}{2}\operatorname{mp}(G) + \frac{1}{2} \right\rfloor$. Therefore, $\gamma_b(G) \leq \left\lfloor \frac{3}{2}\operatorname{mp}(G) + \frac{1}{2} \right\rfloor = \left\lceil \frac{3}{2}\operatorname{mp}(G) \right\rceil$. □

The proof of Proposition 1 has the following algorithmic application.

Proposition 2. *If G is a connected chordal graph, there is a polynomial-time algorithm to construct a multipacking of G of size at least $\left\lceil \frac{2\operatorname{mp}(G)-1}{3} \right\rceil$.*

Proof. If $P = v_0, \ldots, v_d$ is a diametrical path of G, then the set $M = \{v_i : i \equiv 0 \ (mod\ 3), i = 0, 1, \ldots, d\}$ is a multipacking of G of size $\left\lceil \frac{d+1}{3} \right\rceil$ by Theorem 1. We can construct M in polynomial-time since we can find a diametral path of a graph G in polynomial-time. Moreover, from Theorem 1, Theorem 2 and Theorem 3, $\left\lceil \frac{2\operatorname{mp}(G)-1}{3} \right\rceil \leq \left\lceil \frac{2r-1}{3} \right\rceil \leq \left\lceil \frac{d+1}{3} \right\rceil \leq \operatorname{mp}(G)$. □

Example 1. *The connected chordal graph S_3 (Fig. 1) has $\operatorname{mp}(S_3) = 1$ and $\gamma_b(S_3) = 2$. So, here $\gamma_b(S_3) = \left\lceil \frac{3}{2}\operatorname{mp}(S_3) \right\rceil$.*

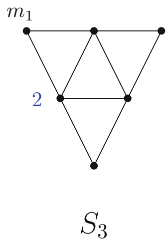

Fig. 1. S_3 is a connected chordal graph with $\gamma_b(S_3) = 2$ and $\text{mp}(S_3) = 1$

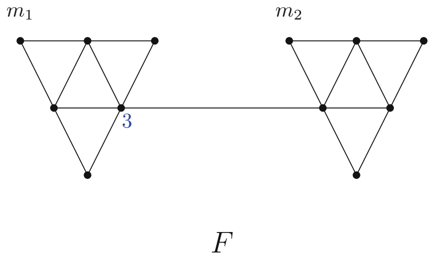

Fig. 2. F is a connected chordal graph with $\gamma_b(F) = 3$ and $\text{mp}(F) = 2$

Example 2. *The connected chordal graph F (Fig. 2) has* $\text{mp}(F) = 2$ *and* $\gamma_b(F) = 3$. *So, here* $\gamma_b(F) = \left\lceil \frac{3}{2}\,\text{mp}(F) \right\rceil$.

Example 3. *The connected chordal graph H (Fig. 3) has* $\text{mp}(H) = 4$ *and* $\gamma_b(H) = 6$. *So, here* $\gamma_b(H) = \left\lceil \frac{3}{2}\,\text{mp}(H) \right\rceil$.

We could not find an example of connected chordal graph with $\text{mp}(G) = 3$ and $\gamma_b(G) = \left\lceil \frac{3}{2}\,\text{mp}(G) \right\rceil = 5$.

3 Unboundedness of the Gap Between Broadcast Domination and Multipacking Numbers of Chordal Graphs

Here we prove that the difference between broadcast domination number and multipacking number of connected chordal graphs can be arbitrarily large. We state the theorem formally below.

Theorem 4. *The difference* $\gamma_b(G) - \text{mp}(G)$ *can be arbitrarily large for connected chordal graphs.*

Consider the graph G_1 as in Fig 4. Let B_1 and B_2 be two isomorphic copies of G_1. Join $b_{1,21}$ of B_1 and $b_{2,1}$ of B_2 by an edge (Fig. 5 and 6). We denote this new graph by G_2 (Fig. 5). In this way, we form G_k by joining k isomorphic copies of G_1: $B_1, B_2, \cdots, B_k$ (Fig. 6). Here B_i is joined with B_{i+1} by joining $b_{i,21}$ and $b_{i+1,1}$. We say that B_i is the i-th block of G_k. B_i is an induced subgraph of G_k as

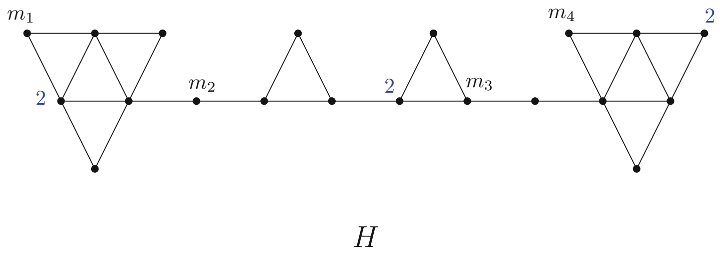

Fig. 3. H is a connected chordal graph with $\gamma_b(H) = 6$ and $\text{mp}(H) = 4$

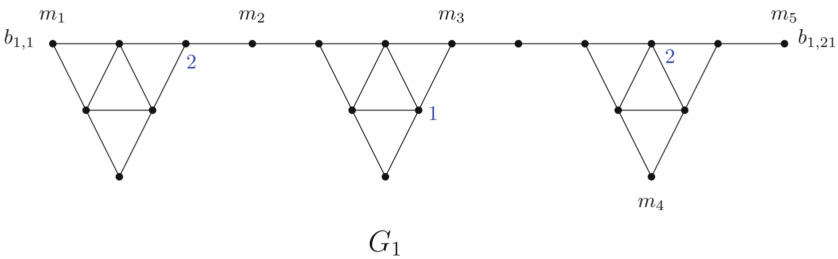

Fig. 4. G_1 is a connected chordal graph with $\gamma_b(G_1) = 5$ and $\text{mp}(G_1) = 5$. $M_1 = \{m_i : 1 \leq i \leq 5\}$ is a multipacking of size 5.

given by $B_i = G_k[\{b_{i,j} : 1 \leq j \leq 21\}]$. Similarly, for $1 \leq i \leq 2k-1$, we define $B_i \cup B_{i+1}$, induced subgraph of G_{2k}, as $B_i \cup B_{i+1} = G_{2k}[\{b_{i,j}, b_{i+1,j} : 1 \leq j \leq 21\}]$. We prove Theorem 4 by establishing that $\gamma_b(G_{2k}) = 10k$ and $\text{mp}(G_{2k}) = 9k$. Then we can say, for all natural numbers k, $\gamma_b(G_{2k}) - \text{mp}(G_{2k}) = k$, so the difference can be arbitrarily large.

3.1 Proof of Theorem 4

Our proof of Theorem 4 is accomplished through a set of lemmas which are stated and proved below. We begin by observing a basic fact about multipacking in a graph. We formally state it in Lemma 1 for ease of future reference.

Lemma 1. *Suppose M is a multipacking in a graph G. If $u, v \in M$ and $u \neq v$, then $d(u, v) \geq 3$.*

Proof. If $d(u, v) = 1$, then $u, v \in N_1[v] \cap M$, then M cannot be a multipacking. So, $d(u, v) \neq 1$. If $d(u, v) = 2$, then there exists a common neighbour w of u and v. So, $u, v \in N_1[w] \cap M$, then M cannot be a multipacking. So, $d(u, v) \neq 2$. Therefore, $d(u, v) > 2$. □

Lemma 2. $\text{mp}(G_{2k}) \geq 9k$*, for each positive integer k.*

Proof. Consider the set $M_{2k} = \{b_{2i-1,1}, b_{2i-1,7}, b_{2i-1,13}, b_{2i-1,18}, b_{2i-1,21}, b_{2i,4}, b_{2i,8}, b_{2i,14}, b_{2i,18} : 1 \leq i \leq k\}$ (Fig. 6) of size $9k$. We want to show that M_{2k} is a

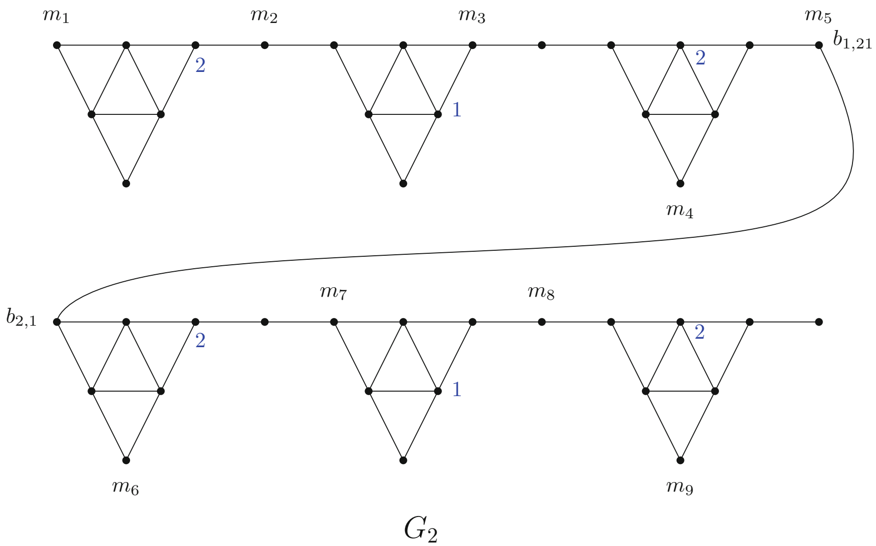

Fig. 5. Graph G_2 with $\gamma_b(G_2) = 10$ and $\text{mp}(G_2) = 9$. $M = \{m_i : 1 \leq i \leq 9\}$ is a multipacking of size 9.

multipacking of G_{2k}. So, we have to prove that, $|N_r[v] \cap M_{2k}| \leq r$ for each vertex $v \in V(G_{2k})$ and for every integer $r \geq 1$. We prove this statement using induction on r. It can be checked that $|N_r[v] \cap M_{2k}| \leq r$ for each vertex $v \in V(G_{2k})$ and for each $r \in \{1, 2, 3, 4\}$. Now assume that the statement is true for $r = s$, we want to prove that, it is true for $r = s+4$. Observe that, $|(N_{s+4}[v] \setminus N_s[v]) \cap M_{2k}| \leq 4$ for every vertex $v \in V(G_{2k})$. Therefore, $|N_{s+4}[v] \cap M_{2k}| \leq |N_s[v] \cap M_{2k}| + 4 \leq s + 4$. So, the statement is true. Therefore, M_{2k} is a multipacking of G_{2k}. So, $\text{mp}(G_{2k}) \geq |M_{2k}| = 9k$. □

Lemma 3. $\text{mp}(G_1) = 5$.

Proof. $V(G_1) = N_3[b_{1,7}] \cup N_2[b_{1,17}]$. Suppose M is a multipacking on G_1 such that $|M| = \text{mp}(G_1)$. So, $|M \cap N_3[b_{1,7}]| \leq 3$ and $|M \cap N_2[b_{1,17}]| \leq 2$. Therefore, $|M \cap (N_3[b_{1,7}] \cup N_2[b_{1,17}])| \leq 5$. So, $|M \cap V(G)| \leq 5$, that implies $|M| \leq 5$. Let $M_1 = \{b_{1,1}, b_{1,7}, b_{1,13}, b_{1,18}, b_{1,21}\}$. Since $|N_r[v] \cap M| \leq r$ for each vertex $v \in V(G_1)$ and for every integer $r \geq 1$, so M_1 is a multipacking of size 5. Then $5 = |M_1| \leq |M|$. So, $|M| = 5$. Therefore, $\text{mp}(G_1) = 5$. □

So, now we have $\text{mp}(G_1) = 5$. Using this fact we prove that $\text{mp}(G_2) = 9$.

Lemma 4. $\text{mp}(G_2) = 9$.

Proof. As mentioned before, $B_i = G_k[\{b_{i,j} : 1 \leq j \leq 21\}]$, $1 \leq i \leq 2$. So, B_1 and B_2 are two blocks in G_2 which are isomorphic to G_1. Let M be a multipacking of G_2 with size $\text{mp}(G_2)$. So, $|M| \geq 9$ by Lemma 2. Since M is a multipacking of

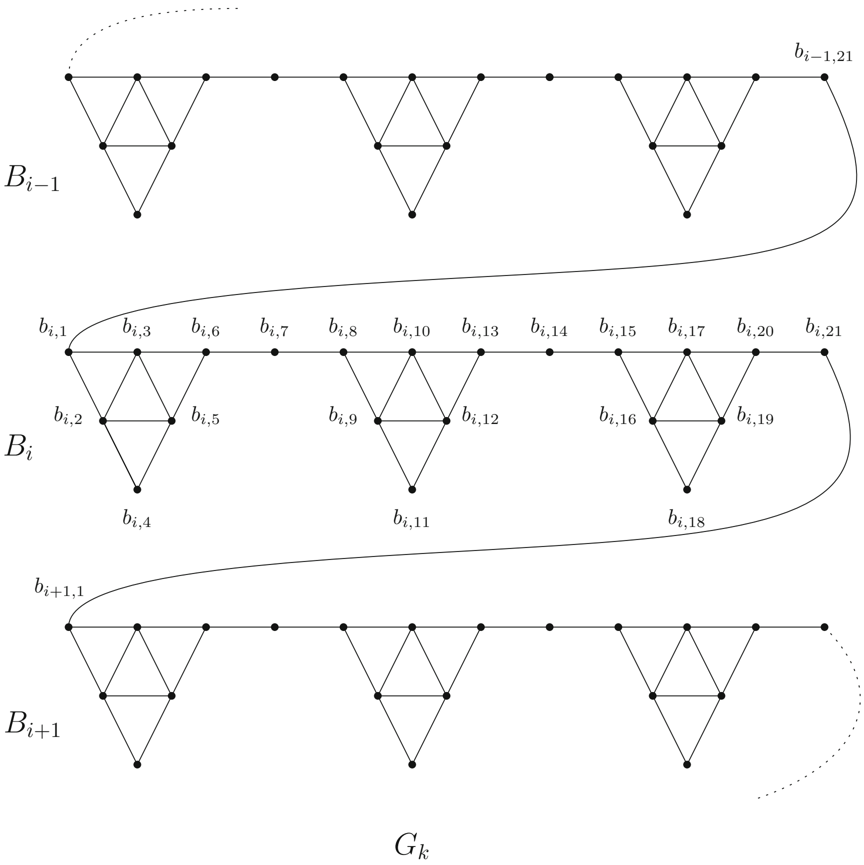

Fig. 6. Partial depiction of graph G_k.

G_2, so $M \cap V(B_1)$ and $M \cap V(B_2)$ are multipackings of B_1 and B_2, respectively. Let $M \cap V(B_1) = M_1$ and $M \cap V(B_2) = M_2$. Since $B_1 \cong G_1$ and $B_2 \cong G_1$, so $\mathrm{mp}(B_1) = 5$ and $\mathrm{mp}(B_2) = 5$ by Lemma 3. This implies $|M_1| \leq 5$ and $|M_2| \leq 5$. Since $V(B_1) \cup V(B_2) = V(G_2)$ and $V(B_1) \cap V(B_2) = \phi$, so $M_1 \cap M_2 = \phi$ and $|M| = |M_1| + |M_2|$. Therefore, $9 \leq |M| = |M_1| + |M_2| \leq 10$. So, $9 \leq |M| \leq 10$.

We establish this lemma by using contradiction on $|M|$. In the first step, we prove that if $|M_1| = 5$, then the particular vertex $b_{1,21} \in M_1$. Using this, we can show that $|M_2| \leq 4$. In this way we show that $|M| \leq 9$.

For the purpose of contradiction, we assume that $|M| = 10$. So, $|M_1|+|M_2| = 10$, and also $|M_1| \leq 5$, $|M_2| \leq 5$. Therefore, $|M_1| = |M_2| = 5$.

Claim 4.1. If $|M_1| = 5$, then $b_{1,21} \in M_1$.

Proof of Claim. Suppose $b_{1,21} \notin M$. Let $S = \{b_{1,7}, b_{1,14}\}$, $S_1 = \{b_{1,r} : 1 \leq r \leq 6\}$, $S_2 = \{b_{1,r} : 8 \leq r \leq 13\}$, $S_3 = \{b_{1,r} : 15 \leq r \leq 20\}$. If $u, v \in S_t$, then $d(u, v) \leq 2$, this holds for each $t \in \{1, 2, 3\}$. So, by Lemma 1, u, v together cannot be in a multipacking. Therefore $|S_t \cap M_1| \leq 1$ for $t = 1, 2, 3$ and $|S \cap M_1| \leq |S| = 2$. Now, $5 = |M_1| = |M_1 \cap [V(G_1) \setminus \{b_{1,21}\}| = |M_1 \cap (S \cup S_1 \cup S_2 \cup S_3)| = |(M_1 \cap S) \cup (M_1 \cap S_1) \cup (M_1 \cap S_2) \cup (M_1 \cap S_3)| \leq |M_1 \cap S| + |M_1 \cap S_1| + |M_1 \cap S_2| + |M_1 \cap S_3| \leq 2 + 1 + 1 + 1 = 5$. Therefore, $|S_t \cap M_1| = 1$ for $t = 1, 2, 3$ and $|S \cap M_1| = 2$, so $b_{1,7}, b_{1,14} \in M_1$. Since $|S_2 \cap M_1| = 1$, there exists $w \in S_2 \cap M_1$. Then $N_2[b_{1,10}]$ contains three vertices $b_{1,7}, b_{1,14}, w$ of M_1, which is not possible. So, this is a contradiction. Therefore, $b_{1,21} \in M_1$. ◁

Claim 4.2. If $|M_1| = 5$, then $|M_2| \leq 4$.

Proof of Claim. Let $S' = \{b_{2,14}, b_{2,21}\}$, $S_4 = \{b_{2,r} : 1 \leq r \leq 6\}$, $S_5 = \{b_{2,r} : 8 \leq r \leq 13\}$, $S_6 = \{b_{2,r} : 15 \leq r \leq 20\}$. By Lemma 1, $|S_t \cap M_2| \leq 1$ for $t = 4, 5, 6$ and also $|S' \cap M_2| \leq |S'| = 2$.

Observe that, if $S_4 \cap M_2 \neq \phi$, then $b_{2,7} \notin M_2$ (i.e. if $b_{2,7} \in M_2$, then $S_4 \cap M_2 = \phi$). [Suppose not, then $S_4 \cap M_2 \neq \phi$ and $b_{2,7} \in M_2$, so, there exists $u \in S_4 \cap M_2$. Then $N_2[b_{2,3}]$ contains three vertices $b_{1,21}, b_{2,7}, u$ of M, which is not possible. This is a contradiction].

Suppose $S_4 \cap M_2 \neq \phi$, then $b_{2,7} \notin M_2$. Now, $5 = |M_2| = |M_2 \cap [V(B_2) \setminus \{b_{2,7}\}]| = |M_2 \cap (S' \cup S_4 \cup S_5 \cup S_6)| = |(M_2 \cap S') \cup (M_2 \cap S_4) \cup (M_2 \cap S_5) \cup (M_2 \cap S_6)| \leq |M_2 \cap S'| + |M_2 \cap S_4| + |M_2 \cap S_5| + |M_2 \cap S_6| \leq 2 + 1 + 1 + 1 = 5$. Therefore $|S_t \cap M_2| = 1$ for $t = 4, 5, 6$ and $|S' \cap M_2| = 2$. Since $|M_2 \cap S_6| = 1$, so there exists $u_1 \in M_2 \cap S_6$. Then $N_2[b_{2,17}]$ contains three vertices $b_{2,14}, b_{2,21}, u_1$ of M_2, which is not possible. So, this is a contradiction.

Suppose $S_4 \cap M_2 = \phi$, then either $b_{2,7} \in M_2$ or $b_{2,7} \notin M_2$. First consider $b_{2,7} \notin M_2$, then $5 = |M_2| = |M_2 \cap (S' \cup S_5 \cup S_6)| = |(M_2 \cap S') \cup (M_2 \cap S_5) \cup (M_2 \cap S_6)| \leq |M_2 \cap S'| + |M_2 \cap S_5| + |M_2 \cap S_6| \leq 2 + 1 + 1 = 4$. So, this is a contradiction. And if $b_{2,7} \in M_2$, then $5 = |M_2| = |M_2 \cap (S' \cup S_5 \cup S_6 \cup \{b_{2,7}\})| = |(M_2 \cap S') \cup (M_2 \cap S_5) \cup (M_2 \cap S_6) \cup (M_2 \cap \{b_{2,7}\})| \leq |M_2 \cap S'| + |M_2 \cap S_5| + |M_2 \cap S_6| + |M_2 \cap \{b_{2,7}\}| \leq 2 + 1 + 1 + 1 = 5$. Therefore $|S_t \cap M_2| = 1$ for $t = 5, 6$ and $|S' \cap M_2| = 2$. Since $|M_2 \cap S_6| = 1$, so there exists $u_2 \in M_2 \cap S_6$. Then $N_2[b_{2,17}]$ contains three vertices $b_{2,14}, b_{2,21}, u_2$ of M_2, which is not possible. So, this is a contradiction. So, $|M_1| = 5 \implies |M_2| \leq 4$. ◁

Recall that for contradiction, we assume $|M| = 10$, which implies $|M_2| = 5$. In the proof of the above claim, we established $|M_2| \leq 4$, which in turn contradicts our assumption. So, $|M| \neq 10$. Therefore, $|M| = 9$. □

Notice that graph G_{2k} has k copies of G_2. Moreover, we have $\mathrm{mp}(G_2) = 9$. If $\mathrm{mp}(G_{2k}) > 9k$, then we will use the Pigeonhole principle to show that $\mathrm{mp}(G_{2k}) = 9k$.

Lemma 5. $\mathrm{mp}(G_{2k}) = 9k$, *for each positive integer* k.

Proof. For $k = 1$ it is true by Lemma 4. Moreover, we know $\mathrm{mp}(G_{2k}) \geq 9k$ by Lemma 2. Suppose $k > 1$ and assume $\mathrm{mp}(G_{2k}) > 9k$. Let $\hat{M}$ be a multipacking of G_{2k} such that $|\hat{M}| > 9k$. Let $\hat{B}_j$ be a subgraph of G_{2k} defined as $\hat{B}_j = B_{2j-1} \cup B_{2j}$ where $1 \leq j \leq k$. So, $V(G_{2k}) = \bigcup_{j=1}^{k} V(\hat{B}_j)$ and $V(\hat{B}_p) \cap V(\hat{B}_q) = \phi$ for all $p \neq q$ and $p, q \in \{1, 2, 3, \ldots, k\}$. Since $|\hat{M}| > 9k$, so by the Pigeonhole principle there exists a number $j \in \{1, 2, 3, \ldots, k\}$ such that $|\hat{M} \cap \hat{B}_j| > 9$. Since $\hat{M} \cap \hat{B}_j$ is a multipacking of $\hat{B}_j$, so $\text{mp}(\hat{B}_j) > 9$. But $\hat{B}_j \cong G_2$ and $\text{mp}(G_2) = 9$ by Lemma 4, so $\text{mp}(\hat{B}_j) = 9$, which is a contradiction. Therefore, $\text{mp}(G_{2k}) = 9k$. □

R. C. Brewster and L. Duchesne [2] introduced fractional multipacking in 2013 (also see [16]). Suppose G is a graph with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $w : V(G) \to [0, \infty)$ is a function. So, $w(v)$ is a weight on a vertex $v \in V(G)$. Let $w(S) = \sum_{u \in S} w(u)$ where $S \subseteq V(G)$. We say w is a *fractional multipacking* of G, if $w(N_r[v]) \leq r$ for each vertex $v \in V(G)$ and for every integer $r \geq 1$. The *fractional multipacking number* of G is the value $\max\limits_{w} w(V(G))$ where w is any fractional multipacking and it is denoted by $mp_f(G)$. A *maximum fractional multipacking* is a fractional multipacking w of a graph G such that $w(V(G)) = mp_f(G)$. If w is a fractional multipacking, we define a vector y with the entries $y_j = w(v_j)$. So,

$$mp_f(G) = \max\{y.\mathbf{1} : yA \leq c, y_j \geq 0\}.$$

So, this is a linear program which is the dual of the linear program $\min\{c.x : Ax \geq \mathbf{1}, x_{i,k} \geq 0\}$. Let,

$$\gamma_{b,f}(G) = \min\{c.x : Ax \geq \mathbf{1}, x_{i,k} \geq 0\}.$$

Using the strong duality theorem for linear programming, we can say that

$$\text{mp}(G) \leq mp_f(G) = \gamma_{b,f}(G) \leq \gamma_b(G).$$

Lemma 6. *If k is a positive integer, then $mp_f(G_k) \geq 5k$.*

Proof. We define a function $w : V(G_k) \to [0, \infty)$ where $w(b_{i,1}) = w(b_{i,6}) = w(b_{i,7}) = w(b_{i,8}) = w(b_{i,13}) = w(b_{i,14}) = w(b_{i,15}) = w(b_{i,20}) = w(b_{i,21}) = \frac{1}{3}$ and $w(b_{i,4}) = w(b_{i,11}) = w(b_{i,18}) = \frac{2}{3}$ for each $i \in \{1, 2, 3, \ldots, k\}$ (Fig. 7). So, $w(G_k) = 5k$. We want to show that w is a fractional multipacking of G_k. So, we have to prove that $w(N_r[v]) \leq r$ for each vertex $v \in V(G_k)$ and for every integer $r \geq 1$. We prove this statement using induction on r. It can be checked that $w(N_r[v]) \leq r$ for each vertex $v \in V(G_k)$ and for each $r \in \{1, 2, 3, 4\}$. Now assume that the statement is true for $r = s$, we want to prove that it is true for $r = s + 4$. Observe that, $w(N_{s+4}[v] \setminus N_s[v]) \leq 4$, $\forall v \in V(G_k)$. Therefore, $w(N_{s+4}[v]) \leq w(N_s[v]) + 4 \leq s + 4$. So, the statement is true. So, w is a fractional multipacking of G_k. Therefore, $mp_f(G_k) \geq 5k$. □

Lemma 7. *If k is a positive integer, then $mp_f(G_k) = \gamma_b(G_k) = 5k$.*

Proof. Define a broadcast f on G_k as $f(b_{i,j}) = \begin{cases} 2 & \text{if } 1 \leq i \leq k \text{ and } j = 6, 17 \\ 1 & \text{if } 1 \leq i \leq k \text{ and } j = 12 \\ 0 & \text{otherwise} \end{cases}$.

Here f is an efficient dominating broadcast and $\sum_{v \in V(G_k)} f(v) = 5k$. So, $\gamma_b(G_k) \leq 5k$, $\forall k \in \mathbb{N}$. So, by the strong duality theorem and Lemma 6, $5k \leq mp_f(G_k) = \gamma_{b,f}(G_k) \leq \gamma_b(G_k) \leq 5k$. Therefore, $mp_f(G_k) = \gamma_b(G_k) = 5k$. □

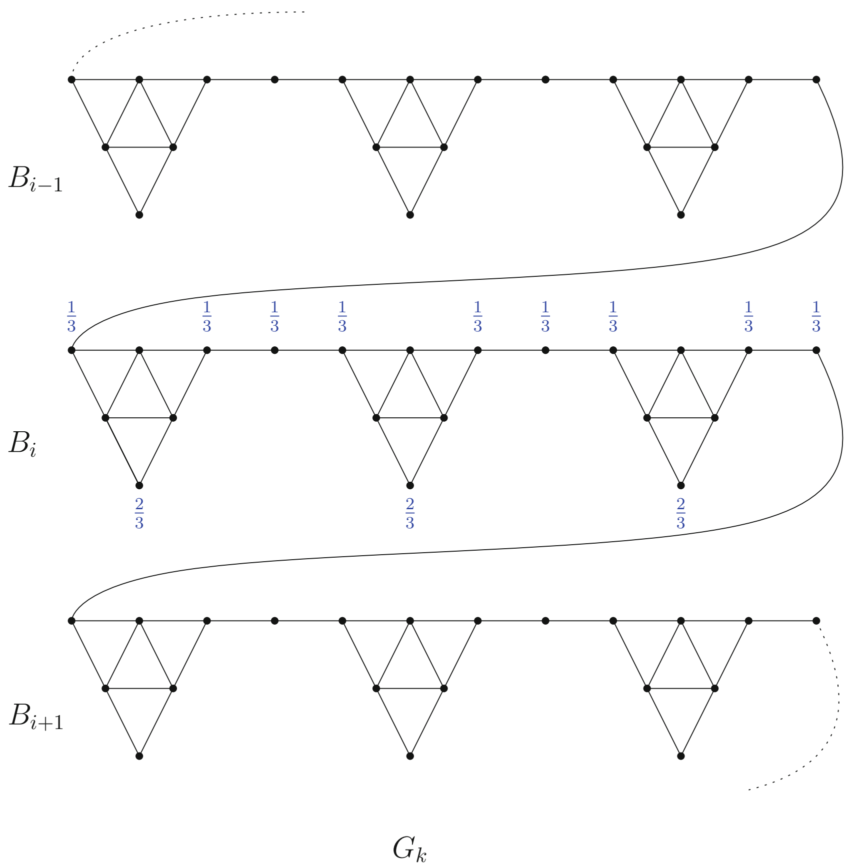

Fig. 7. Fractional multipacking of G_k.

So, $\gamma_b(G_{2k}) = 10k$ by Lemma 7 and $\mathrm{mp}(G_{2k}) = 9k$ by Lemma 5. So, we can say that for all positive integers k, $\gamma_b(G_{2k}) - \mathrm{mp}(G_{2k}) = k$. Therefore, this proves Theorem 4. So, the difference $\gamma_b(G) - \mathrm{mp}(G)$ can be arbitrarily large for connected chordal graphs.

Corollary 1. *The difference $mp_f(G) - \mathrm{mp}(G)$ can be arbitrarily large for connected chordal graphs.*

Proof. We get $mp_f(G_{2k}) = 10k$ by Lemma 7 and $\mathrm{mp}(G_{2k}) = 9k$ by Lemma 5. Therefore, $mp_f(G_{2k}) - \mathrm{mp}(G_{2k}) = k$ for all positive integers k. □

Corollary 2. *For every integer $k \geq 1$, there is a connected chordal graph G_{2k} with $\mathrm{mp}(G_{2k}) = 9k$, $mp_f(G_{2k})/\mathrm{mp}(G_{2k}) = 10/9$ and $\gamma_b(G_{2k})/\mathrm{mp}(G_{2k}) = 10/9$.*

Corollary 3. *For connected chordal graphs G,*

$$\frac{10}{9} \leq \lim_{\mathrm{mp}(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{\mathrm{mp}(G)} \right\} \leq \frac{3}{2}.$$

4 Conclusion

We have shown that the bound $\gamma_b(G) \leq 2\,\mathrm{mp}(G) + 3$ for general graphs G can be improved to $\gamma_b(G) \leq \left\lceil \frac{3}{2}\,\mathrm{mp}(G) \right\rceil$ for connected chordal graphs. It is known that for strongly chordal graphs, $\gamma_b(G) = \mathrm{mp}(G)$, we have shown that this is not the case for connected chordal graphs. Even more, $\gamma_b(G) - \mathrm{mp}(G)$ can be arbitrarily large for connected chordal graphs, as we have constructed infinitely many connected chordal graphs G where $\gamma_b(G)/\,\mathrm{mp}(G) = 10/9$ and $\mathrm{mp}(G)$ is arbitrarily large.

It remains an interesting open problem to determine the best possible value of $\lim_{\mathrm{mp}(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{\mathrm{mp}(G)} \right\}$ for general connected graphs and for chordal connected graphs. This problem could also be studied for other interesting graph classes.

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