Smallest $C_{2\ell+1}$-critical graphs of odd-girth $2k+1$☆

Laurent Beaudou [a], Florent Foucaud [b,c,d], Reza Naserasr [e,*]

[a] *National Research University, Higher School of Economics, 3 Kochnovsky Proezd, Moscow, Russia*
[b] *LIMOS, CNRS UMR 6158, Université Clermont Auvergne, Aubière, France*
[c] *Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence, France*
[d] *Univ. Orléans, INSA Centre Val de Loire, LIFO EA 4022, F-45067 Orléans, France*
[e] *Université de Paris, CNRS, IRIF, F-75006, Paris, France*

ABSTRACT

Given a graph H, a graph G is called H-critical if G does not admit a homomorphism to H, but any proper subgraph of G does. Observe that K_{k-1}-critical graphs are the standard k-(colour)-critical graphs. We consider questions of extremal nature previously studied for k-critical graphs and generalize them to H-critical graphs. After complete graphs, the next natural case to consider for H is that of the odd-cycles. Thus, given integers ℓ and k, $\ell \geq k$, we ask: what is the smallest order of a $C_{2\ell+1}$-critical graph of odd-girth at least $2k+1$? Denoting this value by $\eta(k, C_{2\ell+1})$, we show that $\eta(k, C_{2\ell+1}) = 4k$ for $1 \leq \ell \leq k \leq \frac{3\ell+i-3}{2}$ ($2k = i$ mod 3) and that $\eta(3, C_5) = 15$. The latter means that a smallest graph of odd-girth 7 not admitting a homomorphism to the 5-cycle is of order 15. Computational work shows that there are exactly eleven such graphs on 15 vertices of which only two are C_5-critical.

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1. Introduction

A *k*-critical graph is a graph which is *k*-chromatic, but any of its proper subgraphs is $(k - 1)$-colourable. Extremal questions on critical graphs are a rich source of research in graph theory. Many well-known results and conjectures deal with this subject, see for example [5,9–11,23]. Typical questions are for example:

Problem. What is the smallest possible order of a *k*-critical graph having a certain property, such as low clique number, high girth or high odd-girth?

For example, the existence of graphs of high girth and high chromatic number, proved by Erdős [7], is a starting point of this problem. This fact implies that each of the above questions has a finite answer. The specific question of the smallest 4-critical graph without a triangle has received considerable attention: Grötzsch built a graph on 11 vertices which is triangle-free and not 3-colourable. Harary [14] showed that any such graph must have at least 11 vertices, and Chvátal [4] showed that the Grötzsch graph is the only one on 11 vertices.

In the context of colouring, it is of interest to consider the *odd-girth*, that is, the size of a smallest odd-cycle of a graph, rather than the girth, since every graph with no odd-cycle is 2-colourable. Extending the example of the Grötzsch graph,

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Mycielski [18] introduced a construction, now known as the Mycielski construction, to increase the chromatic number without increasing the clique number. A generalization of this construction is used to build 4-critical graphs of high odd-girth, more precisely the *generalized Mycielski construction on* C_{2k+1}, denoted $M_k(C_{2k+1})$, is a graph of odd-girth $2k + 1$. Several authors (starting with Stiebitz [27] and Payan [24]) showed that $M_k(C_{2k+1})$ is 4-chromatic for any $k \geq 2$ and in fact 4-critical, thus providing an upper bound of $2k^2 + k + 1$ for the minimum order $\eta(k)$ of a 4-critical graph of odd-girth at least $2k + 1$. (We will later generalize the function η using the notion of homomorphism.) We refer to [13,21,28,29] for several other proofs. Among these authors, Ngoc and Tuza [21] asked whether the upper bound $\eta(k) \leq 2k^2 + k + 1$ is essentially optimal. The first quadratic lower bound for $\eta(k)$ was established in [22] by considering a maximum induced bipartite subgraph. Improving on this approach, Jiang in [16] proved $\eta(k) \geq (k-1)^2 + 2$. Recently, Esperet and Stehlík [8] proved that $2k^2 + k + 1$ is the correct bound for the special case of graphs where any two odd-cycles have a common vertex.[1]

The current work is a first step towards generalizing these extremal questions for k-critical graphs to H-critical graphs, defined using the terminology of homomorphisms. A *homomorphism* of a graph G to a graph H is a vertex-mapping that preserves adjacency, i.e., a mapping $\psi : V(G) \to V(H)$ such that if x and y are adjacent in G, then $\psi(x)$ and $\psi(y)$ are adjacent in H. If there exists a homomorphism of G to H, we may write $G \to H$ and we may say that G is H-colourable, or that G maps to H. In the study of homomorphisms, it is usual to work with *the core* of a graph, that is, a minimal subgraph which admits a homomorphism from the graph itself. It is not difficult to show that a core of any graph is unique up to isomorphism. A graph is said to be *a core* if it admits no homomorphism to a proper subgraph. We refer to the book [15] for a reference on these notions.

Homomorphisms generalize proper vertex-colourings. Indeed, a homomorphism of G to K_k is equivalent to a k-colouring of G. However, the extension of the notion of colour-criticality to a homomorphism-based one is not much studied. As defined by Catlin [3], for a graph H, (we may assume H is a core), a graph G is said to be H-*critical* if G does not have a homomorphism to H but any proper subgraph of G does. We thus have the following.

Observation 1. *A graph G is k-critical if and only if it is K_{k-1}-critical.*

This gives a large number of interesting extremal questions. By Observation 1, these questions are well-studied when H is a complete graph. The next most accessible family of graphs H to be considered is the one of odd-cycles. For example, the extremal question of the minimum number of edges of an H-critical graph on n vertices has been the subject of extensive studies for $H = K_n$, see [17] for results and references on this subject. Recently, extensions of this study to C_5-critical graphs [6] and to C_7-critical graphs [25] were performed. We note, furthermore, that the $(2k + 1)$-colouring problem is captured by the C_{2k+1}-colouring problem through a basic graph operation. For further details and extension of the study to signed graphs we refer to [19].

In this work, we are interested in the order of a smallest $C_{2\ell+1}$-critical graph having the homomorphism property that its odd-girth is at least a given value.[2] We generally denote by $\eta(k, H)$ the smallest order of a graph of odd-girth at least k that has no homomorphism to H, and we ask the following.

Problem 2. Given positive integers k, ℓ, what is the smallest order $\eta(k, C_{2\ell+1})$ of a $C_{2\ell+1}$-critical graph of odd-girth at least $2k + 1$?

In this paper, we study Problem 2 when $\ell \geq 2$. As we discuss in Section 2, it follows from a theorem of Gerards [12] that $\eta(k, C_{2\ell+1}) \geq 4k$ whenever $\ell \leq k$, and $\eta(k, C_{2k+1}) = 4k$. We prove (in Section 3) that, surprisingly, $\eta(k, C_{2\ell+1}) = 4k$ whenever $\ell \leq k \leq \frac{3\ell+i-3}{2}$ (with $2k = i \mod 3$). We then prove (in Section 4) that $\eta(3, C_5) = 15$. We conclude with further research questions in the last section. Table 1 summarizes the known bounds for Problem 2 and small values of k and ℓ.

Note that the value of $\eta(3, C_5)$ indeed was the initial motivation of this work. In [1], we use the fact that $\eta(3, C_5) = 15$ to prove that if a graph B of odd-girth 7 has the property that any series–parallel graph of odd-girth 7 admits a homomorphism to B, then B has at least 15 vertices. The 15 vertices solution then strengthen at the same time results on circular and fractional colouring of this family of graphs. One may expect similar properties for other extremal solutions.

2. Preliminaries

This section is devoted to the introduction of useful preliminary notions and results.

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[1] In fact, Ngoc and Tuza [21] equivalently asked for the value of the smallest constant c such that the odd-girth of any 4-chromatic graph of order n is at most $(c + o(1))\sqrt{n}$. In this language, the tentative value $\eta(k) = 2k^2 + k + 1$ translates to a value of $\sqrt{2n - 7/4} + 1/2$ for the odd-girth of a 4-chromatic graph of order n; this is also the formulation used in [8]. The result of Jiang [16] is formulated as a strict upper bound of $2\sqrt{n} + 3$ for the odd-girth of a 4-chromatic graph of order n.

[2] This is a homomorphism property because any graph G has odd-girth at least $2k + 1$ if and only if C_{2k-1} does not map to G.

Table 1
Known values and bounds on the smallest order $\eta(k, C_{2\ell+1})$ of a $C_{2\ell+1}$-critical graph of odd-girth at least $2k+1$. Bold values are proved in this paper.

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
$\ell = 1$	4	11 [14]	**15**–22 [Theorem 11]–[24]	**17**–37 [Corollary 12]–[24]	**20**–56 [Corollary 7]–[24]	27–79 [16–24]	38–106 [16–24]	51–137 [16–24]
$\ell = 2$	3	**8** [Corollary 5]	**15** [Theorem 11]	**17**–37 [Corollary 12]–[24]	**20**–56 [Corollary 7]–[24]	**24**–79 [Corollary 7]–[24]	**28**–106 [Corollary 7]–[24]	**32**–137 [Corollary 7]–[24]
$\ell = 3$	3	5	**12** [Corollary 5]	**16** [Theorem 9]	**20**–56 [Corollary 7]–[24]	**24**–79 [Corollary 7]–[24]	**28**–106 [Corollary 7]–[24]	**32**–137 [Corollary 7]–[24]
$\ell = 4$	3	5	7	**16** [Corollary 5]	**20** [Theorem 9]	**24**–79 [Corollary 7]–[24]	**28**–106 [Corollary 7]–[24]	**32**–137 [Corollary 7]–[24]
$\ell = 5$	3	5	7	9	**20** [Corollary 5]	**24** [Theorem 9]	**28** [Theorem 9]	**32**–137 [Corollary 7]–[24]
$\ell = 6$	3	5	7	9	11	**24** [Corollary 5]	**28** [Theorem 9]	**32** [Theorem 9]
$\ell = 7$	3	5	7	9	11	13	**28** [Corollary 5]	**32** [Theorem 9]
$\ell = 8$	3	5	7	9	11	13	15	**32** [Corollary 5]

Circular chromatic number. We recall some basic notions related to circular colourings. For a survey on the matter, consult [30]. Given two integers p and q with $gcd(p, q) = 1$, the *circular clique* $C(p, q)$ is the graph on vertex set $\{0, \dots, p - 1\}$ with i adjacent to j if and only if $q \le |i - j| \le p - q$.

A homomorphism of a graph G to $C(p, q)$ is called a (p, q)-colouring, and the *circular chromatic number* of G, denoted $\chi_c(G)$, is the smallest rational $\frac{p}{q}$ such that G has a (p, q)-colouring. Since $C(p, 1)$ is the complete graph K_p, we have $\chi_c(G) \le \chi(G)$. On the other hand $C(2\ell + 1, \ell)$ is the cycle $C_{2\ell+1}$. Thus $C_{2\ell+1}$-colourability is about deciding whether $\chi_c(G) \le 2 + \frac{1}{\ell}$.

It is a well-known fact that $C(p, q) \to C(r, s)$ if and only if $\frac{p}{q} \le \frac{r}{s}$ (e.g. see [30]), in particular we will use the fact that $C(12, 5)$ has a homomorphism to $C(5, 2)$, that is, C_5.

Odd-K_4's and a theorem of Gerards. The following notion will be central in our proofs. An *odd-K_4* is a subdivision of the complete graph K_4 where each of the four triangles of K_4 has become an odd-cycle [12]. Furthermore, we call it a $(2k+1)$-*odd-K_4* if each such cycle has length exactly $2k + 1$. Since subdivided triangles are the only odd-cycles of an odd-K_4, the odd-girth of a $(2k+1)$-odd-K_4 is $2k+1$. The following is an easy fact about odd-K_4's whose proof we leave as an exercise.

Proposition 3. *Let K be an odd-K_4 of odd-girth at least $2k+1$. Then, K has order at least $4k$, with equality if and only if K is a $(2k+1)$-odd-K_4. Furthermore, in the last case any two disjoint edges of K_4 are subdivided the same number of times when constructing K.*

A $(2k+1)$-odd-K_4 is, more precisely, referred to as an (a, b, c)-odd-K_4 if three edges of a triangle of K_4 are subdivided into paths of length a, b and c respectively (by Proposition 3 this is true for all four triangles). Note that while the terms "odd-K_4" or "$(2k+1)$-odd-K_4" refer to many non-isomorphic graphs, an (a, b, c)-odd-K_4 $(a + b + c = 2k + 1)$ is unique up to a relabelling of vertices. See [20, Section 4.2.1] for some homomorphism properties of odd-K_4's.

An *odd-K_3^2* is a graph obtained from three disjoint odd-cycles and three disjoint paths (possibly of length 0) joining each pair of cycles [12]. Thus, in such a graph, any two of the three cycles have at most one vertex in common (if the path joining them has length 0). Hence, an odd-K_3^2 of odd-girth at least $2k + 1$ has order at least $6k$.

In a graph G, a *thread* is a path in G where the internal vertices have degree 2 in G. For example, in an odd-K_4 or in an odd-K_3^2, the paths of degree 2-vertices joining two vertices of degree at least 3 are threads.

Theorem 4 (Gerards [12]). *If G has neither an odd-K_4 nor an odd-K_3^2 as a subgraph, then it admits a homomorphism to its shortest odd-cycle.*

We have the following corollary of Proposition 3 and Theorem 4.

Corollary 5. *For any positive integer k, we have $\eta(k, C_{2k+1}) = 4k$.*

Proof. Consider a C_{2k+1}-critical graph G of odd-girth $2k + 1$. It follows from Theorem 4 that G contains either an odd-K_4, or an odd-K_3^2. If it contains the latter, then G has at least $6k$ vertices. Otherwise, G must contain an odd-K_4 of odd-girth at least $2k + 1$, and then by Proposition 3, G has at least $4k$ vertices. This shows that $\eta(k, C_{2k+1}) \ge 4k$.

Moreover, no $(2k+1)$-odd-K_4 admits a homomorphism to C_{2k+1}. Indeed, assume by contradiction that there is such a homomorphism φ of a $(2k+1)$-odd-K_4, G, to C_{2k+1}, and let x, y, z, t be the vertices of degree 3 of G. Consider the $(2k+1)$-cycle C that passes through x, y, z. This cycle must be mapped surjectively to C_{2k+1} by φ. Thus, t must be identified with some vertex of C by φ. If it is identified with a vertex of the xy-thread of G, then the $(2k+1)$-cycle going through x, y, t would not be mapped surjectively to C by φ, a contradiction. The same argument applies to the xz-thread and the yz-thread, showing that t cannot be mapped and thus φ does not exist.

Since any $(2k+1)$-odd-K_4 has order $4k$ by Proposition 3, the latter paragraph shows that $\eta(k, C_{2k+1}) \leq 4k$. □

Since $C_{2\ell+3}$ maps to $C_{2\ell+1}$ and by transitivity of homomorphisms, a graph with no homomorphism to $C_{2\ell+1}$ also has no homomorphism to $C_{2\ell+3}$. Thus:

Observation 6. *Let k, ℓ be two positive integers. We have $\eta(k, C_{2\ell+1}) \geq \eta(k, C_{2\ell+3})$.*

We obtain this immediate consequence of Corollary 5 and Observation 6:

Corollary 7. *For any two integers k, ℓ with $k \geq \ell \geq 1$, we have $\eta(k, C_{2\ell+1}) \geq 4k$.*

3. Rows of Table 1: $\eta(k, C_{2\ell+1})$ for fixed ℓ

In this section, we study the behaviour of $\eta(k, C_{2\ell+1})$ when ℓ is a fixed value, that is, the behaviour of each row of Table 1. Note again that whenever $\ell \geq k+1$, we have $\eta(k, C_{2\ell+1}) = 2k+1$. As mentioned before, the first row (i.e. $\ell = 1$) is about the smallest order of a 4-critical graph of odd-girth $2k+1$ and we know $\eta(k, C_3) = \Theta(k^2)$ [22].

It is not difficult to observe that for fixed ℓ, the function $\eta(k, C_{2\ell+1})$ is strictly increasing, in fact with a little bit of effort we can even show the following.

Proposition 8. *For $k \geq \ell$, we have $\eta(k+1, C_{2\ell+1}) \geq \eta(k, C_{2\ell+1}) + 2$.*

Proof. Let G be a $C_{2\ell+1}$-critical graph of odd-girth $2k+3$ and order $\eta(k+1, C_{2\ell+1})$. Consider any $(2k+3)$-cycle $v_0 \cdots v_{2k+2}$ of G, and build a smaller graph by identifying v_0 with v_2 and v_1 with v_3. It is not difficult to check that the resulting graph has odd-girth exactly $2k+1$ and does not map to $C_{2\ell+1}$ (otherwise, G would), proving the claim. □

While it is likely that for a fixed ℓ, the value of $\eta(k, C_{2\ell+1})$ grows quadratically in terms of k, we show, somewhat surprisingly, that at least just after the threshold of $k = \ell$, the function $\eta(k, C_{2\ell+1})$ only increases by 4 when ℓ increases by 1, implying that Proposition 8 cannot be improved much in this formulation. More precisely, we have the following theorem.

Theorem 9. *For any $k, \ell \geq 3$ and $\ell \leq k \leq \frac{3\ell+i-3}{2}$ (where $2k = i$ mod 3), we have $\eta(k, C_{2\ell+1}) = 4k$.*

To prove this theorem, we give a family of $C_{2\ell+1}$-critical odd-K_4's which are of odd-girth $2k+1$. This is done in the next theorem, after which we give a proof of Theorem 9.

Theorem 10. *Let $p \geq 3$ be an integer. If p is odd, any (a, b, c)-odd-K_4 with $(a, b, c) \in \{(p-1, p-1, p), (p, p, p)\}$ has no homomorphism to C_{2p+1}. If p is even, any $(p-1, p, p)$-odd-K_4 has no homomorphism to C_{2p+1}.*

Proof. Let $(a, b, c) \in \{(p-1, p-1, p), (p, p, p), (p-1, p, p)\}$ and let K be an (a, b, c)-odd-K_4. Let t, u, v, w be the vertices of degree 3 in K with the tu-thread of length a, the uv-thread of length b and the tv-thread of length c. We now distinguish two cases depending on the parity of p and the values of (a, b, c).

Case 1. Assume that p is odd and $(a, b, c) \in \{(p-1, p-1, p), (p, p, p)\}$. By contradiction, we assume that there is a homomorphism h of K to C_{2p+1}. Then, the cycle C_{tvw} formed by the union of the tv-thread, the vw-thread and the tw-thread is an odd-cycle of length $a+b+c$. Therefore, its mapping by h to C_{2p+1} must be *onto*. Thus, u has the same image by h as some vertex u' of C_{tvw}. Note that u' is not one of t, v or w, indeed by identifying u with any of these vertices we obtain a graph containing an odd-cycle of length p or $2p-1$; thus, this identification cannot be extended to a homomorphism to C_{2p+1}. Therefore, u' is an internal vertex of one of the three maximal threads in C_{tvw}. Let C_u be the odd-cycle of length $a+b+c$ containing u and u'. After identifying u and u', C_u is transformed into two cycles, one of them being odd. If $(a, b, c) = (p, p, p)$, then C_u has length $3p$. Then, the two newly created cycles have length at least $p+1$, and thus at most $2p-1$. If $(a, b, c) = (p-1, p-1, p)$, then C_u has length $3p-2$, and the two cycles have length at least p and at most $2p-2$. In both cases, we have created an odd-cycle of length at most $2p-1$. Hence, this identification cannot be extended to a homomorphism to C_{2p+1}, a contradiction.

Case 2. Assume that p is even and $(a, b, c) = (p-1, p, p)$, and that h is a homomorphism of K to C_{2p+1}. Again, the image of C_{tvw} by h is onto, and u has the same image as some vertex u' of C_{tvw}. If $u' = t$, identifying u and u' produces an odd $(p-1)$-cycle, a contradiction. If $u' \in \{v, w\}$, then we get a $(2p-1)$-cycle, a contradiction. Thus, u' is an internal vertex of one of the three maximal threads in C_{tvw}. Let C_u be the odd-cycle of length $3p-1$ containing u and u'. As in Case 1, after identifying u and u', C_u is transformed into two cycles, each of length at least p and at most $2p-1$; one of them is odd, a contradiction. □
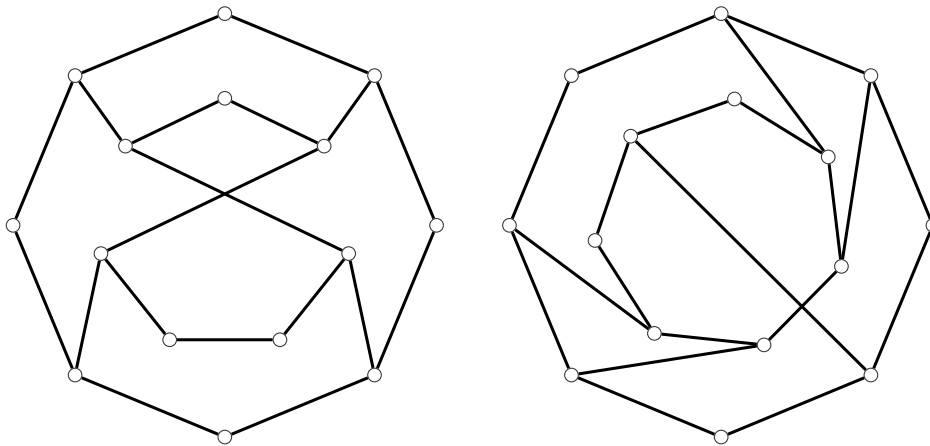
Fig. 1. The two C_5-critical graphs of order 15 and odd-girth 7.

We note that Theorem 10 is tight, in the sense that if p is odd and $(a, b, c) \in \{(p - 1, p - 1, p), (p, p, p)\}$ or if p is even and $(a, b, c) = (p - 1, p, p)$, then an (a, b, c)-odd-K_4 has a homomorphism to C_{2p-1}.

We can now prove Theorem 9.

Proof of Theorem 9. By Corollary 7, we know that $\eta(k, C_{2\ell+1}) \geq 4k$. We now prove the upper bound. Recall that $\eta(k, C_{2\ell+1}) \leq \eta(k, C_{2\ell-1})$. If $2k \equiv 0 \pmod 3$, then $p = \frac{2k+3}{3}$ is an odd integer, and $p \leq \ell$. By Theorem 10, a $(p - 1, p - 1, p)$-odd-K_4, which has order $6p - 6 = 4k$, has no homomorphism to C_{2p+1}, and thus $\eta(k, C_{2\ell+1}) \leq \eta(k, C_{2p+1}) \leq 4k$. Similarly, if $2k \equiv 1 \pmod 3$, then $p = \frac{2k+2}{3}$ is an even integer, and $p \leq \ell$. By Theorem 10, a $(p - 1, p, p)$-odd-K_4, which has order $6p - 4 = 4k$, has no homomorphism to C_{2p+1}, and thus $\eta(k, C_{2\ell+1}) \leq \eta(k, C_{2p+1}) \leq 4k$. Finally, if $2k \equiv 2 \pmod 3$, then $p = \frac{2k+1}{3}$ is an odd integer, and $p \leq \ell$. By Theorem 10, a (p, p, p)-odd-K_4, which has order $6p - 2 = 4k$, has no homomorphism to C_{2p+1}, and thus $\eta(k, C_{2\ell+1}) \leq \eta(k, C_{2p+1}) \leq 4k$. □

4. The value of $\eta(3, C_5)$

We now determine $\eta(3, C_5)$, which is not covered by Theorem 10. By Corollary 7, we know that $\eta(3, C_5) \geq 12$. In fact, we will show that $\eta(3, C_5) = 15$.

Examples of graphs of odd-girth 7 on 15 vertices are given in Fig. 1. In fact, using a computer search, Gordon Royle (private communication, 2016) has verified that these two graphs are the only two C_5-critical graphs of order 15 and odd-girth 7.

One may add a few edges to each of these two graphs without creating a shorter odd cycle. G. Royle reports that there are eleven non isomorphic graphs built in this way. For the most symmetric of all these graphs two different presentations are given in Fig. 2.

This implies that $\eta(3, C_5) \leq 15$. Next, we prove that this upper bound is tight.

Theorem 11. *Any graph G of order at most* 14 *and odd-girth at least* 7 *admits a homomorphism to C_5, and thus $\eta(3, C_5) = 15$.*

Proof. We consider a C_5 on the vertex set $\{0, 1, 2, 3, 4\}$ where vertex i is adjacent to vertices $i + 1$ and $i - 1$ (modulo 5). Thus, in the following, to give a C_5-colouring we will give a colouring using elements of $\{0, 1, 2, 3, 4\}$ where adjacent pairs are mapped into (cyclically) consecutive elements of this set.

Given a graph G and a vertex v of it, we partition $V(G)$ into four sets $\{v\}, N_1(v), N_2(v)$ and $N_{3+}(v)$ where $N_1(v)$ (respectively $N_2(v)$) designates the set of vertices at distance exactly 1 (respectively 2) of v, and $N_{3+}(v)$ the vertices at distance 3 or more of v. Note that because of the odd-girth requirement for G, $N_1(v)$ and $N_2(v)$ are independent sets.

A proper 3-colouring of $G[N_{3+}(v)]$ using colours c_1, c_2 and c_3 is said to be v-*special* if:

  (i) each vertex with colour c_3 is an isolated vertex of $G[N_{3+}(v)]$,
  (ii) no vertex from $N_2(v)$ sees both colours c_1 and c_2.

A key observation is the following: given any graph G, if for some vertex v of G, there exists a v-special colouring of $G[N_{3+}(v)]$, then G maps to C_5. Such a homomorphism is given by mapping c_1-vertices to 0, c_2-vertices to 1 and c_3-vertices to 3, and then extending as follows:

- for any vertex u in $N_2(v)$, if u has a c_1-neighbour, map it to 4; otherwise, map it to 2,
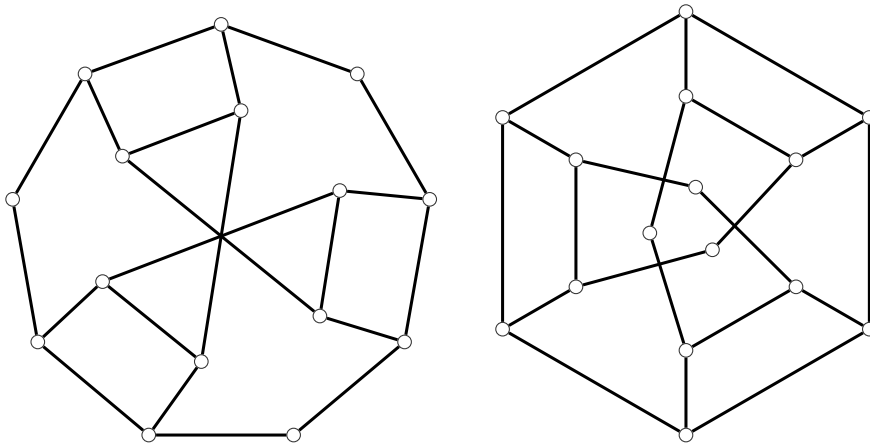
Fig. 2. Two drawings of a symmetric graph of odd-girth 7 which does not map to C_5.

- all vertices of $N_1(v)$ are mapped to 3,
- vertex v is mapped to 2 or 4.

Now, let G be a minimal counterexample to Theorem 11: G has odd-girth at least 7, has order at most 14, but does not map to C_5. Note that G is connected, indeed G does not map to C_5 if and only if one of its components does not map to C_5.

We first collect a few properties of G. The previous paragraph allows us to state our first claim.

Claim 11.A. *For no vertex v of G there is a v-special colouring of $G[N_{3+}(v)]$.*

Since G is a minimal counterexample, it cannot map to a subgraph of itself (which would be a smaller counterexample):

Claim 11.B. *Graph G is a core. In particular, for any two vertices u and v of G, $N(u) \not\subseteq N(v)$.*

Recall that a *walk* between two vertices u and v is a sequence of (not necessarily distinct) vertices starting with u and ending with v, where two consecutive vertices in the sequence are adjacent. A walk between u and v is a *uv-walk*, and a *k-walk* is a walk with $k+1$ vertices counting multiplicity.

Claim 11.C. *For any two distinct vertices u and v of G, there is a uv-walk of length 5.*

Proof of claim. If not, identifying u and v would result in a smaller graph of yet odd-girth 7 which does not map to C_5, contradicting the minimality of G. □

Claim 11.D. *Graph G has no thread of length 4 or more.*

Proof of claim. Once again, by minimality of G, if we remove a thread of length 4, the resulting graph maps to C_5. But since there is a walk of length 4 between any two vertices of C_5, this mapping could easily be extended to G. □

Now we can state a more difficult claim.

Claim 11.E. *There is no vertex of G of degree 4 or more, nor a vertex of degree exactly 3 with a second neighbourhood of size 5 or more.*

Proof of claim. For a contradiction, suppose that a vertex v has degree 4 or more, or has degree 3 and a second neighbourhood of size 5 or more.

By Claim 11.B, the neighbours of v should have pairwise distinct neighbourhoods, so that even if v has degree 4 or more, we must have $|N_2(v)| \geq 4$. Thus, by a counting argument (recall that G has at most 14 vertices), $N_{3+}(v)$ has size at most 5. Since G has odd-girth 7, this means $G[N_{3+}(v)]$ is bipartite.

Suppose $G[N_{3+}(v)]$ has at most one non-trivial connected component. Consider any proper 3-colouring of $G[N_{3+}(v)]$ such that colours c_1 and c_2 are used for the non-trivial connected component, and colour c_3 is used for isolated vertices. Then, no vertex of $N_2(v)$ can see both colours c_1 and c_2, as this would result in a short odd-cycle in G. Thus, any such colouring is v-special. Hence, by Claim 11.A, we derive that $G[N_{3+}(v)]$ has at least two non-trivial components. Since it has order at most 5, it must have exactly two.

Fig. 3. The two 2-colourings of $N_{3+}(v)$ in the proof of Claim 11.E.

Assume now that both non-trivial connected components are isomorphic to K_2. Consider all proper 3-colourings of $G[N_{3+}(v)]$ such that colours c_1 and c_2 are used for the copies of K_2 (the potentially remaining vertex being coloured with c_3). Considering Claim 11.A, one may easily check that if none of these colourings is v-special, then there is a short odd-cycle in G, which is a contradiction.

Hence, $G[N_{3+}(v)]$ is isomorphic to the disjoint union of K_2 and $K_{2,1}$. Let u_1 and u_2 be the vertices of K_2, and u_3, u_4 and u_5 be the vertices of $K_{2,1}$ such that u_4 is the central vertex.

Let φ_1 and φ_2 be two proper 2-colourings of $G[N_{3+}(v)]$ as follows: (i) $\varphi_1(u_1) = \varphi_1(u_3) = \varphi_1(u_5) = c_1$ and $\varphi_1(u_2) = \varphi_1(u_4) = c_2$, and (ii) $\varphi_2(u_2) = \varphi_2(u_3) = \varphi_2(u_5) = c_1$ and $\varphi_2(u_1) = \varphi_2(u_4) = c_2$ (see Fig. 3).

Since φ_1 is not v-special, there is either a vertex t_1 adjacent to u_2 and u_3 (by considering the symmetry of u_3 and u_5), or a vertex t_1' adjacent to u_1 and u_4. Similarly, since φ_2 is not v-special, either there is a vertex t_2 adjacent to u_1 and one to u_3 and u_5, or there is a vertex t_2' adjacent to u_2 and u_4. If t_1' exists, then in the subgraph induced by t_1', t_2 or t_2', whichever exists, and their neighbours we will find an odd-cycle of length at most 5. Similarly, t_2' does not exist. Thus, t_1 and t_2 must exist. Moreover, for a similar reason, t_2 is not a neighbour of u_3 and, therefor, it is a neighbour of u_5.

Next, we show that u_4 has degree at least 3. Suppose not, then it has degree exactly 2. Let φ_3 be a partial C_5-colouring of G defined as follows: $\varphi_3(u_1) = \varphi_3(u_5) = 0$, $\varphi_3(u_2) = 1$, $\varphi_3(u_3) = 3$ and $\varphi_3(u_4) = 4$. Then, no vertex in $N_2(v)$ sees both 0 and 1 (by odd-girth arguments). Thus, we can extend φ_3 to $N_2(v)$ using only colours 2 and 4 on these vertices. Then, all vertices of $N_1(v)$ can be mapped to 3 and v can be mapped to 2 and $G \rightarrow C_5$, a contradiction.

Hence, there exists a vertex t_3 in $N_2(v)$ which is adjacent to u_4. Note that, by the odd-girth condition, t_3 has no other neighbour in $G[N_{3+}(v)]$ and, in particular, it must be distinct from t_1 and t_2. Vertices t_1, t_2 and t_3 are in $N_2(v)$, so there are vertices s_1, s_2 and s_3 in $N_1(v)$ such that s_i is adjacent to t_i for i between 1 and 3. Moreover, vertices t_1, t_2 and t_3 are pairwise connected by a path of length 3. Therefore, their neighbourhoods cannot intersect, so that vertices s_1, s_2 and s_3 are distinct.

Now, consider the partial C_5-colouring φ_3 again. We may extend φ_3 to $N_2(v)$ by assigning colour 0 to neighbours of u_4, colour 4 to neighbours of u_1 and u_5, and colour 2 to the rest. If no vertex of $N_1(v)$ sees both colours 0 and 4 in $N_2(v)$, we may colour $N_1(v)$ with 1 and 3 and colour v with 2, which is a contradiction. Thus, there exists some vertex x in $N_1(v)$ seeing both colours 0 and 4 in $N_2(v)$. The only vertices with colour 0 in $N_2(v)$ are neighbours of u_4 so that x must be at distance 2 from u_4. Since there is no short odd-cycle in G, vertex x cannot be at distance 2 from u_5. Thus, it is at distance 2 from u_1. Let t_4 be the middle vertex of this path from x to u_1. Now t_4 is a neighbour of u_1 which is distinct from t_1, t_2 and t_3. By the symmetry between u_1 and u_2, there must be a fifth vertex t_5 in $N_2(v)$ which is a neighbour of u_2 and distinct from t_1, t_2, t_3 and t_4. Moreover, t_5 has a neighbour y in $N_1(v)$ that is at distance 2 from u_4. We can readily check that y is distinct from x, s_1 and s_2. Thus, $N_1(v)$ has size at least 4 and $N_2(v)$ has size at least 5, which is a contradiction with the order of G (which should be at most 14). This concludes the proof of Claim 11.E. □

Claim 11.F. *G contains no 6-cycle.*

Proof of claim. Suppose, by contradiction, that G contains a 6-cycle $C : v_0, \ldots, v_5$. For a pair v_i and v_{i+2} (addition in indices is done modulo 6) of vertices of C, the 5-walk connecting them (see Claim 11.C) is necessarily a 5-path because of the odd-girth condition, and we denote it by P^i. Furthermore, at most one inner-vertex of P^i may belong to C, and if it does, it must be a neighbour of v_i or v_{i+2} (one can check that otherwise, there is a short odd-cycle in G).

Assume first that none of the six paths P^i ($0 \leq i \leq 5$) has any inner-vertex on C. In this case, by Claim 11.E, we observe that the neighbours of v_i in P^i and P^{i+4} (addition in superscripts is done modulo 6) are the same. Let v_i' be this neighbour of v_i.

Vertices v_i', $i = 0, 1, \ldots, 5$ are all distinct, as otherwise we have a short odd-cycle in G. Let x and y be the two internal vertices of P^0 distinct from v_0' and v_2'. By our assumption, x and y are distinct from vertices of C. We claim that they are also distinct from v_i', $i = 0, \ldots, 5$. Vertex x is indeed distinct from v_0' and v_2' by assumption. It is distinct from v_1', v_3' and v_4' as otherwise there will be a short odd-cycle. By the symmetry of x and y and by the same arguments, y is distinct from v_0', v_1', v_2', v_4', v_5'. Moreover, we cannot simultaneously have $x = v_5'$ and $y = v_3'$, otherwise we get a 5-cycle. Finally, if we have $x = v_5'$ then $\{v_0', v_1, y, v_3, v_4'\} \subseteq N_2(v_5)$ and $d(v_5) \geq 3$, contradicting Claim 11.E. As a result, since $|V(G)| \leq 14$, x and y are internal vertices of all P^i's. But then it easy to find a short odd-cycle.

Hence, we may assume that some path P^i, say P^1 without loss of generality, has (exactly) one inner-vertex on C. Without loss of generality, say $v_1' = v_0$ (recall that it is not possible that a vertex of P^1 that is not a neighbour of v_1 or v_3 is on C). Let $v_0 x_1 x_2 x_3 v_3$ be the 4-path connecting v_0 and v_3 (recall that at most one inner-vertex of P^i belongs to C, so $x_i \notin C$ for $i = 1, 2, 3$). We next assume that P^5 does not have any inner-vertex in C. Then, no vertex of P^5 is a vertex from
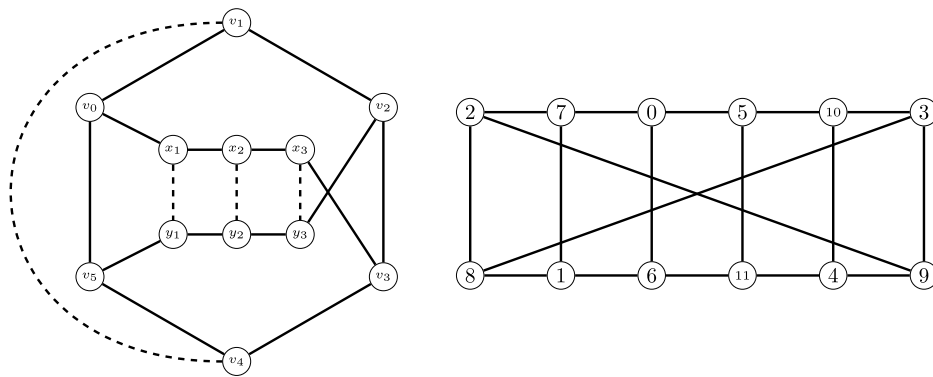
(i) The subgraph G' of G. Dashed edges may or may not exist.

(ii) The graph $C(12, 5)$.

Fig. 4. Graphs used in the proof of Claim 11.F.

$\{x_1, x_2, x_3\}$, for otherwise we have a short odd-cycle in G. But then, v_0 violates Claim 11.E as $\{v'_5, v_4, x_2, v_2, v'_1\} \subseteq N_2(v_0)$. Therefore, an inner-vertex of P^5 lies on C. Considering the odd-girth conditions and Claim 11.E, either $v'_5 = v_4$, or the neighbour of v_1 on P^5 is v_2. As both cases are symmetric with respect to our assumptions so far, we may assume that the second case holds without loss of generality. Thus, we have $P^5 : v_5 y_1 y_2 y_3 v_2 v_1$. Then, P^1 and P^5 are vertex-disjoint, for otherwise we have a short odd-cycle in G.

Let us call G', the 12-vertex subgraph induced by the vertices of C, P^1 and P^5. Notice that G' also contains P^0, P^2, P^3 and P^4. Moreover, the vertices and edges of C, P^1 and P^5 form a 7-odd K_4, more precisely, a $(1, 2, 4)$-odd-K_4. Now, because of the odd-girth of G, and by Claim 11.E, one can check that the only possible third neighbour of v_1 in G', if any, is v_4 (and vice-versa). Furthermore, within the set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ of vertices, the only possible edges are those of the form $x_i y_i$ with $i = 1, 2, 3$. (See Fig. 4(i) for an illustration of G'.) We claim that G' must be a subgraph of $C(12, 5)$. Indeed, even if these four optional edges are present, we obtain an isomorphic copy of $C(12, 5)$: consider the isomorphism from $C(12, 5)$ to this graph, given by $\{0 : v_0, 1 : v_4, 2 : v_2, 3 : v_3, 4 : y_2, 5 : x_1, 6 : v_5, 7 : v_1, 8 : v_3, 9 : y_3, 10 : x_2, 11 : y_1\}$. (See Fig. 4(ii) for an illustration of $C(12, 5)$.) Thus, the circular chromatic number of G' is at most $12/5$, implying that G' has a homomorphism to C_5.

Hence, there is at least one vertex in G, that is not a vertex of G'. Using Claim 11.E, one can check that any vertex of G not in G' can be adjacent only to x_2 or to y_2 (or both) and that there are at most two such vertices, one a neighbour of x_2 another a neighbour of y_2. Without loss of generality (considering the symmetries of the graph), assume that x_2 has an additional neighbour, v. Then, either v is also adjacent to y_2 (then G has order 13), or v and y_2 have a common neighbour, say w. If v is adjacent to both x_2 and y_2, then the only edges induced by the set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ are edges in G' (otherwise we have a short odd-cycle). But then, we exhibit a homomorphism of G to C_5: map x_3, y_1, v to 0; x_2, y_2 to 1; v_1, x_1, y_3 to 2; v_0, v_2, v_4 to 3; v_3 and v_5 to 4. This is a contradiction. Thus, v and y_2 have a common neighbour w, and both v and w are of degree 2. We now create a homomorphic image of G by identifying v with y_2 and w with x_2. Then, this image of G is again a subgraph of $C(12, 5)$, and thus the circular chromatic number of G is at most $12/5$, implying that G has a homomorphism to C_5, a contradiction. This completes the proof of Claim 11.F. □

Claim 11.G. *G contains no 4-cycle.*

Proof of claim. Assume by contradiction that G contains a 4-cycle $C : tuvw$. As in the proof of Claim 11.F, there must be two 5-paths $P_{tv} : t a_1 a_2 a_3 a_4 v$ and $P_{uw} : u b_1 b_2 b_3 b_4 w$ connecting t with v and u with w, respectively. Moreover, these two paths must be vertex-disjoint because of the odd-girth of G. Thus, the union of C, P_{tv} and P_{uw} forms a $(1, 1, 5)$-odd-K_4, K. By assumption on the odd-girth of G, any additional edge inside $V(K)$ must connect an internal vertex of P_{tv} to an internal vertex of P_{uw}. But any such edge would either create a short odd-cycle or a 6-cycle in G, the latter contradicting Claim 11.F. Thus, the only edges in $V(K)$ are those of K. If there is no additional vertex in G, we have two 5-threads in G, contradicting Claim 11.D; thus there is at least one additional vertex in G, say x, and perhaps a last vertex, y. Note that t, u, v and w are already, in K, degree 3-vertices with a second neighbourhood of size 4, thus by Claim 11.E none of t, u, v, w, a_1, a_4, b_1 and b_4 are adjacent to any vertex not in K. Thus $N(x), N(y) \subseteq \{a_2, a_3, b_2, b_3, x, y\}$.

Assume that some vertex not in K (say x) is adjacent to two vertices of K. Then, these two vertices must be a vertex on the tu-path of K (a_2 or a_3, without loss of generality it is a_2) and a vertex on the vw-path of K (b_2 or b_3). By the automorphism of K that swaps v and w and reverses the vw-path, without loss of generality we can assume that the second neighbour of x in K is b_3. Then, we claim that G has no further vertex. Indeed, if there is a last vertex y, since a_2 and b_3 have already three neighbours, y must be adjacent to at least two vertices among $\{a_3, b_2, x\}$. If it is adjacent to
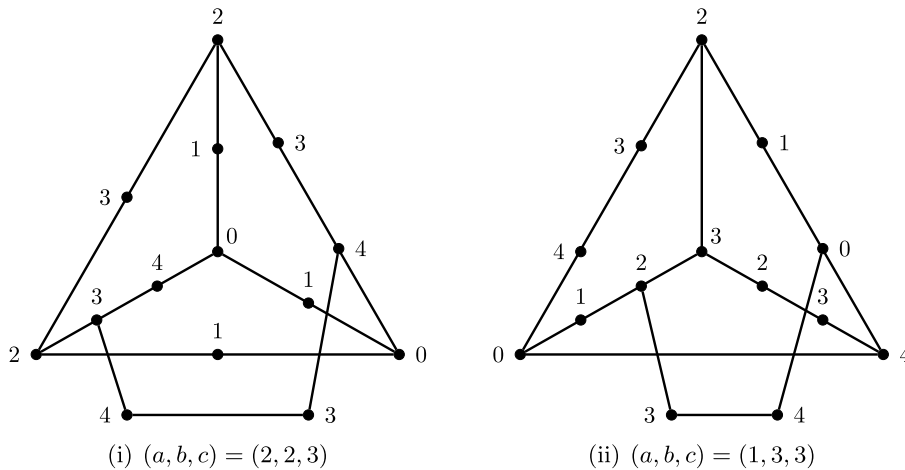
Fig. 5. The two graphs considered in Case 1.

both a_3 and b_2, we have a 6-cycle, contradicting Claim 11.F; otherwise, y is of degree 2 but part of a 4-cycle, implying that G is not a core, a contradiction. Thus, G has order 13 and no further edge. We create a homomorphic image of G by identifying x and a_3. The obtained graph is a subgraph of $C(12, 5)$. Hence, G has circular chromatic number at most $12/5$ and a homomorphism to C_5, a contradiction.

Thus, any vertex not in K has at most one neighbour in K. Since G has no 4-thread, one vertex not in K (say x) is adjacent to one of a_2 or a_3 (without loss of generality, say a_2), and the last vertex, y, is adjacent to one of b_2 and b_3 (as before, by the symmetries of K we can assume it is b_2). Moreover, x and y must be adjacent, otherwise they both have degree 1. Also there is no further edge in G. But then, as before, we create a homomorphic image of G by identifying x with b_2 and y with a_2. The resulting graph is a subgraph of $C(12, 5)$, which again gives a contradiction. This completes the proof of Claim 11.G. □

Claims 11.F and 11.G imply that G has girth at least 7. To complete the proof, we note that since G has no homomorphism to C_5, it also has no homomorphism to C_7. Thus, by Theorem 4, G must contain either an odd-K_3^2 or an odd-K_4 of odd-girth at least 7. Since such an odd-K_3^2 must have at least 18 vertices, G contains an odd-K_4. Let H be such an odd-K_4 of G. Since in fact its *girth* is at least 7, by Proposition 3 it has at least 12 vertices. We consider three cases, depending on the order of H.

Case 1. Assume that $|V(H)| = 12$. By Proposition 3, H is a 7-odd-K_4, in fact, it is an (a, b, c)-odd-K_4 with $a+b+c = 7$. Since H has girth at least 7, we have only two possibilities: (i) $(a, b, c) = (2, 2, 3)$ or (ii) $(a, b, c) = (1, 3, 3)$. Each of these two possible odd-K_4's is of diameter 4. Furthermore, in each case, the pairs at distance 4 are isomorphic (each pair consists of two internal vertices of two disjoint threads of length 3). If G contains one more vertex than H, say x, then either x is of degree at most 1, which contradicts Claim 11.B or it creates a cycle of length at most 6, which is not possible due to the girth of G. Thus G has exactly two more vertices, say x and y and each of them is adjacent to at most one vertex in H. These two neighbours of x and y are at distance 4 in H and x is adjacent to y. There are only two non-isomorphic such possibilities. A homomorphism to C_5 in each of these two possibilities is given in Fig. 5.

Case 2. Assume that $|V(H)| = 13$. Let the lengths of the four odd-cycles of H be l_1, l_2, l_3 and l_4. Using the fact that $l_1 + l_2 + l_3 + l_4 = 2|E(H)| = 30$ we conclude that three of these cycles are of length 7 and the last one is of length 9. Let a, b and c be the lengths of the three maximal threads of H whose union forms the 9-cycle. Let a', b' and c' be the respective lengths of the corresponding pairwise disjoint threads. Then, we have $a' + b' + c = 7$ and $a + b + c = 9$, which implies that $a' + b' = a + b - 2$. Together with the similar equalities $a' + c' = a + c - 2$ and $b' + c' = b + c - 2$, we obtain that $a = a' + 1$, $b = b' + 1$ and $c = c' + 1$. By the symmetries, we may assume that $a \le b \le c$. Considering that there are cycles of lengths $2a' + 2b' + 2$, $2a' + 2c' + 2$ and $2b' + 2c' + 2$ in H, we have either (i) $a' = b' = c' = 2$ or (ii) $a' = 1$, $b' = 2$ and $c' = 3$. In the first case, the graph is of diameter 4 and thus there is no room for a 14th vertex; in the second case, there is a unique pair at distance 5, which leads us to a unique possible graph on 14 vertices, by connecting a 14th vertex to these two diametral vertices. Homomorphisms to C_5 for each of these graphs are given in Fig. 6.

Case 3. Assume that $|V(H)| = 14$. Again, let l_1, l_2, l_3 and l_4 be the lengths of the four odd-cycles of H. Then, we have $l_1 + l_2 + l_3 + l_4 = 2|E(H)| = 32$, and either (I) $l_1 = l_2 = l_3 = 7$ and $l_4 = 11$, or (II) $l_1 = l_2 = 7$ and $l_3 = l_4 = 9$.

For (I), we define a, b, c, a', b' and c' similarly as in Case 2. It is then implied that $a = a' + 2$, $b = b' + 2$ and $c = c' + 2$. Since $a', b', c' \ge 1$ we have $a, b, c, \ge 3$ and since $a + b + c = 11$ we have two possibilities: either $a = b = 3, c = 5$ or $a = 3, b = c = 4$. In the case $(a, b, c) = (3, 3, 5)$, the vertex of degree 3 of H which is not on the face of size 11
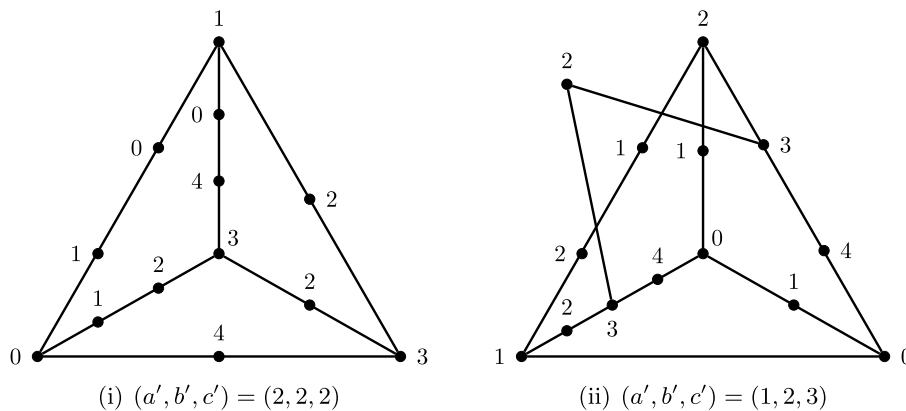
(i) $(a', b', c') = (2, 2, 2)$ (ii) $(a', b', c') = (1, 2, 3)$

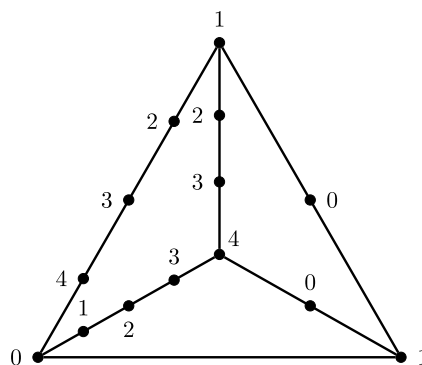**Fig. 6.** The two graphs considered in Case 2.

Fig. 7. The first graph considered in Case 3, with $(a, b, c) = (3, 4, 4)$.

has already five distinct vertices at distance 2, contradicting Claim 11.E. In the case $(a, b, c) = (3, 4, 4)$, H is already of diameter 5, thus H is the same as G. For this case a C_5-colouring is given in Fig. 7.

For (*II*), assuming c' is the length of the common thread of the two cycles of length 9, a similar calculation as in Case 2 implies that $a = a'$, $b = b'$ and $c' = c + 2$. The possibilities are (i) $a = 1$, $b = 3$ and $c = 3$, (ii) $a = 1$, $b = 4$ and $c = 2$, (iii) $a = 2$, $b = 2$ and $c = 3$, (iv) $a = 2$, $b = 3$ and $c = 2$, and (v) $a = b = 3$ and $c = 1$. As in each case, each pair of vertices is on a common cycle of length at most 10, and since the girth is at least 7, G must be isomorphic to H. Then, a mapping to C_5 can readily be found (see Fig. 8). This case completes the proof as there is no counterexample to the statement of the theorem. □

We now deduce the following consequence of Theorem 11 and Proposition 8, that improves the known lower bounds on $\eta(4, C_5)$ and $\eta(4, C_3)$ (noting that for larger values of k, the bound of Corollary 7 is already stronger).

Corollary 12. *We have* $\eta(4, C_3) \geq \eta(4, C_5) \geq 17$.

5. Concluding remarks

In this work, we have started investigating the smallest order of a $C_{2\ell+1}$-critical graph of odd-girth $2k + 1$. We have determined a number of previously unknown values, in particular we showed that a smallest C_5-critical graph of odd-girth 7 is of order 15. In contrast to the result of Chvátal on the uniqueness of the smallest triangle-free 4-chromatic graph [4], we have found more than one such graph: Gordon Royle showed computationally, that there are exactly eleven such graphs (private communication, 2016) and Fig. 1 shows three of them.

Regarding Table 1, we do not know the growth rate in each row of the table. Perhaps it is quadratic; that would be to say that for a fixed ℓ, $\eta(k, C_{2\ell+1}) = \Theta(k^2)$. This is indeed true for $\ell = 1$, as proved in [22].

Our last remark is about Theorem 9. We think that for any given k, Theorem 9 covers all values of k for which $\eta(k, C_{2\ell+1}) = 4k$.

A similar problem of interest is to ask for the order of a smallest $C_{2\ell+1}$-critical graph of girth g. This question has not been studied much. In this formulation, the Grötzsch graph is the smallest C_3-critical graph of girth 4. When girth 5 is

(i) $(a, b, c) = (1, 3, 3)$

(ii) $(a, b, c) = (1, 4, 2)$

(iii) $(a, b, c) = (2, 2, 3)$

(iv) $(a, b, c) = (2, 3, 2)$
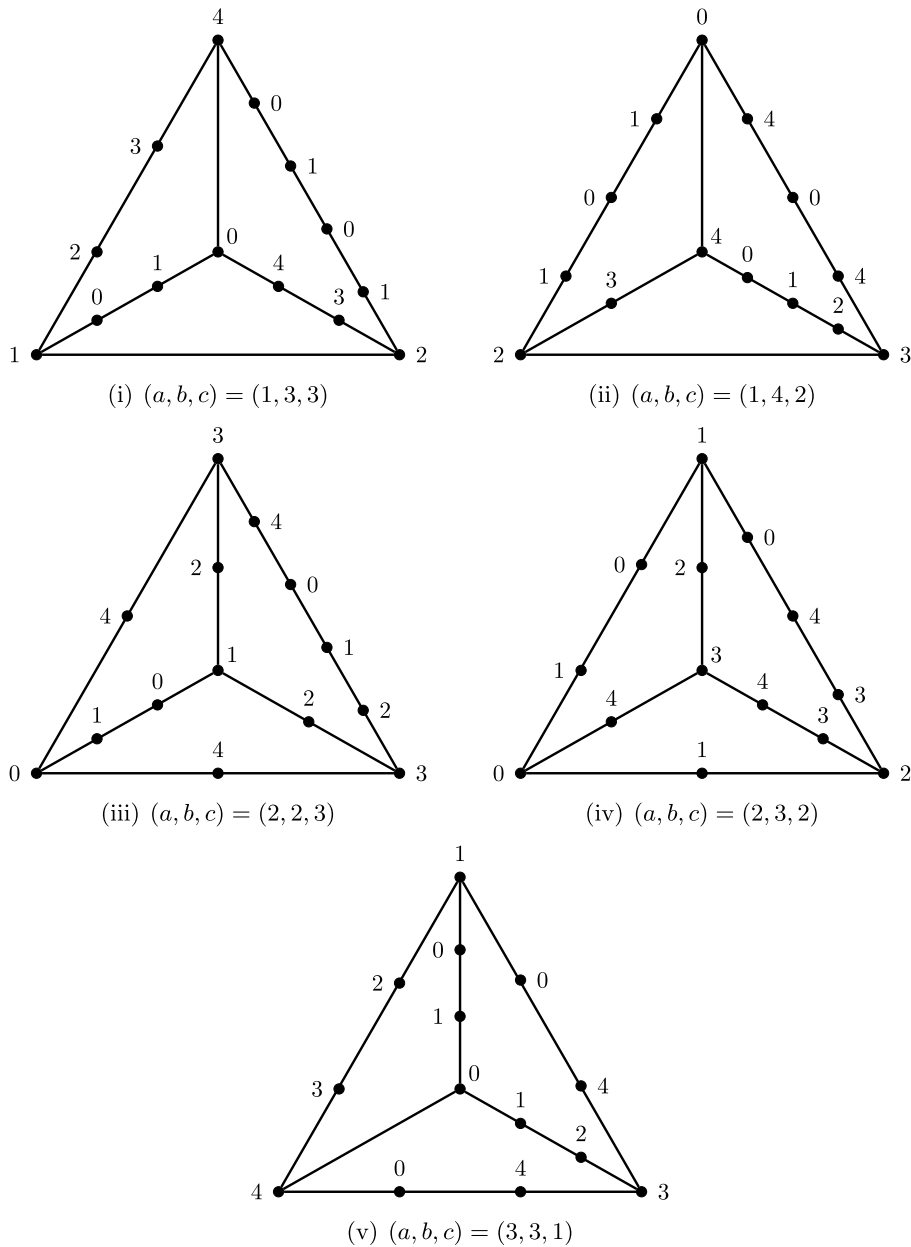
(v) $(a, b, c) = (3, 3, 1)$

Fig. 8. Five graphs considered in Case 3.

considered, then again a computational work of Gordon Royle showed that 21 vertices are needed and again the answer is not unique [26]. See [9] for further studies.

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