Discrete Applied Mathematics 358 (2024) 366–381

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On locating and neighbor-locating colorings of sparse graphs

Dipayan Chakraborty [a,b], Florent Foucaud [a], Soumen Nandi [c], Sagnik Sen [d], D.K. Supraja [c,d,*]

[a] Université Clermont Auvergne, CNRS, Clermont Auvergne INP, Mines Saint-Étienne, LIMOS, 63000 Clermont-Ferrand, France
[b] Department of Mathematics and Applied Mathematics, University of Johannesburg, South Africa
[c] Netaji Subhas Open University, India
[d] Indian Institute of Technology Dharwad, India

ARTICLE INFO

Article history:
Received 15 January 2024
Received in revised form 26 May 2024
Accepted 12 July 2024
Available online xxxx

Keywords:
Locating coloring
Neighbor-locating coloring
Neighbor-locating chromatic number
Sparse graphs
Computational complexity

ABSTRACT

A proper k-coloring of a graph G is a *neighbor-locating k-coloring* if for each pair of vertices in the same color class, the two sets of colors found in their respective neighborhoods are different. The *neighbor-locating chromatic number* $\chi_{NL}(G)$ is the minimum k for which G admits a neighbor-locating k-coloring. A proper k-vertex-coloring of a graph G is a *locating k-coloring* if for each pair of vertices x and y in the same color-class, there exists a color class S_i such that $d(x, S_i) \neq d(y, S_i)$. The locating chromatic number $\chi_L(G)$ is the minimum k for which G admits a locating k-coloring.

Our main results concern the largest possible order of a sparse graph of given neighbor-locating chromatic number. More precisely, we prove that if a connected graph G has order n, neighbor-locating chromatic number k and average degree d, then n is upper-bounded by $\mathcal{O}(d^2k^{\lceil d\rceil+1})$. We also design a family of graphs of bounded maximum degree whose order is close to reaching this upper bound. Our upper bound generalizes two previous bounds from the literature, which were obtained for graphs of bounded maximum degree and graphs of bounded cycle rank, respectively.

Also, we prove that determining whether $\chi_L(G) \leq k$ and $\chi_{NL}(G) \leq k$ are NP-complete for sparse graphs: more precisely, for graphs with average degree at most 7, maximum average degree at most 20 and that are 4-partite.

We also study the possible relation between the ordinary chromatic number, the locating chromatic number and the neighbor-locating chromatic number of a graph.

1. Introduction

Our aim is to study two graph coloring problems from the field of graph identification, namely, *locating coloring* and *neighbor-locating coloring*, with an emphasis on the latter.

Identification problems. The above two problems belong to the general framework of *identification problems*, where one is given a graph (or a hypergraph) and one wishes to distinguish all vertices of the graph by giving each of them a unique attribute. Classically, the problems in this area largely fall into two main categories: (i) local identification problems, and (ii) distance-based identification problems. The study of the former class of problems was initiated by Rényi in the 1960s for hypergraphs, under the name of *separating sets* [31] (also later called *separating systems* [12], *test covers* [29],

* Corresponding author at: Netaji Subhas Open University, India.
E-mail address: dksupraja95@gmail.com (D.K. Supraja).

https://doi.org/10.1016/j.dam.2024.07.012

discriminating codes [14], etc.). The concept was then adapted to graphs under the name of *locating-dominating sets* by Slater in the 1980s [34]. On the other hand, the prominent distance-based identification problem is the *metric dimension* problem for graphs, introduced independently by Harary and Melter [22] and by Slater [33] in the 1970s.

In all these problems, one seeks a (small) set of *solution* vertices (possibly, hyperedges in the case of hypergraphs) and wishes to distinguish the vertices either by their neighborhoods in the solution in the case of the local problems, or by their distances to the solution vertices, in the case of the distance-based problems. These types of problems are very fundamental and have numerous applications in various fields, such as for example, fault-detection in networks [30,35], biological diagnosis [29], machine learning [17], canonical representations of graphs [6,25], coin-weighing problems [32], games [18], learning theory [21], etc.

One of the most fundamental graph problems is the graph coloring problem, as it is essential to model applications such as clustering, resource allocation, etc. Thus, it is a natural approach to combine the concepts of coloring and identification. As one of the earliest instances of this effort, the concept of *locating coloring* was introduced in 2002 by Chartrand et al. [15], providing a coloring version of the aforementioned *distance-based* identification problems like the metric dimension. Here, one seeks a proper coloring of the graph such that each vertex is uniquely identified by its distances to the color classes. In 2014, a coloring version of the above *local* identification problems was introduced by Behtoei and Anbarloei [9] (under the name of *adjacency locating coloring*) and rediscovered by Alcon et al. in 2020 [2] under the name of *neighbor-locating coloring* (see below for the formal definitions). The setting of these two problems is very natural: instead of minimizing the size of a solution set like in the classic identification problems, we wish to assign a color to each vertex in order to partition the vertex set, e.g., to perform resource allocation, and thus we want to minimize the number of colors. Moreover, we also want to be able to uniquely identify the vertices in each color class, with one of the aforementioned applications of identification problems in mind, for example, fault-detection.

While the former concept of locating coloring has been extensively studied since 2002 [5,7–11,15,16,19], our focus of study is the latter (neighbor-locating coloring), which is more recent but has already started gaining some attention in the very recent years [1–3,23,28].

Notation and terminology. Throughout this article, we consider only simple graphs (graphs without loops and multiple edges). Moreover, we will use the standard terminology and notation used in "Introduction to Graph Theory" by West [36].

Given a graph G, a *(proper) k-coloring* is a function $f : V(G) \to S$, where S is a set of k *colors*, such that $f(u) \neq f(v)$ whenever u is adjacent to v. Usually, we will assume the set of k colors S to be equal to $\{1, 2, \ldots, k\}$, unless otherwise stated. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k for which G admits a k-coloring.

Given a k-coloring f of G, its ith color class is the collection S_i of vertices that have received the color i. The *distance* between a vertex x and a set S of vertices is given by $d(x, S) = \min\{d(x, y) : y \in S\}$, where the distance $d(x, y)$ between the vertices x and y is the number of edges in a shortest path connecting x and y. Two vertices x and y are *metric-distinguished* with respect to f if $d(x, S_i) \neq d(y, S_i)$ for some color class S_i. A k-coloring f of G is a *locating k-coloring* if any two distinct vertices are metric-distinguished with respect to f. The *locating chromatic number* of G, denoted by $\chi_L(G)$, is the minimum k for which G admits a locating k-coloring.

Given a k-coloring f of G, suppose that a neighbor y of a vertex x belongs to the color class S_i. In such a scenario, we say that i is a *color-neighbor* of x (with respect to f). The set of all color-neighbors of x is denoted by $N_f(x)$. Two vertices x and y are *neighbor-distinguished* with respect to f if either $f(x) \neq f(y)$ or $N_f(x) \neq N_f(y)$. A k-coloring f is *neighbor-locating k-coloring* if each pair of distinct vertices are neighbor-distinguished. The *neighbor-locating chromatic number* of G, denoted by $\chi_{NL}(G)$, is the minimum k for which G admits a neighbor-locating k-coloring.

The *average degree* of a graph G having n vertices and m edges is the average of the degree of its vertices, which, due to the Handshaking Lemma, is equal to $\frac{2m}{n}$. The average degree of G is a measure of the density of the graph: if it is bounded by a constant, then the graph has a linear number of edges, and may be called sparse. However, a graph may have low average degree and still contain very dense parts. The *maximum average degree* of G is the maximum of the average degrees taken over all the subgraphs of G. This notion serves as a more "uniform" measure of the graph density. Two non-adjacent vertices $x, y \in G$ are *false twins* if $N(x) = N(y)$, where the *open neighborhood of* x, denoted by $N(x)$, is the set of all vertices adjacent to x.

Applications. Neighbor-locating coloring (and locating coloring, with a slight modification) can model the following fault-detection problem. This kind of fault-detection in networks and complex systems is of high practical importance in the industry, see for example the settings of *multi-core C & I cables* [27], and *smart grids* [26]. We wish to monitor a network for faults (or a facility for hazards). The facility is partitioned into several segments (each represented by a color), and each segment consists of multiple nodes where a fault may occur. To every segment, we associate one detector that monitors it for potential faults. To avoid mistakes in the detection, two adjacent nodes cannot be in the same segment. Every detector is able to signal the following two things in case of occurrence of a fault (at exactly one node): (i) the segment where the fault has occurred, (ii) the segments that are adjacent to the faulty node.

Hence, if all nodes in a given segment have different sets of segments in their neighborhood, the information (i) and (ii) from all detectors is sufficient to locate the faulty node. To reduce costs, one wishes to minimize the number of detectors (that is, segments). In such a scenario, the network is modeled by a graph, nodes correspond to vertices, node adjacencies to edges, and segments to color classes. Thus, this fault-detection scenario corresponds to the neighbor-locating coloring problem.

To minimize the number of detectors even further (but at the expense of more powerful detectors), the setting can be slightly modified for (ii) if every detector can measure the smallest *distance* from a node of its segment to the node where the fault has occurred. In that case, this fault-detection scenario corresponds to the locating coloring problem.

Context and contributions. Observe that a neighbor-locating coloring is, in particular, a locating coloring as well. Therefore, we have the following obvious relation among the three parameters [2]:

$$\chi(G) \leq \chi_L(G) \leq \chi_{NL}(G).$$

Note that for complete graphs, all three parameters have the same value, that is, equality holds in the above relation. Nevertheless, the difference between the pairs of values of parameters $\chi(\cdot)$, $\chi_{NL}(\cdot)$ and $\chi_L(\cdot)$, $\chi_{NL}(\cdot)$, respectively, can be arbitrarily large. Moreover, it was proved that for any pair p, q of integers with $3 \leq p \leq q$, there exists a connected graph G_1 with $\chi(G_1) = p$ and $\chi_{NL}(G_1) = q$ [2] and a connected graph G_2 with $\chi_L(G_2) = p$ and $\chi_{NL}(G_2) = q$ [28]. The latter of the two results positively settled a conjecture posed in [2]. We strengthen these results as follows.

Theorem 1. *For all $2 \leq p \leq q \leq r$, except when $p = q = 2$ and $r > 2$, there exists a connected graph $G_{p,q,r}$ satisfying $\chi(G_{p,q,r}) = p$, $\chi_L(G_{p,q,r}) = q$, and $\chi_{NL}(G_{p,q,r}) = r$.*

One fundamental difference between coloring and locating coloring (resp., neighbor-locating coloring) is that the restriction of coloring of G to an (induced) subgraph H is necessarily a coloring, whereas the analogous property is not true for locating coloring (resp., neighbor-locating coloring). Interestingly, we show that the locating chromatic number (resp., neighbor-locating chromatic number) of an induced subgraph H of G can be arbitrarily larger than that of G.

Theorem 2. *For every $k \geq 0$, there exists a graph G_k having an induced subgraph H_k such that $\chi_L(H_k) - \chi_L(G_k) = k$ and $\chi_{NL}(H_k) - \chi_{NL}(G_k) = k$.*

Alcon et al. [2] showed that the number n of vertices of G is bounded above by $k(2^{k-1} - 1)$, where $\chi_{NL}(G) = k$ and G has no isolated vertices, and this bound is tight. This exponential bound is reduced to a polynomial one when G has maximum degree Δ. Indeed it was further shown in [2] that the upper-bound $n \leq k\sum_{j=1}^{\Delta}\binom{k-1}{j} = \mathcal{O}(k^{\Delta+1})$ holds (for graphs with no isolated vertices and when $\Delta \leq k-1$). The tightness of this bound was left open. Alcon et al. [3] gave the upper bound $n \leq \frac{1}{2}(k^3 + k^2 - 2k) + 2(c-1) = \mathcal{O}(k^3)$ for graphs of order n, neighbor-locating chromatic number k and cycle rank c, where the *cycle rank* c of a graph G, is defined as $c = |E(G)| - n + 1$. Further, they also obtained tight upper bounds on the order of trees and unicyclic graphs in terms of the neighbor-locating chromatic number [3], where a unicyclic graph is a connected graph having exactly one cycle.

A connected graph with cycle rank c and order n has $n+c-1$ edges and a graph of order n and maximum degree Δ has at most $\frac{\Delta}{2}n$ edges. Thus, the two latter bounds, which are in terms of cycle rank c and maximum degree Δ respectively, can be seen as two approaches for studying the neighbor-locating coloring for sparse graphs. We generalize this approach by studying graphs with a given *average* degree and neighbor-locating chromatic number k. For such graphs, we prove the following.

Theorem 3. *Let G be a connected graph on n vertices, with neighbor-locating chromatic number k and average degree d. Then, we have $n = \mathcal{O}(d^2k^{\lceil d\rceil+1})$. More precisely:*

(i) if $k \leq \lceil d\rceil$, then $n < \lceil d\rceil k^{\lceil d\rceil-1}$;
(ii) if $k \geq \lceil d\rceil + 1$, then $n \leq k\sum_{i=1}^{\lceil d\rceil}(\lceil d\rceil + 1 - i)\binom{k-1}{i}$. Moreover, any graph G whose order attains the upper bound has maximum degree $\Delta \leq \lceil d\rceil + 1$ and exactly $k\binom{k-1}{i}$ vertices of degree i.

Furthermore, we design a construction that shows that the above upper bound is asymptotically almost tight, as follows.

Theorem 4. *For every integer $\Delta \geq 2$, there exists a connected graph G of maximum degree Δ of order $n = \Omega\left(\Delta\left(\frac{k}{\Delta-1}\right)^{\Delta+1}\right)$, where k is the neighbor-locating chromatic number of G.*

Note that the above lower bound is also $\Omega\left(d^{-d}k^{d+1}\right)$. It implies that our bound from Theorem 3 and the one from [2] are tight up to a multiplicative factor that is a function of Δ or d (when d is an integer), respectively. In other words, if Δ or d is considered a fixed constant, our construction shows that these two bounds are tight up to a constant factor.

A natural question that arises, is whether determining the value of the locating chromatic number and the neighbor-locating chromatic number can be done efficiently on sparse graphs. We show that this is not the case, proving that the associated decision problems are NP-complete even on graphs of bounded maximum average degree.

Theorem 5. *The* L-Coloring *and the* NL-Coloring *problems are NP-complete even when restricted to 4-partite graphs of average degree at most 7 and maximum average degree at most 20.*

Organization of the paper. In Section 2, we study the connected graphs with prescribed values of chromatic number, locating chromatic number, and neighbor-locating chromatic number. We also study the relation between the locating chromatic number (resp., neighbor-locating chromatic number) of a graph and its induced subgraphs. In particular, we prove Theorems 1 and 2 in this section. In Section 3, we provide an upper bound on the number of vertices of a sparse graph in terms of neighbor-locating chromatic number by proving Theorem 3. In Section 4, we prove that the obtained upper bound is almost tight by proving Theorem 4. Finally, in Section 5, we prove that the L-Coloring and the NL-Coloring problems are NP-complete for sparse graphs; more precisely, for graphs that are 4-partite, have average degree at most 7, and maximum average degree at most 20. In particular, we prove Theorem 5.

Note: A preliminary version of this work (without the NP-completeness proofs, without the figures, and with less detailed proofs and statements) appeared in the proceedings of the CALDAM 2023 conference [13].

2. Gaps among $\chi(G)$, $\chi_L(G)$ and $\chi_{NL}(G)$

The first result we would like to prove involves three different parameters, namely, the chromatic number, the locating chromatic number, and the neighbor-locating chromatic number.

Proof of Theorem 1. First of all, let us assume that $p = q = r$. In this case, for $G_{p,q,r} = K_p$, it is trivial to note that $\chi(G_{p,q,r}) = \chi_L(G_{p,q,r}) = \chi_{NL}(G_{p,q,r}) = p$. This completes the case when $p = q = r$.

Second of all, let us handle the case when $p < q = r$. If $2 = p < q = r$, then take $G_{p,q,r} = K_{1,q-1}$. Therefore, we have $\chi(G_{p,q,r}) = 2$ as it is a bipartite graph, and it is known that $\chi_L(G_{p,q,r}) = \chi_{NL}(G_{p,q,r}) = q$ [2,15].

If $3 \le p < q = r$, then we construct $G_{p,q,r}$ as follows: start with a complete graph K_p, on vertices $v_0, v_1, \dots, v_{p-1}$, take $(q-1)$ new vertices $u_1, u_2, \dots, u_{q-1}$, and make them adjacent to v_0. It is trivial to note that $\chi(G_{p,q,r}) = p$ in this case. Moreover, note that we need to assign q distinct colors to $v_0, u_1, u_2, \dots, u_{q-1}$ under any locating or neighbor-locating coloring. On the other hand, $f(v_i) = i + 1$ and $f(u_j) = j + 1$ is a valid locating q-coloring as well as neighbor locating q-coloring of $G_{p,q,r}$. Thus we are done with the case when $p < q = r$.

Thirdly, we will consider the case when $p = q < r$. If $3 = p = q < r$, then let $G_{p,q,r} = C_n$ where C_n is an odd cycle of suitable length, that is, a length which will imply $\chi_{NL}(C_n) = r$. It is known that such a cycle exists [1,9]. As we know that $\chi(G_{p,q,r}) = 3$, $\chi_L(G_{p,q,r}) = 3$ [15], and $\chi_{NL}(G_{p,q,r}) = r$ [1,9], we are done.

If $4 \le p = q < r$, then we construct $G_{p,q,r}$ as follows: start with a complete graph K_p on vertices $v_0, v_1, \dots, v_{p-1}$, and an odd cycle C_n on vertices $u_0, u_1, \dots, u_{n-1}$, and identify the vertices v_0 and u_0. Moreover, we say that the length of the odd cycle C_n is a suitable length, that is, it is of a length which ensures $\chi_{NL}(C_n) = r$. Notice that $\chi(G_{p,q,r}) = p$. A locating coloring f can be assigned to $G_{p,q,r}$ as follows: $f(v_i) = i + 1$, $f(u_j) = a$ for odd integers $1 \le j \le n-1$ and $f(u_l) = b$ for even integers $2 \le l \le n-1$, where $a, b \in \{2, 3, \dots, p\}$. A vertex $v_i \in K_p$ (other than v_0) and a vertex $u_j \in C_n$ such that $f(v_i) = f(u_j)$ are metric-distinguished with respect to f since $d(v_i, S_l) = 1 \neq d(u_j, S_l)$ for at least one $l \in \{2, 3, \dots, p\} \setminus \{a, b\}$. Thus, $\chi_L(G_{p,q,r}) = p$. On the other hand, as the neighborhood of the vertices of the cycle C_n (subgraph of $G_{p,q,r}$) does not change if we consider it as an induced subgraph except for the vertex $v_0 = u_0$. Thus, we will need at least r colors to color C_n while it is contained inside $G_{p,q,r}$ as a subgraph. Assign a neighbor-locating coloring c to $G_{p,q,r}$ as follows: assign p distinct colors to the complete graph K_p. Use p colors from K_p and $r-p$ new colors to provide a neighbor-locating coloring to the odd cycle C_n. A vertex $v_i \in K_p$ (other than v_0) and a vertex $u_j \in C_n$ such that $c(v_i) = c(u_j)$ are neighbor-distinguished with respect to c since v_i has $p-1$ distinct color neighbors whereas u_j can have at most two distinguished color neighbors. Hence $\chi_{NL}(G_{p,q,r}) = r$. Thus, we are done in this case also.

Finally, we are into the case when $p < q < r$. If $2 = p < q < r$, then refer [28] for this case. If $3 = p < q < r$, then we start with an odd cycle C_n on n vertices $v_0, v_1, v_2, \dots, v_{n-1}$. Here, let $k = \frac{r(r-1)(r-2)}{2}$ and

$$n = \begin{cases} k & \text{if } k \text{ is odd,} \\ k-1 & \text{if } k \text{ is even.} \end{cases}$$

It is known that $\chi_{NL}(C_n) = r$ from [1,9]. Take $q-1$ independent vertices $u_1, u_2, \dots, u_{q-1}$ and make all of them adjacent to v_0. This so obtained graph is $G_{p,q,r}$. It is trivial to note that $\chi(G_{p,q,r}) = 3$ in this case. Note that we need to assign q distinct colors to $v_0, u_1, u_2, \dots, u_{q-1}$ under any locating or neighbor-locating coloring. Now, we assign a locating coloring f to $G_{p,q,r}$ as follows:

$$f(v_i) = \begin{cases} 1 & \text{if } i = 0, \\ 2 & \text{if } i \text{ is odd and } 1 \le i \le n-1, \\ 3 & \text{if } i \text{ is even and } 2 \le i \le n-1. \end{cases}$$

Also, $f(u_j) = j + 1$ for all $1 \le j \le q-1$. This gives us $\chi_L(G_{p,q,r}) = q$. On the other hand, as the neighborhood of the vertices of the cycle C_n (subgraph of $G_{p,q,r}$) does not change if we consider it as an induced subgraph except for the vertex v_0. Thus, we will need at least r colors to color C_n while it is contained inside $G_{p,q,r}$ as a subgraph. Assign a neighbor-locating r-coloring c to $G_{p,q,r}$ as follows: assign a neighbor-locating r-coloring to the odd cycle C_n such that each vertex has two distinct color neighbors in case of $n = k$, and all vertices except the two vertices, say v_i and v_j, have two distinct color neighbors in case of $n = k-1$ (refer [1] for such a neighbor-locating r-coloring). Assign distinct colors to the $q-1$ leaf vertices by choosing any $q-1$ colors from r colors (except $c(v_i)$ and $c(v_j)$ in case of $n=k-1$) given to the cycle C_n. A vertex v_i in the cycle and a leaf vertex u_j such that $f(v_i)=f(u_j)$ are neighbor distinguished since v_i has two distinct color neighbors whereas u_j has only one color neighbor. Hence we have $\chi_{NL}(G_{p,q,r})=r$.

If $4 \le p < q < r$, then we start with a path P_n on n vertices, where $n=\frac{r(r-1)(r-2)}{2}$. It is known that $\chi_{NL}(P_n)=r$ from [1,9]. Let $P_n = u_0u_1\cdots u_{n-1}$. Now let us take a complete graph on p vertices $v_0, v_1, \ldots, v_{p-1}$. Identify the two graphs at u_0 and v_0 to obtain a new graph. Furthermore, take $(q-2)$ independent vertices $w_1, w_2, \ldots, w_{q-2}$ and make them adjacent to u_{n-2}. This so obtained graph is $G_{p,q,r}$. It is trivial that $\chi(G_{p,q,r})=p$. Note that under any locating or neighbor-locating coloring, q distinct colors have to be given to the vertices $u_{n-2}, w_1, w_2, \ldots, w_{q-2}, u_{n-1}$. Now, define a locating coloring f of $G_{p,q,r}$ as follows:

$$f(u_i)=\begin{cases}1 & \text{if } i=0 \text{ or } i=n-3,\\ 2 & \text{if } i \text{ is odd and } 1\le i\le n-4,\\ 3 & \text{if } i \text{ is even and } 2\le i\le n-4,\\ 3 & \text{if } i=n-2 \text{ and } n \text{ is odd},\\ 2 & \text{if } i=n-2 \text{ and } n \text{ is even},\\ 2 & \text{if } i=n-1 \text{ and } n \text{ is odd},\\ 3 & \text{if } i=n-1 \text{ and } n \text{ is even}.\end{cases}$$

Further, assign the colors $1, 4, 5, 6, \ldots, q$ to the leaf vertices and $f(v_j)=j+1$ for all $1\le j\le p-1$. Thus, $\chi_L(G_{p,q,r})=q$.

Moreover, the neighborhood of the vertices of the path P_n (subgraph of $G_{p,q,r}$) does not change if we consider it as an induced subgraph except for the vertices u_0 and u_{n-2}. Recall that, for a path P_n on n vertices with $\frac{(r-1)^2(r-2)}{2} < n \le \frac{r^2(r-1)}{2}$, we have $\chi_{NL}(P_n)=r$ [1,9]. As $n-3=\frac{r(r-1)(r-2)-6}{2} > \frac{(r-1)^2(r-2)}{2}$, where $r\ge 6$, at least r colors are required for neighbor-distinguishing the vertices $u_1, u_2, \ldots, u_{n-3}$ in $G_{p,q,r}$.

Assign a neighbor-locating r-coloring c to $G_{p,q,r}$ as follows: assign a neighbor-locating r-coloring to the path P_n such that each vertex (except the end vertices u_0 and u_{n-1}) has two distinct color-neighbors (refer [1] for such a neighbor-locating r-coloring). Choose any $p-1$ distinct colors from r colors (except $c(u_0)$) used in neighbor-locating r-coloring of P_n and assign them to the remaining $p-1$ vertices of the complete graph. Assign distinct colors to the $q-2$ leaf vertices by choosing any $q-2$ colors from r colors of P_n except the colors $c(u_0)$, $c(u_{n-2})$ and $c(u_{n-1})$. A vertex u_i ($i\ne 0, n-2, n-1$) on the path P_n and a leaf vertex w_j such that $c(u_i)=c(w_j)$ are neighbor distinguished since u_i has two distinct color neighbors whereas w_j has only one color neighbor. Hence, we have $\chi_{NL}(G_{p,q,r})=r$. □

Furthermore, we show that, unlike the case of the ordinary chromatic number, an induced subgraph can have an arbitrarily higher locating chromatic number (resp., neighbor-locating chromatic number) than that of the original graph.

Proof of Theorem 2. The graph G_k is constructed as follows. We start with $2k$ disjoint K_1s named $a_1, a_2, \ldots, a_{2k}$ and k disjoint K_2s named $b_1b'_1, b_2b'_2, \ldots, b_kb'_k$. After that, we make all the above mentioned vertices adjacent to a special vertex v to obtain our graph G_k. Notice that v and the a_is must all receive distinct colors under any locating coloring or neighbor-locating coloring. On the other hand, the coloring f given by $f(v)=1$, $f(a_i)=i+1$, $f(b_i)=2i+1$, and $f(b'_i)=2i$ is indeed a locating coloring as well as a neighbor-locating coloring of G_k. Hence we have $\chi_L(G_k)=\chi_{NL}(G_k)=2k+1$.

Now take H_k as the subgraph induced by v, a_is and b_is. It is the graph $K_{1,3k}$. Hence, we have $\chi_L(H_k)=\chi_{NL}(H_k)=3k+1$ [2,15]. □

3. Bounds for sparse graphs

In this section, we study the density of graphs having bounded neighbor-locating chromatic number. The first among those results provides an upper bound on the number of vertices of a graph in terms of its neighbor-locating chromatic number. This, in particular shows that the number of vertices of a graph G is bounded above by a polynomial function of $\chi_{NL}(G)$.

Proof of Theorem 3. We can assume that $k\ge 2$, for otherwise G has only one vertex.
(i) First of all, assume that $k\le \lceil d\rceil$. We know from [2] that $n\le k(2^{k-1}-1)$, and thus:

$$\begin{aligned} n &< \lceil d\rceil 2^{\lceil d\rceil-1}\\ &\le \lceil d\rceil k^{\lceil d\rceil-1}, \end{aligned}$$

and the desired bound holds. Moreover, we clearly have $n=\mathcal{O}(d^2k^{\lceil d\rceil+1})$ in this case.
(ii) For the remainder of the proof, we thus assume that $k\ge \lceil d\rceil+1$ and for convenience, we let $\lceil d\rceil = a$.

Let D_i and d_i denote the set and the number of vertices in G having degree equal to i, respectively, and let D_i^+ and d_i^+ denote the set and the number of vertices in G having degree at least i, respectively, for all $i\ge 1$. As G is connected and hence, does not have any vertex of degree 0, it is possible to write

$$\sum_{v \in V(G)} deg(v) = \sum_{i=1}^{a} i \cdot d_i + \sum_{v \in D_a^+} deg(v) \tag{1}$$

and the number of vertices of G can be expressed as

$$n = (d_1 + d_2 + \cdots + d_a) + d_{a+1}^+ = d_{a+1}^+ + \sum_{i=1}^{a} d_i. \tag{2}$$

As $\sum_{v \in V(G)} deg(v) \leq an$, combining Eqs. (1) and (2) we have

$$\sum_{i=1}^{a} i \cdot d_i + \sum_{v \in D_{a+1}^+} deg(v) \leq a\left(d_{a+1}^+ + \sum_{i=1}^{a} d_i\right) = ad_{a+1}^+ + a\sum_{i=1}^{a} d_i \tag{3}$$

which implies

$$\begin{aligned} d_{a+1}^+ &\leq \sum_{v \in D_{a+1}^+} (deg(v) - a) \\ &= \left(\sum_{v \in D_{a+1}^+} deg(v)\right) - ad_{a+1}^+ \\ &\leq \sum_{i=1}^{a} (a-i)d_i. \end{aligned} \tag{4}$$

The first inequality follows from the fact that there are exactly d_{a+1}^+ terms in the summation $\sum_{v \in D_{a+1}^+} (deg(v) - a)$, where each term is greater than or equal to 1, as $deg(v) \geq a+1$ for all $v \in D_{a+1}^+$. The second inequality can be obtained by rearranging Inequation (3).

Let f be any neighbor-locating k-coloring of G. Consider an ordered pair $(f(u), N_f(u))$, where $deg(u) \leq s$, for some integer $s \leq k-1$. Thus, u may receive one of the k available colors, while its color neighborhood may consist of at most s of the remaining $(k-1)$ colors. Therefore, there are at most $k\sum_{i=1}^{s}\binom{k-1}{i}$ choices for the ordered pair $(f(u), N_f(u))$. Note that for any two vertices u, v of degree at most s, the ordered pairs $(f(u), N_f(u))$ and $(f(v), N_f(v))$ must be distinct. Hence:

$$\sum_{i=1}^{s} d_i \leq k\sum_{i=1}^{s}\binom{k-1}{i}. \tag{5}$$

Since $a \leq k-1$ by assumption, using the above relation, we can derive that

$$\begin{aligned} \sum_{i=1}^{a} (a+1-i)d_i &= \sum_{s=1}^{a}\left(\sum_{i=1}^{s} d_i\right) \\ &\leq \sum_{s=1}^{a}\left(k\sum_{i=1}^{s}\binom{k-1}{i}\right) \\ &= k\sum_{i=1}^{a}(a+1-i)\binom{k-1}{i}. \end{aligned} \tag{6}$$

In the above inequation, the two equalities are algebraic identities, while the inequality is obtained using Inequality (5). Therefore,

$$\begin{aligned} n &= d_{a+1}^+ + \sum_{i=1}^{a} d_i \\ &\leq \sum_{i=1}^{a}(a-i)d_i + \sum_{i=1}^{a} d_i \text{ [using Inequality (4)]} \\ &= \sum_{i=1}^{a}(a+1-i)d_i \end{aligned}$$

$$\leq k\sum_{i=1}^{a}(a+1-i)\binom{k-1}{i} \text{ [using Inequality (6)]}$$
$$< k\sum_{i=1}^{a}\left(a\binom{k-1}{i}\right)$$
$$< ak\sum_{i=1}^{a}k^i$$
$$< a^2k^{a+1}$$
$$\leq (d+1)^2k^{\lceil d\rceil+1}$$
$$= \mathcal{O}(d^2k^{\lceil d\rceil+1}).$$

In particular, we have the desired bound from the first part of (ii). Moreover, we also have the general bound $n = \mathcal{O}(d^2k^{\lceil d\rceil+1})$ in this case. Thus, we are left with only proving the second part of (ii).

For the proof of the second part of (ii), we notice that if the order of a graph G^* fulfilling the constraints of (ii) attains the upper bound, then equality holds in all of the above inequations. In particular, we must have $d^+_{a+1} = \sum_{v\in D^+_{a+1}}(deg(v)-a)$ which implies that G^* cannot have a vertex of degree more than $a+1$. Moreover, we also have the following equality.

$$\sum_{i=1}^{s}d_i = k\sum_{i=1}^{s}\binom{k-1}{i} \text{ for } s = 1, 2, \ldots, a+1$$

which implies that G^* has exactly $k\binom{k-1}{i}$ vertices of degree i. □

The above bound applied to the class of planar graphs (whose average degree is less than 6) gives us the following upper bound.

$$n \leq k\sum_{i=1}^{6}(7-i)\binom{k-1}{i} = \mathcal{O}(k^7).$$

4. Tightness of the obtained bound: proof of Theorem 4

Next, we show the asymptotic tightness of Theorem 3.

The proof of Theorem 4 is contained within a number of observations and lemmas. Also, the proof is constructive, and the constructions depend on particular partial colorings. Therefore, we are going to present a series of graph constructions, their particular colorings, and their structural properties. We are also going to present the supporting observations and lemmas in the following. As the proof is a little involved, we start with the following overview.

The final graph G will be built through constructing a sequence of graphs in a number of iterations. Firstly, we take the base graph G_1 as a path on a certain number of vertices (the number of vertices is decided and declared based on arguments mentioned inside the proof) with neighbor-locating chromatic number s.

We also fix a particular neighbor-locating s-coloring of G_1. Based on this fixed neighbor-locating s-coloring, we will add some vertices and edges to construct G_2. Simultaneously to adding the new vertices and edges, we will extend the neighbor-locating s-coloring to a neighbor-locating coloring with more than s colors (the exact value of increment in s is declared and explained in the proof). Similarly, we will continue to build G_{i+1} from G_i to eventually construct the relevant example G.

Lemma 6. *For two positive integers p, q with $p < q$, consider a $(p \times q)$ matrix whose ijth entry is $m_{i,j}$. Let M be a complete graph whose vertices are the entries of the matrix. Then there exists a matching of M satisfying the following conditions:*

(i) The endpoints of an edge of the matching are from different columns.
(ii) Let e_1 and e_2 be two edges of the matching. If one endpoint of e_1 and e_2 are from the same column, then the other endpoints of them must belong to distinct columns.
(iii) The matching saturates all but at most one vertex of M per column.

Proof. The matching consists of edges of the type $m_{(2i-1),j}m_{2i,i+j}$ for all $i \in \{1, 2, \ldots, \lfloor \frac{p}{2} \rfloor\}$ and $j \in \{1, 2, \ldots, q\}$. Note that throughout this proof, we will consider the addition and subtraction operations on the indices modulo q, where the representatives of the integers modulo q are $1, 2, \ldots, q$. We will show that this matching satisfies all the above-listed conditions.

First observe that, a typical edge of the matching is of the form $m_{(2i-1),j}m_{2i,i+j}$. That means the endpoints of the edge in question are from column j and column $i+j$, respectively. As

$$0 < i \leq \left\lfloor \frac{p}{2} \right\rfloor < p < q,$$

we must have $j \neq i+j$. Thus the condition (i) from the statement is verified.

Next suppose that there are two edges of the type $m_{(2i-1),j}m_{2i,i+j}$ and $m_{(2i'-1),j'}m_{2i',i'+j'}$. If $m_{(2i-1),j}$ and $m_{(2i'-1),j'}$ are from the same column, that is, $j = j'$, then we must have $i \neq i'$ as they are different vertices. This will imply that the other endpoints $m_{2i,i+j}$ and $m_{2i',i'+j'}$ are from different columns as $i + j \neq i' + j = i' + j'$. On the other hand, if $m_{(2i-1),j}$ and $m_{2i',i'+j'}$ are from the same column, then we have $j = i' + j'$. Moreover, if $j' = i + j$, then it will imply

$$j = i' + j' = i' + i + j.$$

This is only possible if q divides $(i + i')$, which is not possible as

$$0 < i + i' \leq 2\left\lfloor \frac{p}{2} \right\rfloor \leq p < q.$$

Therefore, we have verified condition (*ii*) of the statement as well.

Notice that, the matching saturates all the vertices of M when p is even, whereas it saturates all except the vertices in the pth row of the matrix when p is odd. This verifies condition (*iii*) of the statement. □

Corollary 7. *For two positive integers p, q with $p < q$, let G be a graph with an independent set M of size $(p \times q)$, where $M = \{m_{ij} : 1 \leq i \leq p, 1 \leq j \leq q\}$. Moreover, let ϕ be a proper $(s' + q)$-coloring of G satisfying the following conditions:*

(i) all s' colors are assigned to the vertices in $G \setminus M$,
(ii) for any vertex x of G, $s' + 1 \leq \phi(x) \leq s' + q$ if and only if $x \in M$,
(iii) for any two vertices x, y of G, x and y are neighbor-distinguished unless both belong to M,
(iv) for all i, j with $1 \leq i \leq p$ and $1 \leq j \leq q$, $\phi(m_{ij}) = s' + j$.

Then it is possible to find a spanning supergraph G' of G by adding a matching between the vertices of M which will make ϕ a neighbor-locating $(s' + q)$-coloring of G'.

Proof. First of all build a matrix whose ijth entry corresponds to the vertex m_{ij}. After that, build a complete graph whose vertices are entries of this matrix. Now using Lemma 6, we can find a matching of this complete graph that satisfies the three conditions mentioned in the statement of Lemma 6. We construct G' by including exactly the edges corresponding to the edges of the matching, between the vertices of M. We want to show that after adding these edges and obtaining G', indeed ϕ is a neighbor-locating $(s' + q)$-coloring of G'.

Notice that by the definition of ϕ, $(s' + q)$ colors are used. So it is enough to show that the vertices of G' are neighbor-distinguished with respect to ϕ. To be precise, it is enough to show that two vertices x, y from M are neighbor-distinguished with respect to ϕ in G' because of condition (*ii*) of the statement. If for some $x, y \in M$ we have $\phi(x) = \phi(y)$, then that means x, y are from the same column of M. Therefore, according to the conditions of the matching, x, y must have neighbors from separate columns of M, that is, they have neighbors of different colors. This is enough to make x, y neighbor-distinguished. □

Let us recall a result from [1,9] which we shall use in the construction, indeed, G_1 will be defined as a path (see point (iii) of the construction below).

Theorem 8 (*[1,9]*). *Let $k \geq 4$ be an integer and P_n be a path on n vertices. If $\frac{(k-1)^2(k-2)}{2} < n \leq \frac{k^2(k-1)}{2}$, then $\chi_{NL}(P_n) = k$.*

The construction of G_{i+1} from G_i: Now we are ready to present our iterative construction. However, given the involved nature of it, we need some specific nomenclatures to describe it. For convenience, we will list down some points to describe the whole construction.

(i) An *i-triplet* is a 3-tuple of the type (G_i, ϕ_i, X_i) where G_i is a graph, ϕ_i is a neighbor-locating (is)-coloring of G_i, X_i is a set of $(i + 1)$-tuples of vertices of G_i, each tuple having distinct elements. Also, X_i disjointly covers the vertices of G_i, that is, each vertex of G_i appears exactly once in one of the $(i + 1)$-tuples of X_i.

(ii) We will assume a partition Y_i of X_i where two elements $(x_1, x_2, \ldots, x_{i+1})$ and $(x'_1, x'_2, \ldots, x'_{i+1})$ of X_i are put in the same cell of the partition if

$$\{\phi_i(x_j) : j = 1, 2, \ldots, i + 1\} = \{\phi_i(x'_j) : j = 1, 2, \ldots, i + 1\}.$$

That is, the partition is based on the set of colors used on the vertices belonging to the $(i+1)$-tuples. Moreover, the cells of the partition are given by $Y_i = \{X_{i1}, X_{i2}, \ldots, X_{ik_i}\}$ where k_i denotes the number of cells in Y_i. In Lemma 9, we will show that each cell of such partition has less than s vertices. For now, we will accept it as a fact and carry on with the construction.

(iii) Let us describe the 1-triplet (G_1, ϕ_1, X_1) explicitly. Here G_1 is the path $P_t = v_1v_2 \cdots v_t$ on t vertices where $t = 4\left\lfloor \frac{s^2(s-1)}{8} \right\rfloor$. As

$$\frac{(s-1)^2(s-2)}{2} < 4\left\lfloor \frac{s^2(s-1)}{8} \right\rfloor \leq \frac{s^2(s-1)}{2},$$

we must have $\chi_{NL}(P_t) = s$ (by Theorem 8).

Let ϕ_1 be any neighbor-locating s-coloring of G_1 and

$$X_1 = \{(v_{i-1}, v_{i+1}) : i \equiv 2, 3 \pmod 4\}.$$

Clearly each 2-tuple in X_1 has distinct elements and X_1 disjointly covers the vertices of G_1. Therefore, (G_1, ϕ_1, X_1) satisfies (i).

(iv) Suppose an i-triplet (G_i, ϕ_i, X_i) is given. We will (partially) describe a way to construct an $(i+1)$-triplet from it. To do so, first we will construct an intermediate graph G'_{i+1} as follows: for each $(i+1)$-tuple $(x_1, x_2, \ldots, x_{i+1}) \in X_i$ we will add a *new vertex* x_{i+2} adjacent to each vertex from the $(i+1)$-tuple. Moreover, $(x_1, x_2, \ldots, x_{i+1}, x_{i+2})$ is designated as an $(i+2)$-tuple in G'_{i+1}. After that, we will take s copies of G'_{i+1} and call this so-obtained graph G''_{i+1}. Furthermore, we will extend ϕ_i to a function ϕ_{i+1} by assigning the color $(is + j)$ to the new vertices from the jth copy of G'_{i+1}. The copies of the $(i+2)$-tuples are the $(i+2)$-tuples of G''_{i+1}. Note that the set X_{i+1} of all $(i+2)$-tuples disjointly cover the vertices of G''_{i+1}.

(v) Recall the notion of partition from (ii). As ϕ_i and X_i will remain unchanged when we add some edges to construct G_{i+1} (we know in hindsight) from G''_{i+1}, we can already speak about the partition

$$Y_{i+1} = \{X_{(i+1)1}, X_{(i+1)2}, \ldots, X_{(i+1)k_{i+1}}\}$$

of X_{i+1}. Recall that the last vertex of an $(i+2)$-tuple is a new vertex of G''_{i+1}. Observe that two new vertices of G''_{i+1} have the same color if and only if they belong to the same copy of G'_{i+1}. Thus, the vertices of the $(i+2)$-tuples of a particular cell $X_{(i+1)r}$ must belong to the same copy of G'_{i+1} in G''_{i+1}. Thus, if $|X_{1r}| < s$ for all $r \in \{1, 2, \ldots, k_1\}$, then $|X_{ir}| < s$ for all i and for all r. In Lemma 9, we will show that each cell of Y_1 has less than s vertices. For now, we will accept it as a fact and carry on with the construction.

(vi) Here we are going to construct a matrix M using some of the new vertices. Let us assume that $X_{(i+1)r}$ is a cell of the partition Y_{i+1} whose vertices belong to the 1st copy of G'_{i+1} in G''_{i+1}. Let us take the last entries (new vertices) of the $(i+2)$-tuples belonging to $X_{(i+1)r}$ and place them in a column (without repetition). This will be the first column of our matrix M. The lth column of the matrix can be obtained by replacing the entries of the 1st column by their copies from the lth copy of G'_{i+1} in G''_{i+1}. This matrix M is a $(p \times q)$ matrix where $p = |X_{ir}|$ and $q = s$. We have $p < q$ assuming Lemma 9.

(vii) Let us delete all the new vertices from G''_{i+1} except for the ones in M. This graph has the exact same properties of the graph G from Corollary 7, where M plays the role of the independent set. Thus, it is possible to add a matching and extend the coloring (like in Corollary 7). We do that for each cell $X_{(i+1)r}$ of the partition Y_{i+1} whose vertices belong to the 1st copy of G'_{i+1} in G''_{i+1}. After adding all such matchings, the graph we obtain is G_{i+1}. See Figs. 1 and 2 for reference.

Lemma 9. *We have $|X_{1r}| < s$, where X_{1r} is any cell of the partition Y_1.*

Proof. Any vertex (other than the end vertices) in G_1 has two color neighbors say i and j (i is possibly equal to j). Having fixed the two color neighbors, this vertex will have at most $s-1$ choices of colors. Thus $|X_{1r}| < s$. □

Lemma 10. *The function ϕ_{i+1} is a neighbor-locating $(i+1)s$-coloring of G_{i+1}.*

Proof. The function ϕ_{i+1} is constructed from ϕ_i, alongside constructing the triplet G_{i+1} from G_i. While constructing, we use the same steps from that of Corollary 7. Thus, the newly colored vertices become neighbor-distinguished in G_{i+1} under ϕ_{i+1}. □

The above two lemmas validate the correctness of the iterative construction of G_is. However, it remains to show how G_is help us prove our result. To do so, let us prove certain properties of G_is.

Lemma 11. *The graph G_i has maximum degree $(i+1)$.*

Proof. We will prove this by induction. As we have started with a path, our G_1 has maximum degree 2. This proves the base case. Suppose that G_i has maximum degree $(i+1)$ for all $i \leq j$. This is our induction hypothesis. Observe that, in the iteration step for constructing the graph G_{i+1} from G_i, the degree of an old vertex (or its copy) can increase at most by 1, while a new vertex of G_{i+1} is adjacent to exactly $(i+1)$ old vertices and at most one new vertex. Hence, a new vertex in G_{i+1} can have degree at most $(i+2)$. □

Finally, we are ready to prove Theorem 4.

Proof of Theorem 4. We consider the graph G to be $G_{\Delta-1}$ as in our construction. By Lemma 11, the maximum degree of G is Δ. Let k be the neighbor-locating chromatic number of G. Observe that $\chi_{NL}(G_1) = s$ and recall that $s \geq 4$ due to Theorem 8. In each iteration, s new colors are added, hence we have $k = \chi_{NL}(G) \leq (\Delta - 1)s$. First, let us count the number of vertices in G. Let n_i denote the number of vertices in the graph G_i. By the construction, the base graph G_1 has

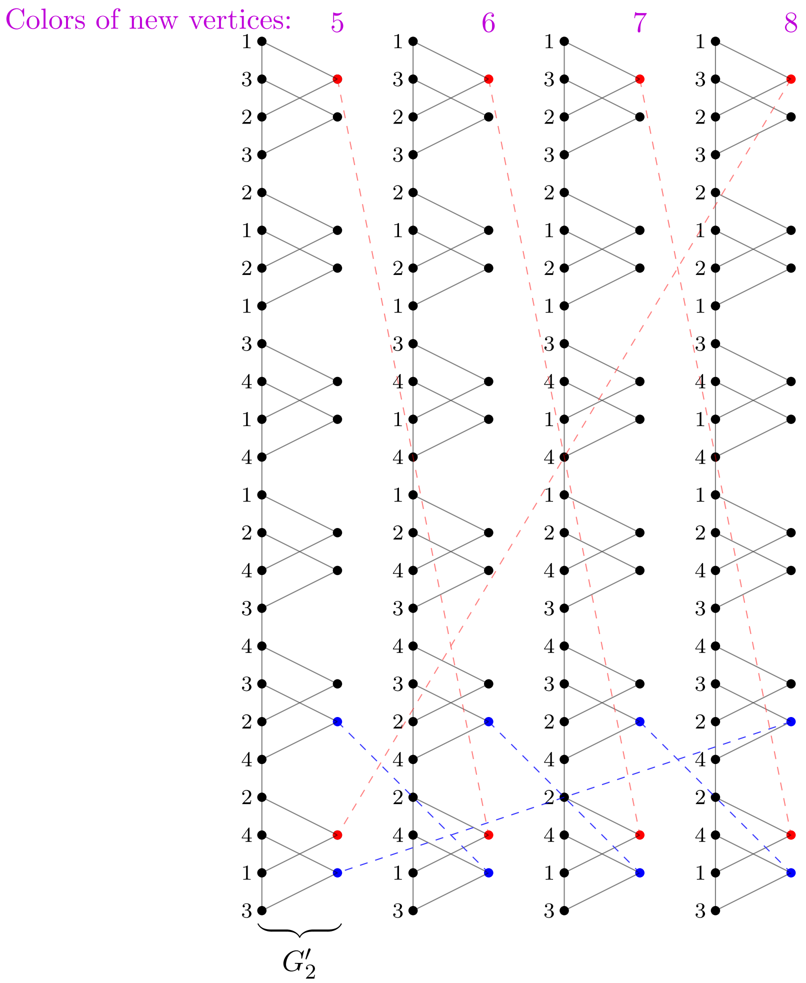

Fig. 1. Construction of G_2 from $G_1 = P_{24}$. Here $\chi_{NL}(P_{24}) = 4$, the red and blue edges are the two sets of newly added matchings. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$n_1 = t = 4\left\lfloor \frac{s^3-s^2}{8} \right\rfloor$ number of vertices. Further, $\frac{n_1}{2}$ new vertices are added to each copy of G_1 to obtain the vertices of G_2. So, $n_2 = s(n_1 + \frac{n_1}{2}) = \frac{3sn_1}{2}$. Further, $\frac{n_2}{3}$ new vertices are added to each of the s copies of G_2 to obtain the vertices of G_3. This gives $n_3 = s(n_2 + \frac{n_2}{3}) = \frac{4}{3}sn_2 = \frac{4}{3}\frac{3}{2}s^2 n_1 = \frac{4}{2}s^2 n_1$. Proceeding in this manner, we have in general, $n_i = \frac{(i+1)}{2}s^{i-1}n_1$. Therefore, putting $i = \Delta - 1$ and using the fact that $n_1 \geq \frac{s^3-s^2}{2} - 4$ it is easy to see that the number of vertices in $G = G_{\Delta-1}$ is

$$
\begin{aligned}
n_{\Delta-1} \geq \frac{\Delta}{4}(s^{\Delta+1} - s^{\Delta} - 8s^{\Delta-2}) &= \frac{\Delta}{4}s^{\Delta+1}\left(1 - \frac{1}{s} - \frac{8}{s^3}\right) \\
&= \frac{\Delta}{4}s^{\Delta+1}\left(1 - \frac{(s^2+8)}{s^3}\right) \\
&\geq \frac{5}{32}\Delta s^{\Delta+1} \text{ (as } s \geq 4) \\
&\geq \frac{5}{32}\Delta\left(\frac{k}{\Delta-1}\right)^{\Delta+1} \\
&= \Omega\left(\Delta\left(\frac{k}{\Delta-1}\right)^{\Delta+1}\right).
\end{aligned}
$$

This establishes the proof. □

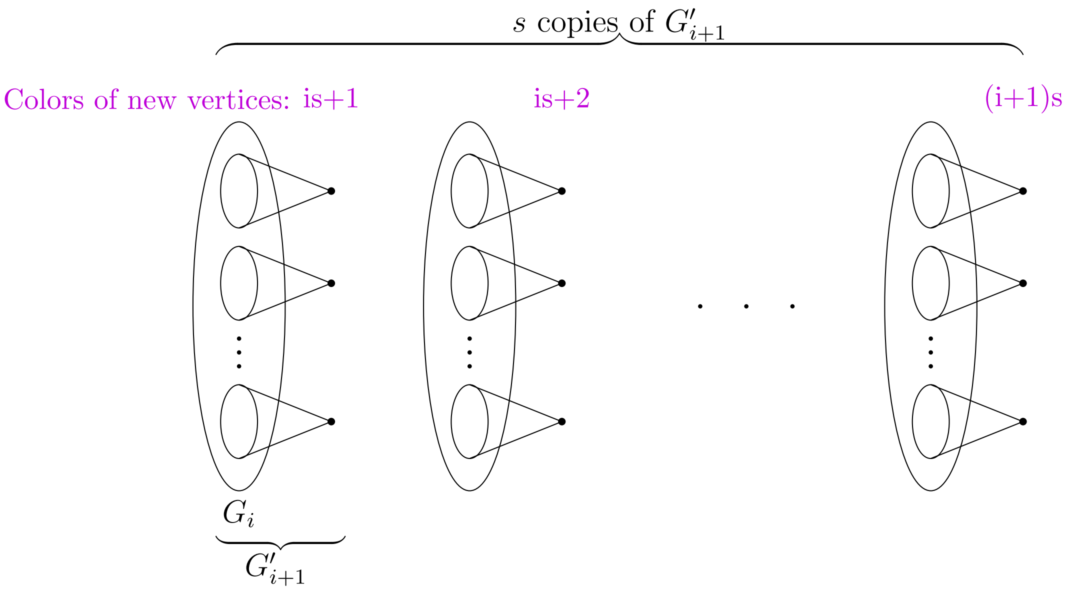

Fig. 2. Construction of G''_{i+1} from G_i.

5. Complexity of locating coloring and neighbor-locating coloring for sparse graphs

In this section, we will show that the locating coloring and the neighbor-locating coloring problems are NP-complete even when restricted to families of sparse graphs. For the sake of precision, let us formally define the 3-coloring, the locating coloring and the neighbor-locating coloring problems.

3-Coloring
Instance: A graph G.
Question: Does there exist a proper 3-coloring of G?

L-Coloring
Instance: A graph G and a positive integer k.
Question: Does there exist a locating k-coloring of G?

NL-Coloring
Instance: A graph G and a positive integer k.
Question: Does there exist a neighbor-locating k-coloring of G?

It is well-known that the 3-Coloring problem is NP-complete [24]. Moreover, the problem remains NP-complete even when restricted to the family of planar graphs having maximum degree 4.

Theorem 12 (*[20]*)**.** *The* 3-Coloring *problem is NP-complete even for planar graphs of maximum degree* 4.

To prove Theorem 5, we provide a reduction from the 3-Coloring problem. The proof involves construction of a graph G^* from a given connected graph G and a few lemmas to analyze its properties.

Construction of G^*: Let G be a connected graph on the vertices $u_1, u_2, \ldots, u_n$. Take a copy of G and call it as G' with the vertices $u'_1, u'_2, \ldots, u'_n$. If u_i is adjacent to u_j, then u'_i is made adjacent to u_j and u_i is made adjacent to u'_j for $i, j \in \{1, 2, \ldots, n\}$.

Next we construct the gadgets X_i for all $i \in \{1, 2, \ldots, n\}$. The gadget X_i consists of a vertex called x_i and two independent sets $A_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,n-1}\}$ and $B_i = \{x'_{i,1}, x'_{i,2}, \ldots, x'_{i,n+2}\}$. Moreover, x_i is adjacent to all the vertices of A_i and B_i. When we say that the gadget X_i is *attached* to a vertex z, we mean that the vertex z is made adjacent to all the vertices in A_i. After that for each $i \in \{1, 2, \ldots, n\}$, the gadget X_i is attached to the vertices u_i and u'_i.

Finally, take another independent set $Y = \{y_1, y_2, y_3\}$ having three vertices. For each $i \in \{1, 2, \ldots, n\}$, we will attach every vertex of Y to the gadgets X_i, and make it adjacent to the vertices x_is as well. See Figs. 3 and 4 for pictorial references.

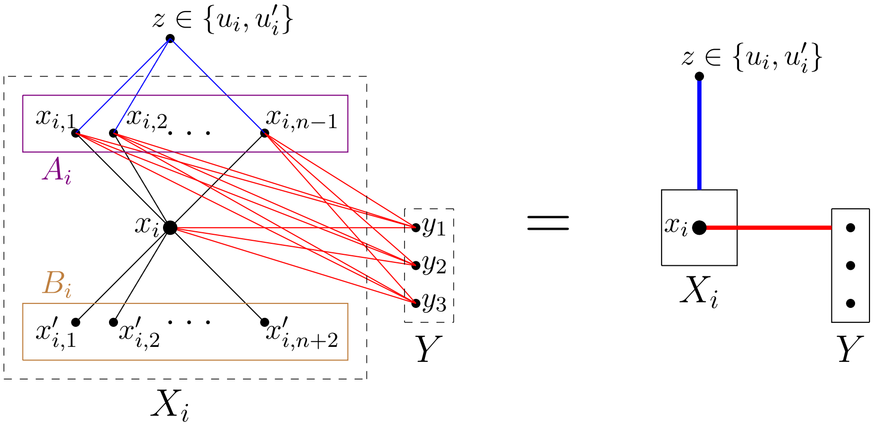

Fig. 3. The gadget X_i from the construction of G^* from G and its connections.

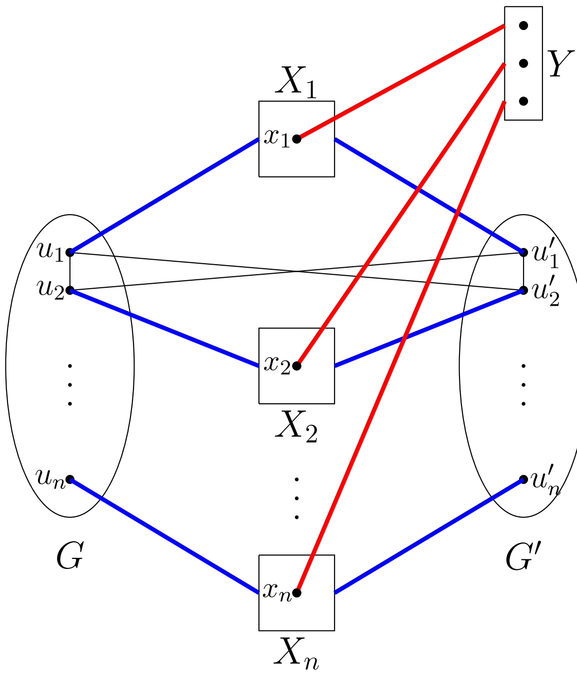

Fig. 4. A schematic diagram for the construction of G^*.

Lemma 13. *Let G be a connected graph on n vertices. If G admits a 3-coloring, then the graph G^* admits a neighbor-locating $(n+3)$-coloring.*

Proof. Let G be a connected graph on n vertices which admits a 3-coloring f. We want to extend f to a neighbor-locating $(n+3)$-coloring f^* of G^*. In this case, we will use $K = \{1, 2, 3, c_1, c_2, c_3, \dots, c_n\}$ as the set of $(n+3)$ colors for f^*. We are going to define f^* first and then show that it is a neighbor-locating coloring.

As mentioned before, f^* is an extension of f, and hence the colors assigned to the vertices of G under f are retained. In other words, we have

$$f^*(u_i) = f(u_i)$$

for all $i \in \{1, 2, \dots, n\}$. Moreover, we assign the color c_i to the vertices u'_i and x_i, that is,

$$f^*(u'_i) = f^*(x_i) = c_i.$$

To the vertices y_1, y_2, and y_3, we assign the colors 1, 2, and 3, respectively. That is,

$$f^*(y_j) = j$$

for all $j \in \{1, 2, 3\}$.

This leaves us with assigning the colors to the vertices of the independent sets A_is and B_is. Notice that for each $i \in \{1, 2, \ldots, n\}$, the set A_i has exactly $(n-1)$ vertices, each of them adjacent to exactly the six vertices $x_i, y_1, y_2, y_3, u_i, u_i'$, and hence are pairwise false twins. Furthermore, notice that the vertices x_i, y_1, y_2, y_3 have received four distinct colors under f^*, namely, c_i, 1, 2, 3, respectively. Thus, to maintain the conditions of a neighbor locating-coloring in hindsight, we will assign distinct colors to the vertices of A_i using the colors from the set $K \setminus \{c_i, 1, 2, 3\}$. To be precise, the value of f^* for the vertices of A_i is decided to be any valid (fixed) solution of the following set theoretic equation:

$$\{f^*(x_{i,1}), f^*(x_{i,2}), \ldots, f^*(x_{i,n-1})\} = \{c_1, c_2, \ldots, c_n\} \setminus \{c_i\}.$$

Similarly, the set B_i has exactly $(n+2)$ vertices, each of them adjacent to exactly one vertex, namely, x_i, and hence are pairwise false twins. To maintain the conditions of a neighbor-locating coloring in hindsight, as $f^*(x_i) = c_i$, we will assign distinct colors to the vertices of B_i using the colors from the set $K \setminus \{c_i\}$. To be precise, the value of f^* for the vertices of B_i is decided to be any valid (fixed) solution of the following set theoretic equation:

$$\{f^*(x'_{i,1}), f^*(x'_{i,2}), \ldots, f^*(x'_{i,n+2})\} = K \setminus \{c_i\}.$$

Next, we will show that f^* is a neighbor-locating $(n+3)$-coloring of G^*. To do so, we will show that the set of vertices having the same color are non-adjacent as well as neighbor-distinguished.

- First we will deal with the vertices that received the color j for some $j \in \{1, 2, 3\}$. Notice that the only vertices that received the color j are some vertices of the original graph G, the vertex y_j from Y, and exactly one vertex of B_i, say b_i, for each $i \in \{1, 2, \ldots, n\}$.
  As f^* is an extension of the 3-coloring f of G, any two vertices of G having the same color are non-adjacent. They are non-adjacent to the vertices of Y as well. Moreover, none of the vertices from B are adjacent to any vertex of G or Y. Therefore, the vertices of G^* having the color j under f^* are all independent.
  To observe that they are also neighbor-distinguished, note that a vertex with color j in G, say u_p, has all c_is, except when $i = p$, as its color-neighbors. Moreover, u_p has at least one color-neighbor from $\{1, 2, 3\}$ as G is connected. That means, the set of color-neighbors of u_p is
  $$N_{f^*}(u_p) = \big(\{c_1, c_2, \ldots, c_n\} \setminus \{c_p\}\big) \cup \big(S \setminus \{f(u_p)\}\big)$$
  where S is a non-empty proper subset of $\{1, 2, 3\}$. The vertex y_j is adjacent to all the vertices of the A_is, and the vertex x_i, and thus has all c_is as its color-neighbors. As y_j does not have any other neighbors apart from the ones mentioned above, the set of color neighbors of y_j is exactly
  $$N_{f^*}(y_j) = \{c_1, c_2, \ldots, c_n\}.$$
  Furthermore, the vertex b_i is adjacent only to the vertex x_i, which implies that the set of color-neighbors of b_i is
  $$N_{f^*}(b_i) = \{c_i\}.$$
  These observations readily imply that the vertices having the color j are pairwise neighbor-distinguished.
- Next we will deal with the vertices that received the color c_i for some $i \in \{1, 2, \ldots, n\}$. Notice that the only vertices that received the color c_i are u_i' from G', the vertex x_i from the gadget X_i, and exactly one vertex of A_p (resp., B_p), say a_p (resp., b_p), for all $p \neq i$.
  From the construction of G^*, we know that u_i' and x_i are non-adjacent. Moreover, u_i' and x_i are both non-adjacent to the vertices of A_p and B_p, as long as $p \neq i$. Furthermore, there is no edge between the vertices of the sets A_p and B_q for all $p, q \in \{1, 2, \ldots, n\}$. Hence, we have shown that the vertices of G^* that received the color c_j are independent.
  To observe that they are also neighbor-distinguished, note that the vertex u_i' is adjacent to some vertices of G as G is connected. However, u_i' is not adjacent to those vertices of G that have received the color $f(u_i)$ due to the construction. The vertex u_i is also adjacent to some vertices of G', and all the vertices of A_i. As all the vertices of G', except u_i', are colored using the set $\{c_1, c_2, \ldots, c_n\} \setminus \{c_i\}$, and as all the colors of the set $\{c_1, c_2, \ldots, c_n\} \setminus \{c_i\}$ are used for the vertices of A_i, we can say that the set of color-neighbors of u_i' is given by
  $$N_{f^*}(u_i') = (\{c_1, c_2, \ldots, c_n\} \setminus \{c_i\}) \cup (S \setminus \{f(u_i)\})$$
  where S is a non-empty proper subset of $\{1, 2, 3\}$. The vertex x_i is adjacent to exactly the vertices of A_i, B_i, and Y. That means, the set of color-neighbors of x_i is given by
  $$N_{f^*}(x_i) = K \setminus \{c_i\}.$$
  Furthermore, b_p has only one color-neighbor, which is c_p, whereas the set of color-neighbors of a_p is
  $$N_{f^*}(a_p) = \{1, 2, 3, c_p\}.$$
  These observations readily imply that the vertices having the color c_i are pairwise neighbor-distinguished.

This proves that f^* is indeed a neighbor-locating $(n+3)$-coloring of G^*. □

Lemma 14. *Let G be a connected graph on n vertices. If G^* admits a neighbor-locating $(n+3)$-coloring, then G admits a 3-coloring.*

Proof. Suppose that G^* admits a neighbor-locating $(n+3)$-coloring f using the set of colors $K = \{1, 2, 3, c_1, c_2, \dots, c_n\}$. Since the vertices y_1, y_2 and y_3 have the same open neighborhood, under any neighbor-locating $(n+3)$-coloring f of G^*, we have to assign three distinct colors to these three vertices. Without loss of generality, we may assume $f(y_1) = 1$, $f(y_2) = 2$ and $f(y_3) = 3$.

For each $i \in \{1, 2, \dots, n\}$, there are $(n+2)$ vertices of degree one in B_i, all adjacent to x_i. Thus, the vertices of B_i must receive $(n+2)$ distinct colors. Notice that, $f(x_i) \notin \{1, 2, 3\}$ as it is adjacent to all the vertices of Y. Hence, x_i must receive one of the colors, say $c \in \{c_1, c_2, \dots, c_n\}$. Furthermore, the set of all color-neighbors of x_i is given by $N_f(x_i) = K \setminus \{c\}$. So, to be neighbor distinguished, x_i and x_j must receive different colors whenever $i \neq j$. This will force the vertices from $\{x_1, x_2, \dots, x_n\}$ to receive distinct colors from $\{c_1, c_2, \dots, c_n\}$. Thus, without loss of generality, we may assume $f(x_i) = c_i$ for all $i \in \{1, 2, \dots, n\}$.

As the $(n-1)$ vertices of A_i are adjacent to the vertex x_i, which has received the color c_i, and all vertices of Y, which have received the colors 1,2 and 3, they cannot receive a color from the set $\{1, 2, 3, c_i\}$. Moreover, as the vertices of A_i are false twins, they must receive distinct colors. This implies that the vertices of A_i must receive (all) the colors from the set $\{c_1, c_2, \dots, c_n\} \setminus \{c_i\}$.

Since u_i and u'_i are adjacent to all the vertices of A_i, they cannot receive a color from the set $\{c_1, c_2, \dots, c_n\} \setminus \{c_i\}$. In other words, u_i and u'_i must receive colors from the set $\{1, 2, 3, c_i\}$ only. As u_i and u'_i are false twins, they cannot receive the same color. Therefore, one of them must receive a color from the set $\{1, 2, 3\}$. If u_i receives the color c_i and $f(u'_i) \in \{1, 2, 3\}$, then we swap the colors of u_i and u'_i so that we have $f(u'_i) = c_i$ and $f(u_i) \in \{1, 2, 3\}$. As u_i and u'_i are false twins, this does not affect the neighbor-locating coloring of G^*. Hence, the restriction of f to the induced subgraph G will provide a 3-coloring of G. □

Lemma 15. *Let G be a connected graph on n vertices. If G admits a 3-coloring, then the graph G^* admits a locating $(n+3)$-coloring.*

Proof. Since every neighbor-locating coloring is also a locating coloring, the proof follows from Lemma 13. □

Lemma 16. *Let G be a connected graph on n vertices. If G^* admits a locating $(n+3)$-coloring, then G admits a 3-coloring.*

Proof. Note that, under any locating coloring, false twins must receive distinct colors as they have the same distance to every vertex. Suppose that G^* admits a locating $(n+3)$-coloring f using the set of colors $K = \{1, 2, 3, c_1, c_2, \dots, c_n\}$. As the vertices y_1, y_2 and y_3 are pairwise false twins, they must receive distinct colors, say $f(y_1) = 1, f(y_2) = 2$ and $f(y_3) = 3$.

There are $(n+2)$ vertices of degree one in B_i for all $i \in \{1, 2, \dots, n\}$, which are pairwise false twins. So, they must receive $(n+2)$ distinct colors. Further, $f(x_i) \notin \{1, 2, 3\}$ as x_i is adjacent to all the vertices of Y. Let $c \in \{c_1, c_2, \dots, c_n\}$ be the color given to x_i, where c does not appear in B_i. If x_i and x_j $(i \neq j)$ receive the same color c, then they are not metric-distinguished as they are at distance one from all other color classes. Hence, x_i and x_j must receive distinct colors. This implies that the vertices $\{x_1, x_2, \dots, x_n\}$ must receive distinct colors from $\{c_1, c_2, \dots, c_n\}$. Without loss of generality, let $f(x_i) = c_i$ for all $i \in \{1, 2, \dots, n\}$.

Note that the $(n-1)$ vertices of A_i are adjacent to the vertex x_i, which has received the color c_i, and all vertices of Y, which have received the colors 1,2 and 3. So, they cannot receive a color from the set $\{1, 2, 3, c_i\}$. Moreover, as the vertices of A_i are pairwise false twins, they must receive distinct colors. This implies that the vertices of A_i must receive (all) the colors from the set $\{c_1, c_2, \dots, c_n\} \setminus \{c_i\}$.

As u_i and u'_i are adjacent to all the vertices of A_i, they cannot receive a color from the set $\{c_1, c_2, \dots, c_n\} \setminus \{c_i\}$. The only set of colors allowed for u_i and u'_i are $\{c_i, 1, 2, 3\}$. Moreover, as u_i and u'_i are false twins, they must receive distinct colors. Therefore, one of them is forced to receive a color from the set $\{1, 2, 3\}$. If u_i receives the color c_i and $f(u'_i) \in \{1, 2, 3\}$, then we swap the colors of u_i and u'_i so that we have $f(u'_i) = c_i$ and $f(u_i) \in \{1, 2, 3\}$. As u_i and u'_i are false twins, this does not affect the neighbor-locating coloring of G^*. Thus, restricting the coloring f to the induced subgraph G gives a 3-coloring of G. □

Lemma 17. *If G is a connected planar graph with maximum degree 4, then G^* has average degree at most 7.*

Proof. Let G be a connected planar graph with maximum degree 4 on n vertices. Then G has $m \leq 2n$ edges. Let us first count the number of vertices n^* in G^*. There are n vertices in each of G and G', $(2n+2)$ vertices in each of the n gadgets X_i, and 3 vertices in the set Y. Thus, we have

$$n^* = (n+n) + n(2n+2) + 3 = 2n^2 + 4n + 3.$$

Next, let us count the number of edges m^* in G^*. There are m edges in each of G and G', $2m$ edges between the vertices of G and G', $(n-1)$ edges between each vertex u_i (resp., u'_i) and the gadget X_i, $(2n+1)$ edges in each of the gadgets X_i, and $3n$ edges between each X_i and Y. Thus, we have

$$m^* = (m+m) + 2m + 2n(n-1) + n(2n+1) + 3n^2 = 7n^2 - n + 4m \leq 7n^2 + 7n.$$

Therefore, G^* is a graph with average degree at most 7. □

Lemma 18. *If G is a connected planar graph with maximum degree 4, then G^* has maximum average degree at most 20.*

Proof. Let G be a connected planar graph with maximum degree 4 on n vertices. We will observe an edge decomposition of G^*.

Let G_1^* be the subgraph of G^* induced by the vertices of G and G'. As G has maximum degree 4, G_1^* has maximum degree 8. Therefore, the maximum average degree of G_1^* is 8 or less.

Let G_2^* be the graph obtained from G^* by deleting the vertices of G' and Y, and the edges of G. This is a triangle-free planar graph, and thus has maximum average degree less than 4.

Let G_3^* be the graph obtained by taking the vertices of G' and the X_is, and the vertex y_1. Moreover, G_3^* also has the edges between the vertices of G' and the X_is, as well as the vertex y_1 and the X_is. Even this is a triangle-free planar graph, and thus has maximum average degree less than 4.

Let G_4^* be the graph obtained by taking the vertices y_2, y_3, and the vertices of the X_is. Moreover, G_4^* also have the edges between the vertices y_2, y_3 and the X_is. This is also a triangle-free planar graph, and thus has maximum average degree less than 4.

Notice that, the edges of the subgraphs G_1^*, G_2^*, G_3^*, and G_4^* together give all the edges of G^*. Thus, we can say that the maximum average degree of G^* is at most 20. □

Lemma 19. *If G is a connected planar graph with maximum degree 4, then the graph G^* is 4-partite.*

Proof. By the Four-Color Theorem [4], every planar graph is 4-colorable. Thus, there is a 4-coloring, say f, of the graph G. We want to extend f to a 4-coloring f^* of G^*. As admitting a 4-coloring and being 4-partite are the same, we will be done if we can extend f as mentioned above.

As mentioned before, f^* is an extension of f, and hence the colors assigned to the vertices of G under f are retained. Moreover, for a vertex of G, we assign the same color to its false twin in G'. In other words, we have

$$f^*(u_i') = f^*(u_i) = f(u_i)$$

for all $i \in \{1, 2, \dots, n\}$.

Next, if $f^*(u_i) = 1$, then we will assign $f^*(x_i) = 1$. On the other hand, if $f^*(u_i) \neq 1$, then we will assign $f^*(x_i) = 2$. Furthermore, we will assign the color 2 (resp., 1) to all the vertices of A_i and B_i if $f^*(x_i) = 1$ (resp., $f^*(x_i) = 2$). Finally, we assign the color 3 to all the vertices of Y. Notice that this is a 4-coloring of G^*. □

Proof of Theorem 5. It is easy to verify whether a given coloring is a neighbor-locating coloring (resp. locating coloring), so the problem is in NP.

On the other hand, Lemmas 13, 14 show that the NL-Coloring problem is NP-hard and Lemmas 15, 16 show that the L-Coloring problem is NP-hard for the graphs of the type G^* where G is a connected graph. Moreover, as the 3-Coloring problem remains NP-hard even when restricted to the family of connected planar graphs having maximum degree at most 4, and as Lemmas 17, 18, and 19 show that under such conditions, G^* has average degree at most 7, maximum average degree at most 20, and is a 4-partite graph, the proof follows. □

6. Conclusions

In this article, we have studied the neighbor-locating coloring of sparse graphs. Initially, we studied how big the gaps can be between the related parameters $\chi(G)$, $\chi_L(G)$ and $\chi_{NL}(G)$. Later, we have obtained an upper bound on the number of vertices of a sparse graph in terms of neighbor-locating chromatic number. Also, we have proved that the bound is tight by providing constructions of graphs which almost achieve the bound. Moreover, we have proved that the L-Coloring and the NL-Coloring problems are NP-complete for sparse graphs with average degree at most 7, maximum average degree at most 20 and 4-partite. Based on our work, and in general relevant to the topic, we would like to provide a list of open problems.

Question 1. *What is a tight bound for the maximum order of a planar graph with neighbor-locating chromatic number k? Is the bound $n = \mathcal{O}(k^7)$ tight?*

Question 2. *Under what condition does a graph have its neighbor-locating chromatic number equal to its locating chromatic number (resp., chromatic number)?*

Question 3. *How much can the neighbor-locating chromatic number increase or decrease after deleting a vertex (resp., an edge) of a graph?*

Question 4. *Are the* L-Coloring *and the* NL-Coloring *problems NP-hard for other restricted classes of sparse graphs, for example planar graphs, or graphs of bounded maximum degree (for example subcubic graphs)?*

Question 5. *Can the lower bound obtained from the construction from Theorem 4 be further improved and made closer to the upper bound from Theorem 3?*

Data availability

No data was used for the research described in the article.

Acknowledgments

We would like to thank Prof. Koteswararao Kondepu, Department of Computer Science & Engineering, IIT Dharwad for his valuable suggestions regarding possible applications. This work is partially supported by the following projects: "MA/IFCAM/18/39", "SRG/2020/001575", "MTR/2021/000858", "NBHM/RP-8 (2020)/Fresh" and "NSOU Project No. Reg/0520 dated 07.06.2024". Research by the first and the second authors is supported by the French government IDEX-ISITE initiative CAP 20-25 (ANR-16-IDEX-0001), the International Research Center "Innovation Transportation and Production Systems" of the I-SITE CAP 20-25, and the ANR project GRALMECO (ANR-21-CE48-0004).

References

[1] L. Alcon, M. Gutierrez, C. Hernando, M. Mora, I.M. Pelayo, The neighbor-locating-chromatic number of pseudotrees, 2019, arXiv preprint arXiv:1903.11937.
[2] L. Alcon, M. Gutierrez, C. Hernando, M. Mora, I.M. Pelayo, Neighbor-locating colorings in graphs, Theoret. Comput. Sci. 806 (2020) 144–155.
[3] L. Alcon, M. Gutierrez, C. Hernando, M. Mora, I.M. Pelayo, The neighbor-locating-chromatic number of trees and unicyclic graphs, Discuss. Math. Graph Theory 43 (3) (2023) 659–675.
[4] K. Appel, W. Haken, J. Koch, Every planar map is four colorable, Illinois J. Math. 21 (1977) 439–567.
[5] H. Assiyatun, D.K. Syofyan, E.T. Baskoro, Calculating an upper bound of the locating-chromatic number of trees, Theoret. Comput. Sci. 806 (2020) 305–309.
[6] L. Babai, On the complexity of canonical labeling of strongly regular graphs, SIAM J. Comput. 9 (1) (1980) 212–216.
[7] E.T. Baskoro, A. Asmiati, Characterizing all trees with locating-chromatic number 3, Electron. J. Graph Theory Appl. (EJGTA) 1 (2) (2013) 109–117.
[8] E.T. Baskoro, D.I.D. Primaskun, Improved algorithm for the locating-chromatic number of trees, Theoret. Comput. Sci. 856 (2021) 165–168.
[9] A. Behtoei, M. Anbarloei, The locating chromatic number of the join of graphs, Bull. Iranian Math. Soc. 40 (6) (2014) 1491–1504.
[10] A. Behtoei, B. Omoomi, On the locating chromatic number of Kneser graphs, Discrete Appl. Math. 159 (18) (2011) 2214–2221.
[11] A. Behtoei, B. Omoomi, On the locating chromatic number of the Cartesian product of graphs, Ars Combin. 126 (2016) 221–235.
[12] B. Bollobás, A.D. Scott, On separating systems, European J. Combin. 28 (4) (2007) 1068–1071.
[13] D. Chakraborty, F. Foucaud, S. Nandi, S. Sen, D.K. Supraja, New bounds and constructions for neighbor-locating colorings of graphs, in: Algorithms and Discrete Applied Mathematics: 9th International Conference, CALDAM 2023, Gandhinagar, India, February 9–11, 2023, Proceedings, Springer, 2023, pp. 121–133.
[14] E. Charbit, I. Charon, G.D. Cohen, O. Hudry, A. Lobstein, Discriminating codes in bipartite graphs: bounds, extremal cardinalities, complexity, Adv. Math. Commun. 2 (4) (2008) 403–420.
[15] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, P. Zhang, The locating-chromatic number of a graph, Bull. Inst. Combin. Appl. 36 (2002) 89–101.
[16] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, P. Zhang, Graphs of order n with locating-chromatic number $n-1$, Discrete Math. 269 (1–3) (2003) 65–79.
[17] B.S. Chlebus, S.H. Nguyen, On finding optimal discretizations for two attributes, in: Proceedings of the First International Conference on Rough Sets and Current Trends in Computing, Vol. 1424, Springer Berlin Heidelberg, Berlin, Heidelberg, 1998, pp. 537–544.
[18] V. Chvátal, Mastermind, Combinatorica 3 (3) (1983) 325–329.
[19] M. Furuya, N. Matsumoto, Upper bounds on the locating chromatic number of trees, Discrete Appl. Math. 257 (2019) 338–341.
[20] M.R. Garey, D.S. Johnson, L. Stockmeyer, Some simplified NP-complete problems, in: Proceedings of the Sixth Annual ACM Symposium on Theory of Computing, 1974, pp. 47–63.
[21] S.A. Goldman, R.L. Rivest, R.E. Schapire, Learning binary relations and total orders, SIAM J. Comput. 22 (5) (1993) 1006–1034.
[22] F. Harary, R. Melter, On the metric dimension of a graph, Ars Combin. 2 (1976) 191–195.
[23] C. Hernando, M. Mora, I.M. Pelayo, L. Alcón, M. Gutierrez, Neighbor-locating coloring: graph operations and extremal cardinalities, Electron. Notes Discrete Math. 68 (2018) 131–136.
[24] R.M. Karp, Reducibility among combinatorial problems, Complex. Comput. Comput. 1 (1972) 85–103.
[25] J.H. Kim, O. Pikhurko, J.H. Spencer, O. Verbitsky, How complex are random graphs in first order logic? Random Structures Algorithms 26 (1–2) (2005) 119–145.
[26] Z. Krivohlava, S. Chren, B. Rossi, Failure and fault classification for smart grids, Energy Inform. 5 (1) (2022) 33.
[27] C.-K. Lee, Y.-J. Shin, Multi-core cable fault diagnosis using cluster time-frequency domain reflectometry, in: 2018 IEEE International Instrumentation and Measurement Technology Conference (I2MTC), IEEE, 2018, pp. 1–6.
[28] D.A. Mojdeh, On the conjectures of neighbor locating coloring of graphs, Theoret. Comput. Sci. 922 (2022) 300–307.
[29] B.M.E. Moret, H.D. Shapiro, On minimizing a set of tests, SIAM J. Sci. Stat. Comput. 6 (4) (1985) 983–1003.
[30] N. Rao, Computational complexity issues in operative diagnosis of graph-based systems, IEEE Trans. Comput. 42 (4) (1993) 447–457.
[31] A. Rényi, On random generating elements of a finite Boolean algebra, Acta Sci. Math. Szeged 22 (1961) 75–81.
[32] A. Sebő, E. Tannier, On metric generators of graphs, Math. Oper. Res. 29 (2) (2004) 383–393.
[33] P.J. Slater, Leaves of trees, in: Proceedings of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, in: Congressus Numerantium, Vol. 14, 1975, pp. 549–559.
[34] P.J. Slater, Dominating and reference sets in a graph, J. Math. Phys. Sci. 22 (4) (1988) 445–455.
[35] R. Ungrangsi, A. Trachtenberg, D. Starobinski, An implementation of indoor location detection systems based on identifying codes, in: Intelligence in Communication Systems, IFIP International Conference, INTELLCOMM 2004, Bangkok, Thailand, November 23-26, 2004, Proceedings, in: Lecture Notes in Computer Science, Vol. 3283, Springer, 2004, pp. 175–189.
[36] D.B. West, Introduction to Graph Theory, Vol. 2, Prentice hall Upper Saddle River, 2001.