

Identifying open codes in trees and 4-cycle-free graphs of given maximum degree

Dipayan Chakraborty^{a,b,c,d,*}, Florent Foucaud^a, Michael A. Henning^b

^a Université Clermont Auvergne, CNRS, Clermont Auvergne INP, Mines Saint-Étienne, LIMOS, 63000 Clermont-Ferrand, France

^b Department of Mathematics and Applied Mathematics, University of Johannesburg, South Africa

^c Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon

^d Aix-Marseille Université, CNRS, École Centrale Méditerranée, LIS, Marseille, France



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ABSTRACT

An identifying open code of a graph G is a set S of vertices that is both a separating open code (that is, $N_G(u) \cap S \neq N_G(v) \cap S$ for all distinct vertices u and v in G) and a total dominating set (that is, $N(v) \cap S \neq \emptyset$ for all vertices v in G). Such a set exists if and only if the graph G is open twin-free and isolate-free; and the minimum cardinality of an identifying open code in an open twin-free and isolate-free graph G , its identifying open code number, is denoted by $\gamma^{\text{IOC}}(G)$.

We study the identifying open code number of a graph, in relation with its order and its maximum degree. For Δ a fixed integer at least 3, we show that if G is a connected graph of order $n \geq 5$ that contains no 4-cycle and is open twin-free with maximum degree bounded above by Δ , then $\gamma^{\text{IOC}}(G) \leq \left(\frac{2\Delta-1}{2\Delta}\right)n$, unless G is obtained from a star $K_{1,\Delta}$ by subdividing every edge exactly once. Moreover, we show that the bound is best possible by constructing graphs that reach the bound when $\Delta = 3$, and nearly best possible by another construction when $\Delta \geq 4$, with identifying open code numbers $\left(\frac{2\Delta-4}{2\Delta-3}\right)n$.

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1. Introduction

The goal of this paper is to investigate the possible size of a smallest identifying open code in a graph of given maximum degree. Identifying open codes in graphs define a concept that is simultaneously part of two rich areas in discrete mathematics: domination problems and identification problems.

1.1. Definitions

A set S of vertices in a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S . Further if every vertex in G is adjacent to some other vertex in S , then S is a *total dominating set*, abbreviated TD-set of G . The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TD-set of G . A TD-set of cardinality $\gamma_t(G)$ is called a γ_t -set of G . A vertex v *totally dominates* another vertex u if they are adjacent vertices. More generally, if X and Y are subsets of vertices in G , then the set X *totally dominates* the set Y in G if every vertex in Y is adjacent to at least one vertex in X . In particular, if X totally dominates $V(G)$, then X is a TD-set of G . For recent books on domination in graphs, we refer the reader to [18–21].

* Correspondence to: École Centrale Méditerranée, Technopôle Château-Gombert, 38 rue Frédéric Joliot-Curie, 13013 Marseille, France.
E-mail addresses: dipayancha@gmail.com (D. Chakraborty), florent.foucaud@uca.fr (F. Foucaud), mahenning@uj.ac.za (M.A. Henning).

For graph theory notation and terminology, we generally follow [21]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of a graph G , denoted $n(G)$ and $m(G)$, is $|V(G)|$ and $|E(G)|$, respectively. Two vertices u and v of G are *adjacent* if $uv \in E(G)$, and are called *neighbors*. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v , while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of a vertex v in G is the number of neighbors v in G , and is denoted by $\deg_G(v)$, and so $\deg_G(v) = |N_G(v)|$. An *isolated vertex* in G is a vertex of degree 0. A graph is *isolate-free* if it contains no isolated vertices. A *leaf* in G is a vertex of degree 1 in G , and a *support vertex* in G is a vertex with at least one leaf neighbor. A *strong support vertex* in G is a vertex with at least two leaf neighbors. The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n . The *complete graph* on n vertices is denoted by K_n , while the *complete bipartite graph* with one partite set of size n and the other of size m is denoted by $K_{n,m}$. A *star* is a complete bipartite graph $K_{1,n}$. A *cubic graph* is graph in which every vertex has degree 3, while a *subcubic graph* is a graph with maximum degree at most 3. A *subcubic tree* is a tree with maximum degree at most 3. The *diameter* of a graph G , denoted $\text{diam}(G)$, is the maximum distance among all pairs of vertices of G . In particular, the diameter of a tree T is the length of a longest path in T .

A *rooted tree* T distinguishes one vertex r called the *root*. Let T be a tree rooted at vertex r . For each vertex $v \neq r$ of T , the *parent* of v is the neighbor of v on the unique (r, v) -path, while a *child* of v is any other neighbor of v . The root r does not have a parent in T and all its neighbors are its children. A *descendant* of v is a vertex x such that the unique (r, x) -path contains v . Thus, every child of v is a descendant of v . A *grandchild* of v is a descendant of v at distance 2 from v . Let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v , and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v , denoted T_v , is the subtree of T induced by the set $D[v]$. For $k \geq 1$ an integer, we let $[k]$ denote the set $\{1, \dots, k\}$ and we let $[k]_0 = [k] \cup \{0\} = \{0, 1, \dots, k\}$.

A graph G has a TD-set if and only if it is isolate-free. Hence throughout this paper all graphs are isolate-free, unless otherwise stated. A *separating open code* in G is a set S of vertices such that $N_G(u) \cap S \neq N_G(v) \cap S$ for all distinct vertices u and v in G . An *identifying open code* of G , abbreviated IO-code, is a set S that is both a separating open code and a TD-set. Two vertices in G are *open twins* if they have the same open neighborhood. A graph is *open twin-free* if it has no open twins. An isolate-free graph has an IO-code if and only if it is open twin-free. The minimum cardinality of an IO-code in an open twin-free graph G is denoted by $\gamma^{\text{IOC}}(G)$ and called the *identifying open code number* of G (IO-code number for short).

1.2. Previous work

An IO-code has also been called an *open (neighborhood) locating–dominating set* [29,30] or an *identifying code with nontransmitting faulty vertices* [23] in the literature. We follow the terminology from [22]. The problem of “identifying open codes” was introduced by Honkala, Laihonen and Ranto [23] in the context of coding theory for binary hypercubes and further studied, for example, in [1–3,7,8,10,11,14,15,17,22,25,26,28–30]. This concept is closely related to the extensively studied concepts of identifying codes and locating–dominating sets in graphs, and is part of a vast literature on general identification problems in discrete structures, with many theoretical and practical applications. For a survey on (open) identifying codes and locating (total) domination in graphs we refer the reader to the book chapter by Lobstein, Hudry, and Charon [27], and for an online bibliography maintained by Jean and Lobstein on these topics, see [24].

Regarding upper bounds on the IO-code number, the graphs G of order n with $\gamma^{\text{IOC}}(G) = n$ have been characterized by Foucaud, Ghareghani, Roshany-Tabrizi and Sharifani in [10] as the family of *half-graphs* defined in [9] (the half-graph H_k is the bipartite graph with both parts of size k , where each part can be ordered so that the open neighborhoods of consecutive vertices differ by exactly one vertex). In [30], Seo and Slater characterized the trees T of order n with $\gamma^{\text{IOC}}(T) = n - 1$.

Such upper bounds have also been studied for classic identifying codes, where the open neighborhood is replaced by the closed neighborhood in their definition. In particular, some works have explored the best possible upper bound for the optimal size of an identifying code, depending on the maximum degree and the order of the graph [4–6,13,16]. Those works have inspired the current article.

Further motivation comes from the work by Henning and Yeo [22], who initiated the study of upper bounds on the IO-code number of graphs of given maximum degree, focusing on regular graphs. They proved, using an interplay with transversal of hypergraphs, that $\gamma^{\text{IOC}}(G) \leq \frac{3}{4}n$ for every open twin-free connected cubic graph G and this bound is tight for the complete graph K_4 and the hypercube of dimension 3. They also suggested that perhaps, more generally, $\gamma^{\text{IOC}}(G) \leq \left(\frac{\Delta}{\Delta+1}\right)n$ holds for every connected open twin-free Δ -regular graph G of order n . Our goal is to investigate what happens when the graph is non-regular. We will see that, interestingly, the above conjectured bound for the IO-code number of regular graphs must be modified for non-regular graphs.

1.3. Main result

Let G be a connected graph of order $n \geq 2$ that is open twin-free. If $n = 2$, then $G \cong P_2$ and $\gamma^{\text{IOC}}(G) = 2 = n$. If $n = 3$, then $G \cong K_3$ and $\gamma^{\text{IOC}}(G) = 2 = \frac{2}{3}n$. If $n = 4$, then $G \cong P_4$ and $\gamma^{\text{IOC}}(G) = 4 = n$. Our aim is to obtain a best possible upper bound on IO-codes in open twin-free connected graphs G of order $n \geq 5$ that contains no 4-cycles and have bounded maximum degree.

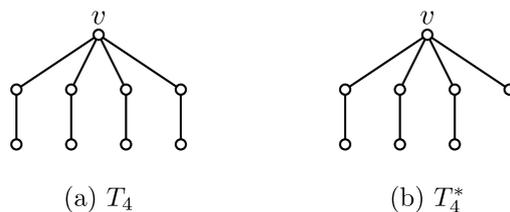


Fig. 1. The subdivided star $T_4 = S(K_{1,4})$.

For $\Delta \geq 3$, a subdivided star $T_\Delta = S(K_{1,\Delta})$ is the tree obtained from a star $K_{1,\Delta}$ with universal vertex v , say, by subdividing every edge exactly once. The resulting tree T_Δ has order $n = 2\Delta + 1$ and maximum degree $\Delta(T_\Delta) = \Delta$. For example, when $\Delta = 4$ the subdivided star $T_4 = S(K_{1,4})$ is illustrated in Fig. 1(a). For $\Delta \geq 3$, a reduced subdivided star T_Δ^* is the tree obtained from a subdivided star $K_{1,\Delta}$ by removing exactly one leaf. The resulting tree T_Δ^* has order $n = 2\Delta$ and maximum degree $\Delta(T_\Delta) = \Delta$. For example, when $\Delta = 4$ the reduced subdivided star T_4^* is illustrated in Fig. 1(b). In both the subdivided and the reduced subdivided tree, the vertex v is called the central vertex of the tree.

Our main result is to prove that, not only for trees but also for graphs with no 4-cycles, among all graphs of given maximum degree, the subdivided stars described above require the largest fraction of their vertex set in any optimal IO-code. More precisely, we prove the following statement.

Theorem 1. For $\Delta \geq 3$ a fixed integer, if $G \not\cong T_\Delta$ is an open twin-free connected graph of order $n \geq 5$ that contains no 4-cycles and satisfies $\Delta(G) \leq \Delta$, then

$$\gamma^{\text{IOC}}(G) \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

On the other hand, when $G \cong T_\Delta$, then we have $\gamma^{\text{IOC}}(G) = \left(\frac{2\Delta}{2\Delta+1}\right)n$.

Moreover, we show that the above bound is nearly tight for every value of $\Delta \geq 3$ by providing a construction of graphs with IO-code numbers equal to $\left(\frac{2\Delta-4}{2\Delta-3}\right)n$. Furthermore, when $\Delta = 3$, we give another construction that provides infinitely many connected graphs for which the bound is attained.

1.4. Organization of the paper

We first prove Theorem 1 for trees in Section 2: this is in fact the essential part of the proof. We then prove Theorem 1 in full generality in Section 3. We provide constructions to show the tightness of the bound in Section 4, and we conclude in Section 5 with directions for future research.

2. Trees

In this section, we prove Theorem 1 for trees. Towards this goal, we first define some special families of trees that will be essential in the proof.

2.1. A special family \mathcal{T} of trees

In this section, we define a special family of trees that we will often need when proving our main result. In order to define our special family of trees, we introduce some additional notations. Let r be a specified vertex in a tree T . We define next several types of attachments at the vertex r that we use to build larger trees. In all cases, we call the vertex of the attachment that is joined to r the link vertex of the attachment.

- For $i \in [4]$, an attachment of Type- i at r is an operation that adds a path P_i to T and joins one of its leaves to r .
- An attachment of Type-5 at r is an operation that adds a path P_4 to T and joins one of its support vertices to r .
- An attachment of Type-6 at r is an operation that adds a star $K_{1,3}$ with one edge subdivided and joins one of its leaves that is adjacent to the vertex of degree 3 to r .

Let \mathbb{N} be the set of all non-negative integers and let

$$\mathbb{N}_*^6 = \{\mathbf{k} = (k_1, k_2, k_3, k_4, k_5, k_6) \in \mathbb{N}^6 : k_i \in \{0, 1\} \text{ and } \mathbf{k} \neq (1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (1, 1, 0, 0, 0, 0)\}. \tag{1}$$

From now on, let \mathbf{k} denote a vector in \mathbb{N}_*^6 , let k_i denote the i th coordinate of \mathbf{k} and let $k = \sum_{i=1}^6 k_i$. We define $T(r; \mathbf{k})$ to be the tree obtained from a trivial tree K_1 whose vertex is named r by applying k_i attachments of Type- i at vertex r

for each $i \in [6]$. Notice that $\text{deg}_T(r) = k$. We now define the set $\mathcal{T} = \{T(r; \mathbf{k}) : \mathbf{k} \in \mathbb{N}_*^6\}$ to be our special family of trees. We note that the forbidden configurations of the vector \mathbf{k} in the definition of \mathbb{N}_*^6 exclude the trees P_2, P_3 and P_4 from the family \mathcal{T} , as in our main result, we only consider graphs of order at least 5.

The tree $T(r; 1, 3, 2, 3, 2, 2)$, for example, is illustrated in Fig. 2(a). For $k \geq 2$, the subdivided star T_k is isomorphic to the tree $T(r; 0, k, 0, 0, 0, 0) \in \mathcal{T}$ and also to the tree $T(r; 1, 0, 1, 0, 0, 0) \in \mathcal{T}$ for $k = 2$ (see Fig. 2(c)). For $k \geq 3$, the reduced subdivided star T_k^* is isomorphic to the tree $T(r; 1, k - 1, 0, 0, 0, 0) \in \mathcal{T}$ and also to the tree $T(r; 1, 0, 0, 0, 1, 0)$ for $k = 2$ (see Fig. 2(c)).

Definition 2. We define the canonical set C of a tree $T = T(r; \mathbf{k}) \in \mathcal{T}$ as follows. For $\mathbf{k} \notin \{(1, 0, 1, 0, 0, 0), (1, 0, 0, 0, 1, 0)\}$, that is, when T is neither T_2 nor T_3^* with r being a degree 2 support vertex of T (as illustrated in Figs. 2(b) and (c), respectively), we define the canonical set C of T as follows.

- Initially, we add the vertex r to the set C .
- If $k_1 = 1$ and $k_2 \geq 1$, then we add to C all vertices that belong to attachments of Type-2 at r , but we do not add to C the vertex that belongs to the attachment of Type-1 at r .
- If $k_1 = 0$ and $k_2 \geq 1$, then we add to C all vertices that belong to attachments of Type-2 at r , except for one leaf at distance 2 from r in T in exactly one attachment of Type-2.
- We add to C all vertices that belong to attachments of Type-3 at r that are not leaves at distance 3 from r in T .
- We add to C all vertices that belong to attachments of Type-4 at r that are not leaves at distance 4 from r in T .
- We add to C all vertices of attachments of Type-5 at r except for the leaves at distance 2 from r in T .
- We add to C all vertices of attachments of Type-6 at r except for the leaves at distance 3 from r in T .

For $\mathbf{k} \in \{(1, 0, 1, 0, 0, 0), (1, 0, 0, 0, 1, 0)\}$, that is, when T is either T_2 or T_3^* with r being a degree 2 support vertex of T (as in Figs. 2(b) and (c), respectively), we define the canonical set C of T as follows.

- For $\mathbf{k} = (1, 0, 1, 0, 0, 0)$, add to C all vertices of T except for the leaf of T at distance 3 of r .
- For $\mathbf{k} = (1, 0, 1, 0, 0, 0)$, add to C all vertices of T except for the leaf of T at distance 2 of r .

For notational convenience, we shall abbreviate from now on the notations for the trees of the form $T(r; (1, 0, 1, 0, 0, 0))$ as $T(r; 1, 0, 1, 0, 0, 0)$. For the trees $T(r; 1, 3, 2, 3, 2, 2)$, $T(r; 1, 0, 1, 0, 0, 0)$ and $T(r; 1, 0, 0, 0, 1, 0)$, for example, the shaded vertices in Fig. 2(a), (b) and (c), respectively, indicate the canonical set C of the tree. Notice that the two trees $T(r; \mathbf{k})$ for $\mathbf{k} \in \{(0, 2, 0, 0, 0, 0), (1, 0, 1, 0, 0, 0)\}$ (both isomorphic to T_2) have the same canonical sets as prescribed in Definition 2. This property also holds for the trees $T(r; \mathbf{k})$ for $\mathbf{k} \in \{(1, 2, 0, 0, 0, 0), (1, 0, 0, 0, 1, 0)\}$ (both isomorphic to T_3^*).

Observation 3. The canonical set of any tree $T = T(r; \mathbf{k}) \in \mathcal{T}$ is also an IO-code of T .

We next determine the IO-code numbers of (reduced) subdivided stars T_k and T_k^* .

Proposition 4. For $k \geq 2$ a fixed integer and a tree T of order n , the following properties hold.

- (a) If $T \cong T_k$, then $\gamma^{\text{IOC}}(T) = \binom{2k}{2k+1} n$.
- (b) If $k \geq 3$ and $T \cong T_k^*$, then $\gamma^{\text{IOC}}(T) = \binom{2k-1}{2k} n$.

Moreover, in both cases, the canonical set of T is an optimal IO-code of T .

Proof (a). Let $T \cong T_k$ have order n where $k \geq 2$, and let v be the central vertex of T (of degree k). Thus, $n = 2k + 1$. Let S be an IO-code of T . Since S is a TD-set of T , in order to totally dominate all the leaves in T the set S contains all support vertices of T . Since S is a separating open code, in order to identify the support vertices of T the set S contains all, except possibly one leaf. If S contains all leaves of T , then $|S| \geq 2k$. Suppose that S does not contain all leaves of T . By our earlier observations, in this case there is exactly one leaf, u say, that does not belong to S . Let w be the support vertex adjacent to u . Thus, $N_T(w) = \{u, v\}$. In order to totally dominate the vertex w , we infer that $v \in S$, implying once again that $|S| \geq 2k$. Since S is an arbitrary IO-code of T , this implies that $\gamma^{\text{IOC}}(T) \geq 2k$. Now, assuming $T \cong T(r; 0, k, 0, 0, 0, 0) \in \mathcal{T}$, let C be the canonical set of the tree T as in Definition 2, that is, C is the set of all vertices of T except for one leaf of T . Then, by Observation 3, the set C is an IO-code and it satisfies $\gamma^{\text{IOC}}(T) \leq |C| = 2k$. Consequently, $\gamma^{\text{IOC}}(T) = 2k = \binom{2k}{2k+1} n$.

(b) Let $T \cong T_k^*$ have order n where $k \geq 3$, and let v be the central vertex of T (of degree k) and let x denote the leaf neighbor of v . Thus, $n = 2k$. Let S be an IO-code of T . Since S is a TD-set of T , the set S contains all support vertices of T . Since S is an IO-code of T , in order to identify the leaf x from all other neighbors of v , the set S contains all leaves of T at distance 2 from v . Thus, S contains all vertices of T , except possibly for the leaf x of T . Thus, $|S| \geq 2k - 1$. Since S is an arbitrary IO-code of T , this implies that $\gamma^{\text{IOC}}(T) \geq 2k - 1$. On the other hand, assuming $T \cong T(r; 1, k - 1, 0, 0, 0, 0)$, let C be the canonical set of T as in Definition 2, that is, C consists of all vertices of T except for the leaf neighbor x of v . Again, by Observation 3, the set C is an IO-code of T . Moreover, we have $\gamma^{\text{IOC}}(T) \leq |C| = 2k - 1$. Consequently, $\gamma^{\text{IOC}}(T) = 2k - 1 = \binom{2k-1}{2k} n$. \square

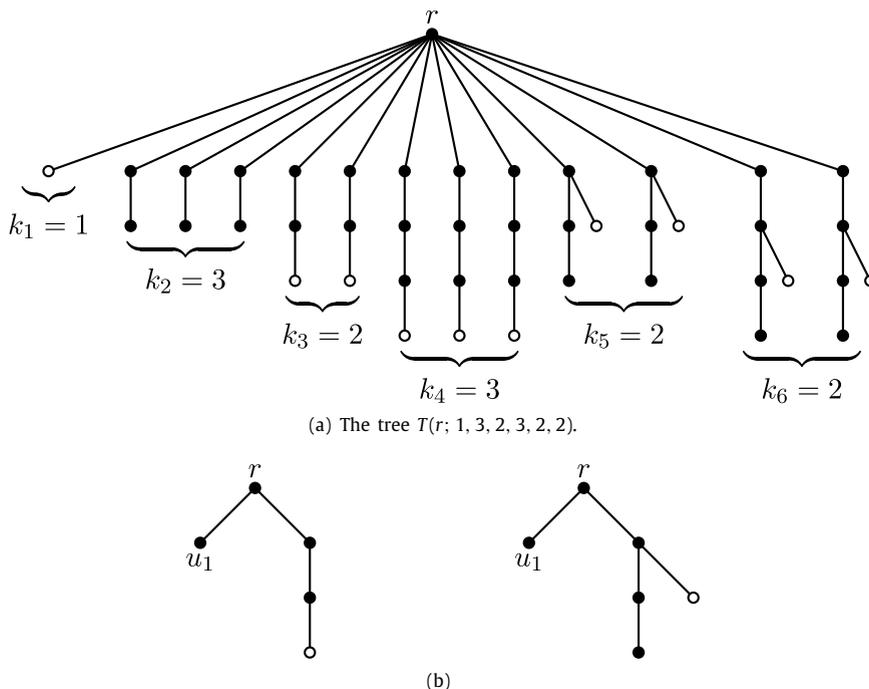


Fig. 2. The canonical sets of a general tree and the trees T_2 and T_3^* in \mathcal{T} .

We next prove that the bound of Theorem 1 holds for the trees in \mathcal{T} .

Proposition 5. Let $\Delta \geq 3$ be a fixed integer and let $T \cong T(r; \mathbf{k}) \in \mathcal{T}$ be a tree of order $n \geq 5$, maximum degree $\Delta(T) \leq \Delta$ such that $\deg_T(r) = k \geq 2$. If T is not isomorphic to the subdivided star T_Δ , then

$$\gamma^{\text{loc}}(T) \leq \left(\frac{2\Delta - 1}{2\Delta} \right) n.$$

Proof. By supposition, $\Delta \geq \Delta(T) \geq \deg_T(r) = k$. Therefore, if $T \cong T_k$ for some $k \leq \Delta - 1$, then the result follows from Proposition 4. Notice that T cannot be isomorphic to the reduced subdivided star $T_2^* \cong P_4$, since $T \in \mathcal{T}$ and the latter does not contain the path P_4 . On the other hand, if T is isomorphic to the reduced subdivided star T_k^* for $k \geq 3$, then again the desired upper bound follows from Proposition 4. Hence, we may assume that the tree T is neither isomorphic to T_k nor to T_k^* for all $2 \leq k \leq \Delta$. Therefore, in particular, T is neither the tree $T(r; 1, 0, 0, 0, 1, 0)$ as in Fig. 2(b) nor the tree $T(r; 1, 0, 0, 0, 1, 0)$ as in Fig. 2(c).

Let C be the canonical set of $T \cong T(r; \mathbf{k})$ as in Definition 2. Let A be the set of vertices of T consisting of the vertex r and all vertices that belong to attachments of Type-1 or of Type-2 at r , and let $B = V(T) \setminus A$. Further, let $a = |A|$ and $b = |B|$, and so $n = a + b$. Since $T \not\cong T_k$ and $T \not\cong T_k^*$, we must have $k_1 + k_2 \leq k - 1$. We note that $1 \leq a \leq 2k - 1$. Let $C_1 = C \cap A$. Let us first assume that $a = 1$. In this case, we have $k_1 + k_2 = 0$ and so, the tree T has no attachments of Type-1 or Type-2. Now, by the construction of the canonical set C , we have $|C| \leq \frac{5}{6} \leq \left(\frac{2\Delta - 1}{2\Delta} \right) n$ and hence, the result holds in this case. Let us therefore assume now that $a \geq 2$. Since neither $\mathbf{k} = (1, 0, 1, 0, 0, 0)$ nor $\mathbf{k} = (1, 0, 0, 0, 1, 0)$, by construction of the canonical set C , we have $|C_1| = a - 1 \leq \left(\frac{2k - 2}{2k - 1} \right) a$. Further, let $C_2 = C \cap B$, and so $|C_2| \leq \frac{4}{5}b$. Since C is an IO-code of T by Observation 3 and $\Delta \geq 3$, we infer that

$$\gamma^{\text{loc}}(T) \leq |C| = |C_1| + |C_2| \leq \left(\frac{2k - 2}{2k - 1} \right) a + \frac{4}{5}b < \left(\frac{2\Delta - 1}{2\Delta} \right) n. \quad \square$$

2.2. Theorem 1 for trees

In this section, we prove the following Theorem 7 which we will need when proving our main result, namely Theorem 1, for graphs that contain no 4-cycles. First, we need a lemma.

Lemma 6. Let T be an open twin-free tree on n vertices and let T_1 and T_2 be open twin-free trees obtained by removing an edge from T . Moreover, let S_i be an IO-code of T_i for each $i \in \{1, 2\}$ and let $S = S_1 \cup S_2$. Then we have the following.

- (a) The set S is an IO-code of T .
- (b) If $|S_i| \leq \left(\frac{2\Delta-1}{2\Delta}\right) |V(T_i)|$, then $|S| \leq \left(\frac{2\Delta-1}{2\Delta}\right) n$.

Proof. Let the edge removed from the tree T be v_1v_2 . Moreover, by a possible renaming of the vertices, let $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$.

(a) Suppose, to the contrary, that there exists a pair x, y of vertices of T which are not identified by S . Then we must have, without loss of generality, $x = v_1$ and $y \in V(T_2)$ with $v_2y \in E(T_2)$. Since S_1 is a TD-set of T_1 , we must have $v_1z \in E(T_1)$ for some vertex $z \in V(T_1) \cap S_1$. Now since S does not identify the pair x, y , it implies that $yz \in E(T)$ as well which further implies that zxv_2yz is a 4-cycle in T which contradicts the fact that the latter is a tree. Hence, S is an IO-code of T .

(b) Since the sets S_1 and S_2 are disjoint, we therefore have

$$|S| = |S_1| + |S_2| \leq \left(\frac{2\Delta-1}{2\Delta}\right) (|V(T_1)| + |V(T_2)|) = \left(\frac{2\Delta-1}{2\Delta}\right) n. \quad \square$$

Theorem 7. For $\Delta \geq 3$ a fixed integer, if $T \not\cong T_\Delta$ is an open twin-free tree of order $n \geq 5$ that satisfies $\Delta(T) \leq \Delta$, then

$$\gamma^{\text{IOC}}(T) \leq \left(\frac{2\Delta-1}{2\Delta}\right) n.$$

On the other hand, when $T \cong T_\Delta$, then we have $\gamma^{\text{IOC}}(T) = \left(\frac{2\Delta}{2\Delta+1}\right) n$.

Proof. Let $\Delta \geq 3$ be a fixed integer. We proceed by induction on the order $n \geq 5$ of a tree T that is open twin-free and satisfies $\Delta(T) \leq \Delta$. Since T has order $n \geq 5$ and is twin-free, we note that $\text{diam}(T) \geq 4$ and T has no strong support vertices, that is, every support has exactly one leaf neighbor. We proceed further with a series of claims.

Claim 8. If $\text{diam}(T) = 4$, then one of the following holds.

- (a) $T \cong T_\Delta$ and $\gamma^{\text{IOC}}(T) = \left(\frac{2\Delta}{2\Delta+1}\right) n$.
- (b) $T \not\cong T_\Delta$ and $\gamma^{\text{IOC}}(T) \leq \left(\frac{2\Delta-1}{2\Delta}\right) n$.

Proof of claim. Suppose that $\text{diam}(T) = 4$. Since T has no strong support vertex, either $T \cong T_k$ for some k where $2 \leq k \leq \Delta$ or $T \cong T_k^*$ for some k where $3 \leq k \leq \Delta$. If $T \cong T_\Delta$, then by Proposition 4(a), we have $\gamma^{\text{IOC}}(T) = \left(\frac{2\Delta}{2\Delta+1}\right) n$, and so statement (a) of the claim holds. If $T \cong T_k$ for some k where $2 \leq k \leq \Delta - 1$, then by Proposition 4(a),

$$\gamma^{\text{IOC}}(T) = \left(\frac{2k}{2k+1}\right) n \leq \left(\frac{2\Delta-2}{2\Delta-1}\right) n < \left(\frac{2\Delta-1}{2\Delta}\right) n.$$

If $T \cong T_k^*$ for some k where $3 \leq k \leq \Delta$, then by Proposition 4(b),

$$\gamma^{\text{IOC}}(T) = \left(\frac{2k-1}{2k}\right) n \leq \left(\frac{2\Delta-1}{2\Delta}\right) n,$$

and so statement (b) of the claim holds. \square

By Claim 8, we may assume that $\text{diam}(T) \geq 5$, for otherwise the desired result holds. In what follows, let $e = v_1v_2$ be an edge of T and let F_i be the component of $T - e$ that contains the vertex v_i for $i \in [2]$. Further, let F_i have order n_i where $i \in [2]$, and so $n = n_1 + n_2$. We note that T is obtained from the trees F_1 and F_2 by adding back the edge e .

Claim 9. If $F_1 \cong T_k$ and v_1 is the central vertex of degree k in F_1 for some k where $2 \leq k \leq \Delta - 1$, then

$$\gamma^{\text{IOC}}(T) \leq \left(\frac{2\Delta-1}{2\Delta}\right) n.$$

Proof of claim. Suppose that $F_1 \cong T_k$ and v_1 is the central vertex of degree k in F_1 for some k where $2 \leq k \leq \Delta - 1$. Then we have $F_1 \cong T(v_1; 0, k, 0, 0, 0, 0) \in \mathcal{T}$. Let S_1 denote the canonical set of the tree F_1 . Thus, S_1 contains all vertices of F_1 , except for exactly one leaf, u_1 say. In particular, we note that S_1 contains the vertex v_1 and all support vertices of F_1 . We first consider the case when $n_2 \leq 4$. Since $T \not\cong T_\Delta$, we have $n_2 \neq 2$. Moreover, for $n_2 \in \{1, 3, 4\}$, the tree T is isomorphic to some tree in the special family \mathcal{T} . Therefore, in the case that $n_2 \leq 4$, we are done by Proposition 5. Hence, we may assume that $n_2 \geq 5$.

Let us now assume that F_2 is open twin-free. Suppose that $F_2 \cong T_\Delta$. In this case, either v_2 is a support vertex in F_2 (as illustrated in Fig. 3(a)) or v_2 is a leaf in F_2 (as illustrated in Fig. 3(b)). In both cases, let $S_1 = V(F_1) \setminus \{u_1\}$. If v_2 is a support vertex in F_2 , then let S_2 consist of all vertices of F_2 except for exactly two leaves, one of which is the leaf neighbor of v_2 in F_2 . If v_2 is a leaf in F_2 , then let S_2 consist of all vertices of F_2 except for exactly one leaf which is different from the vertex v_2 . Let $S = S_1 \cup S_2$. In the first case when v_2 is a support vertex in F_2 , the resulting set S is illustrated by the shaded vertices in Fig. 3(a). In the second case when v_2 is a leaf in F_2 , the resulting set S is illustrated by the shaded

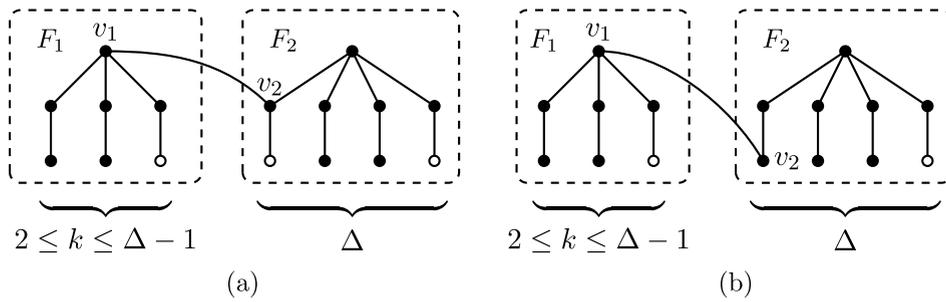


Fig. 3. Possible trees in the proof of Claim 9.

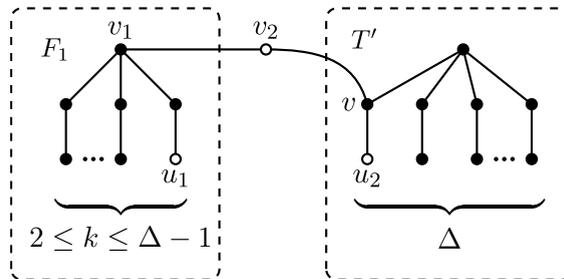


Fig. 4. A possible tree in the proof of Claim 9.

vertices in Fig. 3(b). In both cases, the set S is an IO-code of T by Lemma 6(a). Thus, since $|S| = |S_1| + |S_2| \leq 2k + 2\Delta$ and $n = (2k + 1) + (2\Delta + 1)$, and since $k \leq \Delta - 1$, we infer that

$$\gamma^{\text{IOC}}(T) \leq |S| \leq \left(\frac{2k + 2\Delta}{2k + 2\Delta + 2}\right)n = \left(\frac{k + \Delta}{k + \Delta + 1}\right)n \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

Hence we may assume that $F_2 \not\cong T_\Delta$. Applying the inductive hypothesis to the open twin-free tree F_2 of order $n_2 \geq 5$, there exists an IO-code, S_2 say, of F_2 such that $\gamma^{\text{IOC}}(F_2) = |S_2| \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n_2$. Let $S = S_1 \cup S_2$. Since the vertex v_2 in F_2 is totally dominated by some other vertex of F_2 in S_2 , the set S is an IO-code of the tree T . By Proposition 5, we have $|S_1| \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n_1$, and so

$$\gamma^{\text{IOC}}(T) \leq |S| = |S_1| + |S_2| \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n_1 + \left(\frac{2\Delta - 1}{2\Delta}\right)n_2 = \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

Let us now assume that F_2 contains open twins. Since the tree T contains no open twins, we infer that v_2 is a leaf in F_2 and is an open twin in F_2 with some other leaf, say u_2 , in F_2 . Let v be the common neighbor of v_2 and u_2 in F_2 . Let $T' = F_2 - v_2$, and let T' have order n' , and so $n' = n_2 - 1 \geq 4$. Since T is open twin-free, so too is the tree T' .

Suppose that $n' = 4$, and so $n_2 = 5$. In this case, the tree $T \cong T(v_1; 0, k, 0, 0, 0, 1) \in \mathcal{T}$ and therefore, we are done by Proposition 5. Hence we may assume that $n' \geq 5$. Suppose that $T' \cong T_\Delta$. In this case, the vertex v is a support vertex in the tree T' and the tree T is illustrated in Fig. 4. Let u_2 denote the leaf neighbor of the vertex v . We note that $n_1 = 2k + 1$ and $n_2 = 2\Delta + 2$, and so $n = 2k + 2\Delta + 3$. We now let S_2 consist of all vertices in T' , except for the leaf u_2 . Thus, $|S_2| = 2\Delta$. Further, we let $S = S_1 \cup S_2$, and so the set S is indicated by the shaded vertices in Fig. 4. Then by Lemma 6, the set S is an IO-code of the tree, and so

$$\gamma^{\text{IOC}}(T) \leq |S| = 2k + 2\Delta = \left(\frac{2k + 2\Delta}{2k + 2\Delta + 3}\right)n \leq \left(\frac{4\Delta - 2}{4\Delta + 1}\right)n < \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

Hence, we may assume that $T' \not\cong T_\Delta$. Recall that $n' \geq 5$. Applying the inductive hypothesis to the open twin-free tree T' of order $n' \geq 5$, there exists an IO-code, S' say, of T' such that $\gamma^{\text{IOC}}(T') = |S'| \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n'$. Now let $S = S_1 \cup S'$. Since u_2 is a leaf in T' , the IO-code S' of T' contains the support vertex v , which has two neighbors in S . Therefore, the set S is an IO-code of the tree T , and so

$$\gamma^{\text{IOC}}(T) \leq |S| = |S_1| + |S'| \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n_1 + \left(\frac{2\Delta - 1}{2\Delta}\right)n' < \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

This completes the proof of Claim 9. \square

By Claim 9, we may assume that the removal of any edge from T does not produce a component T' isomorphic to T_k , where $2 \leq k \leq \Delta - 1$ and where the central vertex of the subdivided star T' is incident with the deleted edge, for otherwise the desired result follows. As a consequence of Claim 9, we have the following property of the tree T .

Claim 10. *If T contains an edge whose removal produces a component isomorphic to T_Δ , then*

$$\gamma^{\text{IOC}}(T) \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

Proof of claim. Suppose that T contains an edge $f = xy$ whose removal produces a component, T' say, isomorphic to T_Δ . Renaming x and y if necessary, we may assume that $x \in V(T')$. Let v be the central vertex of the subdivided star in T' , and so v has degree Δ in T' (and in the original tree T). If x is a support vertex of T' , then we infer from Claim 9 with $v_1 = v$ and $v_2 = x$ that $\gamma^{\text{IOC}}(T) \leq \left(\frac{2\Delta-1}{2\Delta}\right)n$. If x is a leaf of T' , then we let w denote the common neighbor of v and x , and we infer from Claim 9 with $v_1 = v$ and $v_2 = w$ that $\gamma^{\text{IOC}}(T) \leq \left(\frac{2\Delta-1}{2\Delta}\right)n$. \square

By Claim 10, we may assume that the removal of any edge from T does not produce a component isomorphic to T_Δ , for otherwise the desired result follows.

For the rest of the proof, let $v_0v_1v_2 \dots v_d$ to be a fixed longest path in T . Then v_0 and v_d are leaves in T . Let us also assume from here on that the tree T is rooted at the leaf v_d . Since $\text{diam}(T) \geq 5$ by assumption, we must have $d \geq 5$. This implies that, for all $i \in [5]$, the vertex v_i is the ancestor of v_{i-1} in T . If v is a vertex in T , then we denote by T_v the maximal subtree rooted at v , and so T_v consists of v and all descendants of the vertex v . Since T has no strong support vertex, we note that $\text{deg}_T(v_1) = 2$.

Claim 11. *If $\text{deg}_T(v_2) \geq 4$, or if $\text{deg}_T(v_i) \geq 3$ for $i \in \{3, 4\}$, then*

$$\gamma^{\text{IOC}}(T) \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

Proof of claim. We first assume that $\text{deg}_T(v_2) \geq 4$. Then, the maximal subtree T_{v_2} of T rooted at v_2 is isomorphic to a tree in the special family \mathcal{T} . By Claim 9, we may assume that $T_{v_2} \not\cong T_k$ for any $k \leq \Delta - 1$. Hence, T_{v_2} must be isomorphic to the reduced subdivided tree T_k^* . Let T_{v_2} be on n_1 vertices and let C be the canonical set of T_{v_2} . Then, by Observation 3, the set C is an IO-code of the tree T_{v_2} and by Proposition 4, we have $|C| \leq \left(\frac{2\Delta-1}{2\Delta}\right)n_1$. Moreover, let T' be the component of $T - v_2v_3$ that contains the vertex v_3 , and let T' have order n' . If $n' \leq 4$, then T is isomorphic to a tree in the special family \mathcal{T} and thus, we are done by Proposition 5. Let us therefore assume that $n' \geq 5$. By Claim 10, we also assume that T' is not isomorphic to T_Δ .

First, we assume that T' is open twin-free. Therefore, applying the inductive hypothesis to the open twin-free tree T' , there exists an IO-code, S' say, of T' such that $\gamma^{\text{IOC}}(T') = |S'| \leq \left(\frac{2\Delta-1}{2\Delta}\right)n'$. Let $S = C \cup S'$. Then by Lemma 6, the set S is an IO-code of the tree T with

$$\gamma^{\text{IOC}}(T) \leq |S| \leq \left(\frac{2\Delta - 1}{2\Delta}\right)n.$$

We now suppose that T' contains open twins. Since the tree T contains no open twins, we infer that v_3 is a leaf in T' and is an open twin in T' with some other leaf, say u_3 , in T' . We note that v_4 is the common neighbor of v_3 and u_3 . Let $T'' = T' - v_3$, and let T'' have order n'' , and so $n'' = n' - 1 \geq 4$. If $n'' = 4$, since v_3 and u_3 are open twins in T'' , it implies that $T \cong T(v_2; 1, k - 2, 0, 0, 0, 1) \in \mathcal{T}$ for some $k = \text{deg}_T(v_2) - 1 \leq \Delta - 1$. Hence, in this case, we are again done by Proposition 5. Let us therefore assume that $n'' \geq 5$. Since T is open twin-free, so too is the tree T'' . By Claim 10, we infer that $T'' \not\cong T_\Delta$. Applying the inductive hypothesis to the open twin-free tree T'' of order $n'' \geq 5$, there exists an IO-code, S'' say, of T'' such that $\gamma^{\text{IOC}}(T'') = |S''| \leq \left(\frac{2\Delta-1}{2\Delta}\right)n''$. Let $S = C \cup S''$. We note that since u_3 is a leaf, the vertex $v_4 \in S''$ and so the vertex v_3 has two neighbors in the set S and thus is identified by S . Hence, the set $S = C \cup S''$ is an IO-code of T , and so

$$\begin{aligned} \gamma^{\text{IOC}}(T) \leq |S| &= |C| + |S''| \\ &\leq \left(\frac{2\Delta-1}{2\Delta}\right)n_1 + \left(\frac{2\Delta-1}{2\Delta}\right)n'' \\ &< \left(\frac{2\Delta-1}{2\Delta}\right)(n_1 + 1) + \left(\frac{2\Delta-1}{2\Delta}\right)n'' \\ &= \left(\frac{2\Delta-1}{2\Delta}\right)(n - n'') + \left(\frac{2\Delta-1}{2\Delta}\right)n'' \\ &= \left(\frac{2\Delta-1}{2\Delta}\right)n. \end{aligned}$$

This proves the result in the case that $\text{deg}_T(v_2) \geq 4$. We can therefore assume from now on that $\text{deg}_T(v_2) \leq 3$, or else, the result holds. In fact, more strongly, we can also assume that whenever $\text{deg}_T(v_2) = 3$, the subtree T_{v_2} is not isomorphic to the subdivided star T_2 . This is due to our previous assumption using Claim 9 that $T_{v_2} \not\cong T_k$ for any $k \leq \Delta - 1$.

We now look at the case that $\text{deg}_T(v_i) \geq 3$ for all $i \in \{3, 4\}$. As before, let n_1 be the number of vertices of T_{v_1} , the subtree of T rooted at v_1 . Then T_{v_3} (respectively, T_{v_4}) is isomorphic to a tree $T(r; \mathbf{k})$ in \mathcal{T} such that $k_3 + k_5 \geq 1$ (respectively,

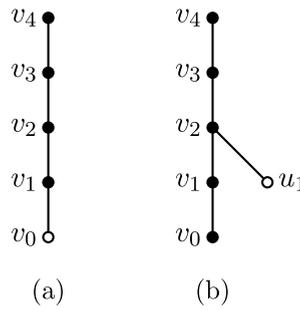


Fig. 5. Two possible subtrees in T .

$k_4 + k_6 \geq 1$). This implies that T_{v_i} is neither a subdivided star nor a reduced subdivided star. Let C be the canonical set of T_{v_i} . Then by [Observation 3](#), the set C is an IO-code of the tree T_{v_i} . Moreover, by [Proposition 5](#), we have $|C| \leq \left(\frac{2\Delta-1}{2\Delta}\right)n_1$. Now, taking T' to be the component of $T - v_3v_4$ (respectively, $T - v_4v_5$) containing the vertex v_4 (respectively, v_5), the result follows in this case by the same analysis on the tree T' as in the case $\deg_T(v_2) \geq 4$. \square

Hence, by [Claim 11](#), we can assume from here on that $2 \leq \deg_T(v_2) \leq 3$ and $\deg_T(v_i) = 2$ for all $i \in \{3, 4\}$, or else, the desired result is achieved. We now look at the maximal subtree T_{v_4} of T rooted at v_4 . By our earlier assumptions, either T_{v_4} is a path P_5 with v_4 as one of the ends of the path as illustrated in [Fig. 5\(a\)](#), or T_{v_4} is a reduced subdivided star T_3^* as illustrated in [Fig. 5\(b\)](#) with v_2 as the central vertex of the reduced subdivided star. In the latter case, we let u_1 be the leaf neighbor of v_2 .

If T_{v_4} is a path P_5 with v_4 as one of the ends of the path, then we let $S_1 = \{v_1, v_2, v_3, v_4\}$. In this case, the set S_1 , indicated by the shaded vertices in [Fig. 5\(a\)](#), is an IO-code of T_{v_4} . If T_{v_4} is a reduced subdivided star T_3^* , then we let $S_1 = \{v_0, v_1, v_2, v_3, v_4\}$. In this case, the set S_1 , indicated by the shaded vertices in [Fig. 5\(b\)](#), is an IO-code of T_{v_4} .

Let T' be the component of $T - v_4v_5$ that contains the vertex v_5 , and let T' have order n' . If $n' \leq 4$, then the tree T is determined and is isomorphic to a tree in \mathcal{T} , and hence, we infer the desired upper bound by [Proposition 5](#). We may therefore assume that $n' \geq 5$.

Suppose now that T' is open twin-free. By our earlier assumptions, we infer that $T' \not\cong T_\Delta$. Applying the inductive hypothesis to the open twin-free tree T' , there exists an IO-code, S' say, of T' such that $\gamma^{\text{IOC}}(T') = |S'| \leq \left(\frac{2\Delta-1}{2\Delta}\right)n'$. Let $S = S_1 \cup S'$. Then by [Lemma 6\(a\)](#), the set S is an IO-code of the tree T . If T' is a path P_5 , then $n_1 = 5$ and $|S_1| = 4 = \frac{4}{5}n_1 = \frac{4}{5}(n - n')$. If T' is a reduced subdivided star T_3^* , then $n_1 = 6$ and $|S_1| = 5 = \frac{5}{6}n_1 = \frac{5}{6}(n - n')$. In both cases, $|S_1| \leq \frac{5}{6}(n - n') \leq \left(\frac{2\Delta-1}{2\Delta}\right)(n - n')$. Therefore, using [Lemma 6\(b\)](#), we have

$$\gamma^{\text{IOC}}(T) \leq |S| \leq \left(\frac{2\Delta-1}{2\Delta}\right)n.$$

Suppose now that T' contains open twins. Since the tree T contains no open twins, we infer that v_5 is a leaf in T' and is an open twin in T' with some other leaf, say u_5 , in T' . We note that v_6 is the common neighbor of v_5 and u_5 . Let $T'' = T' - v_5$, and let T'' have order n'' , and so $n'' = n' - 1 \geq 4$. If $n'' = 4$, then T is isomorphic to $T(v_4; 0, 0, 0, 1, 0, 1)$ if $T_{v_4} \cong P_5$ and to $T(v_4; 0, 0, 0, 0, 0, 2)$ if $T_{v_4} \cong T_3^*$. In both cases, the result follows by [Proposition 5](#). Therefore, let us assume that $n'' \geq 5$. Since T is open twin-free, so too is the tree T'' . By [Claim 10](#), we infer that $T'' \not\cong T_\Delta$. Applying the inductive hypothesis to the open twin-free tree T'' of order $n'' \geq 5$, there exists an IO-code, S'' say, of T'' such that $\gamma^{\text{IOC}}(T'') = |S''| \leq \left(\frac{2\Delta-1}{2\Delta}\right)n''$. Let $S = S_1 \cup S''$. We note that since u_5 is a leaf, the vertex $v_6 \in S''$, and so v_5 is identified by the set S . Therefore, the set S is an IO-code of the tree T . If T_{v_4} is a path P_5 , then $n_1 = 5$ and $|S_1| = 4 = \frac{2}{3}(n_1 + 1) = \frac{2}{3}(n - n'')$. If T_{v_4} is a reduced subdivided star T_3^* , then $n_1 = 6$ and $|S_1| = 5 = \frac{5}{7}(n_1 + 1) = \frac{5}{7}(n - n'')$. In both cases, $|S_1| \leq \frac{5}{7}(n - n'')$. Therefore,

$$\begin{aligned} \gamma^{\text{IOC}}(T) \leq |S| &= |S_1| + |S''| \\ &\leq \frac{5}{7}(n - n'') + \left(\frac{2\Delta-1}{2\Delta}\right)n'' \\ &< \left(\frac{2\Delta-1}{2\Delta}\right)(n - n'') + \left(\frac{2\Delta-1}{2\Delta}\right)n'' \\ &= \left(\frac{2\Delta-1}{2\Delta}\right)n. \end{aligned}$$

This completes the proof of [Theorem 7](#). \square

3. Proof of [Theorem 1](#)

In this section, we present a proof of our main result, namely [Theorem 1](#). We first present the following preliminary result.

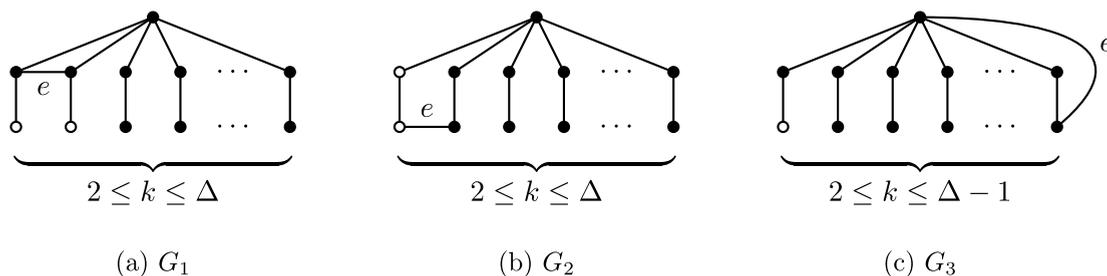


Fig. 6. Possible graphs $G \cong T_k + e$ as in Proposition 12.

Proposition 12. For $\Delta \geq 3$ a fixed integer, if G is a graph of order n that is isomorphic to a graph obtained from a subdivided star T_k , where $2 \leq k \leq \Delta$, by adding an edge e to T_k in such a way that $G \cong T_k + e$ contains no 4-cycle and satisfies $\Delta(G) \leq \Delta$, then

$$\gamma^{IOC}(G) \leq \left(\frac{2\Delta - 1}{2\Delta} \right) n.$$

Proof. It can be verified that the graph G in the statement of the proposition must be isomorphic to exactly one of the three graphs G_1, G_2 and G_3 portrayed in Fig. 6. In each of these cases, we have $n = 2k + 1$.

Suppose that $G \cong G_1$. In this case, $\gamma^{IOC}(G) \leq 2k - 1$, where the shaded vertices in Fig. 6(a) form an IO-code of G . Thus, $\gamma^{IOC}(G) \leq \left(\frac{2k-1}{2k+1} \right) n < \left(\frac{2\Delta-1}{2\Delta} \right) n$.

Suppose that $G \cong G_2$. If $k = 2$, then $G \cong C_5$, $n = 5$ and $\gamma^{IOC}(G) = 4 = \frac{4}{5}n < \left(\frac{2\Delta-1}{2\Delta} \right) n$. If $3 \leq k \leq \Delta$, then $\gamma^{IOC}(G) \leq 2k - 1$, where the shaded vertices in Fig. 6(b) form an IO-code of G . Thus, $\gamma^{IOC}(G) \leq \left(\frac{2k-1}{2k+1} \right) n < \left(\frac{2\Delta-1}{2\Delta} \right) n$.

Suppose that $G \cong G_3$. In this case, $k \leq \Delta - 1$ noting that $\Delta(G) \leq \Delta$. Therefore we have $\gamma^{IOC}(G) \leq 2k - 1$, where the shaded vertices in Fig. 6(c) form an IO-code of G . Thus, $\gamma^{IOC}(G) \leq \left(\frac{2k-1}{2k+1} \right) n < \left(\frac{2\Delta-1}{2\Delta} \right) n$. \square

We are now in a position to present a proof of our main result. Recall its statement.

Theorem 1. For $\Delta \geq 3$ a fixed integer, if $G \not\cong T_\Delta$ is an open twin-free connected graph of order $n \geq 5$ that contains no 4-cycles and satisfies $\Delta(G) \leq \Delta$, then

$$\gamma^{IOC}(G) \leq \left(\frac{2\Delta - 1}{2\Delta} \right) n,$$

On the other hand, when $G \cong T_\Delta$, then we have $\gamma^{IOC}(G) = \left(\frac{2\Delta}{2\Delta+1} \right) n$.

Proof. We proceed by induction on the size $m \geq 4$ of the connected graph G . If $m = 4$, then either $G \cong P_5$ or G is isomorphic to a paw, that is a complete graph on three vertices (equivalently, a K_3) with a leaf adjacent to one of the vertices of the K_3 . Then, $\gamma^{IOC}(G) = 4$ if $G \cong P_5$ and $\gamma^{IOC}(G) = 3$ if G is isomorphic to a paw. In both cases, we have $\gamma^{IOC}(G) \leq \left(\frac{2\Delta-1}{2\Delta} \right) n$. This establishes the base case. For the inductive hypothesis, let $m \geq 5$ and assume that if G' is a connected graph G' of order $n' \geq 5$ and size less than m that contains no 4-cycle and is open twin-free and satisfies $\Delta(G') \leq \Delta$, then $\gamma^{IOC}(G') \leq \left(\frac{2\Delta-1}{2\Delta} \right) n'$, unless $G' \cong T_\Delta$.

We now consider the connected graph G of order $n \geq 5$ and size m that contains no 4-cycles and is open twin-free and satisfies $\Delta(G) \leq \Delta$. If the graph G is a tree, then the desired result follows from Theorem 7. Hence we may assume that the connected graph G contains a cycle. If $m = 5$, then the graph G is isomorphic to one of the graphs G_1, G_2 and G_3 illustrated in Fig. 6 and with $k = 2$, and the desired result holds by Proposition 12. Hence we may assume that $m \geq 6$.

Let C be an induced cycle in G and let $e = uv$ be an edge of the cycle C . Let $G' = G - e$ be the connected graph obtained from G by deleting the cycle edge e . The resulting graph G' has order $n \geq 5$ and maximum degree $\Delta(G') \leq \Delta(G) \leq \Delta$. Since G contains no 4-cycles, neither does the graph G' . If $G' \cong T_\Delta$, then the graph G satisfies the statement of Proposition 12. Hence we may assume that $G' \not\cong T_\Delta$, for otherwise the desired result follows by Proposition 12.

Suppose that G' is open twin-free. Applying the inductive hypothesis to the graph G' , there exists an IO-code S' of G' such that $|S'| \leq \left(\frac{2\Delta-1}{2\Delta} \right) n$. We claim that S' is also an IO-code of G . Suppose, to the contrary, that S' is not an IO-code of G . In this case, at least one of u and v must be in the set S' . Renaming u and v if necessary, we may assume that $u \in S$ and that the vertex v is not identified by S' in G . Let x be the vertex in G such that v and x are identified by S' in G' but are not identified by S' in G . We note that the vertex x is neighbor of u different from v . Since S' is an IO-code of G' , in order for S' to totally dominate the vertex v in G' , there exists a vertex $w \in S'$ such that $w \neq u$ and $wv \in E(G)$. Since the set S' does not identify the pair v and x in the graph G' , we infer that $w \neq x$ and $wx \in E(G)$. This implies that G has a 4-cycle, namely, $uvwxu$, a contradiction. Therefore, S' is also an IO-code of G , and so $\gamma^{IOC}(G) \leq |S'| \leq \left(\frac{2\Delta-1}{2\Delta} \right) n$.

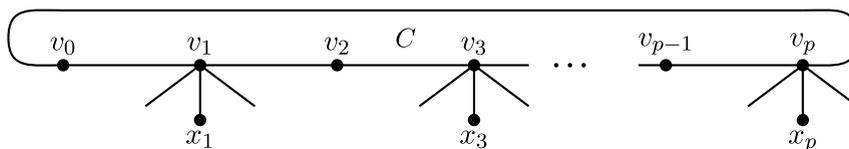


Fig. 7. A possible structure of the graph G in the proof of Theorem 1.

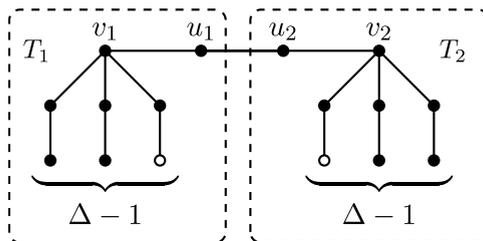


Fig. 8. A tree T of order n satisfying $\gamma^{loc}(T) = (\frac{2\Delta-1}{2\Delta})n$.

Hence we may assume that G' has open twins, for otherwise the desired result follows. More generally, we may assume that for every cycle edge f of G , the graph $G - f$ has open twins. Since G' has open twins, at least one of u and v is an open twin in G' . Renaming vertices if necessary, we may assume that v is an open twin in G' . Let x be an open twin of v in G' . Since G' has no 4-cycles, we infer that the open twins x and v must be of degree 1 in G' . Let w be the common neighbor of x and v . Thus, w is a support vertex in G' with v and x as leaf neighbors. Since e is a cycle edge of G , we note that the vertex w belongs to the cycle C . Thus, $\deg_G(w) \geq 3$, $\deg_{G'}(x) = 1$, $\deg_{G'}(v) = 1$ and $\deg_G(v) = 2$. Moreover, we cannot have $x = u$, or else, the cycle C is $uvwu$. In other words, C is a 3-cycle in G . However, since $\deg_G(w) \geq 3$, it implies that removing the cycle edge uw from G does not produce open twins which contradicts our earlier assumption.

Let $f = vw \in E(G)$ and let $G'' = G - f$. We note that the edge f belongs to the cycle C . By our earlier assumptions, the removal of the cycle edge f creates open twins. By our earlier observations, open twins in G'' must be of degree 1 in G'' . Since both u and w have degree at least 2 in G'' , we therefore infer that the vertex v is an open twin in G'' . Let y be an open twin of v in G'' . Analogously as in the previous case when v is an open twin in the graph G' , we infer that the open twins v and y must be of degree 1 in G'' . Thus, the vertex u is a support vertex in G'' with v and y as leaf neighbors.

Continuing the same analysis for every edge of C , we infer that, if $C : v_0v_1v_2 \dots v_pv_0$, then renaming vertices of C if necessary, the vertex v_i is a support vertex in G for each odd $i \in [p]$ and the vertex v_i has degree 2 in G for each even $i \in [p]$. Thus, we note p is an odd integer and, since G contains no 4-cycles, p must be at least 5. Therefore, C is a cycle of even length $p + 1 \geq 6$. Let x_i be a leaf neighbor of the vertex v_i for each odd $i \in [p]$ (see Fig. 7). We note that for every odd $i \in [p]$, the vertex v_i has two neighbors on the cycle C (and these neighbors are distinct from the leaf x_i), and so v_i has degree at least 3 in G . Moreover, $m \geq p + p/2 = 3p/2 \geq 9$.

We now consider the graph $G_0 = G - v_0$ obtained by deleting the vertex v_0 from G . The graph G_0 is a connected graph satisfying $\Delta(G_0) \leq \Delta(G) \leq \Delta$. Since G has no 4-cycles, the graph G_0 also has no 4-cycles. Recall that the neighbors v_1 and v_p of v_0 are of degree at least 3 in G and are support vertices with leaf neighbors x_1 and x_p , respectively. Since the graph G is open twin-free, we therefore infer that the graph G_0 has no open twins. Let G_0 have size m_0 , and so $m_0 = m(G_0) = m - 2$. By our earlier observations, $m \geq 9$, and so $m_0 \geq 7$. Moreover since $p \geq 5$, the structure of the graph G implies that $G_0 \not\cong T_\Delta$ (noting that $x_1v_1v_2v_3x_3$ is a path in G_0 where x_1 and x_3 are leaves in G_0 , the vertex v_2 has degree 2 in G_0 , and v_3 is a vertex of degree at least 3 in G_0). Hence, by our induction hypothesis, there exists an IO-code S_0 of G_0 such that $|S_0| \leq (\frac{2\Delta-1}{2\Delta})(m_0 - 1) < (\frac{2\Delta-1}{2\Delta})n$.

We show next that S_0 is also an IO-code of G . Since v_1 and v_p are support vertices in G_0 , we note that $v_1, v_p \in S_0$. To show that S_0 is an IO-code of G , we only need to show that S_0 identifies the vertex v_0 from every other vertex of G . However, since G has no 4-cycles, this is indeed true, as v_0 is the only vertex of G with $N_G(v_0) \cap S_0 = \{v_1, v_p\}$. Hence, the result is true in this case as well. This completes the proof of Theorem 1. \square

4. Tight examples

For $\Delta \geq 3$ a fixed integer, let T_1 and T_2 be two vertex-disjoint copies of the reduced subdivided star T_Δ^* . Let v_i be the central vertex in T_i (of degree Δ) and let u_i be the leaf neighbor of v_i for $i \in [2]$. Let $T_{1,2}$ be the tree obtained from the union of T_1 and T_2 by adding the edge u_1u_2 . The resulting tree T is illustrated in Fig. 8 and satisfies $\gamma^{loc}(T) = (\frac{2\Delta-1}{2\Delta})n$. These examples show that the upper bound in Theorem 1 is best possible for every fixed value of $\Delta \geq 3$.

A *subcubic graph* is a graph with maximum degree at most 3. In the special case when $\Delta = 3$, by Theorem 1 if G is a subcubic graph of order $n \geq 5$ that is open twin-free and contains no 4-cycles, then $\gamma^{loc}(G) \leq \frac{5}{6}n$. We construct

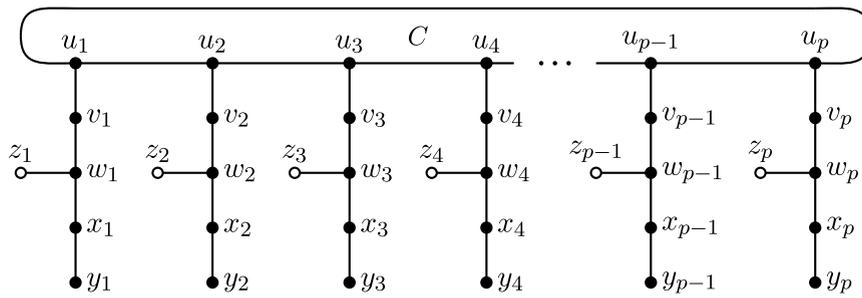


Fig. 9. A subcubic graph G_p of order n satisfying $\gamma^{IO}(G_p) = \frac{5}{6}n$.

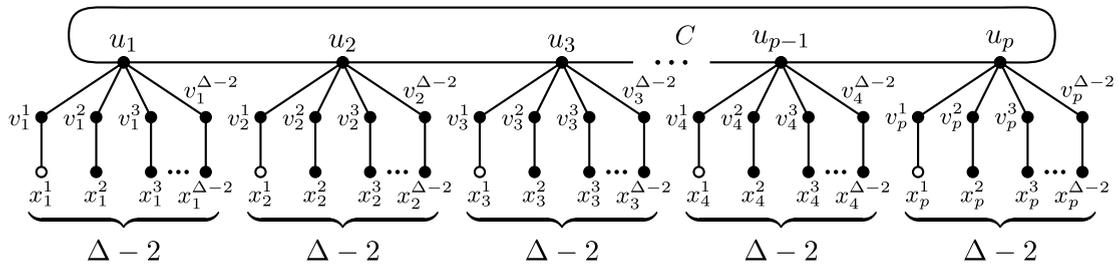


Fig. 10. An open twin-free graph $G_{p,\Delta}$ of order $n_{p,\Delta}$, with no 4-cycles and with maximum degree $\Delta(G) \geq 4$ satisfying $\gamma^{IO}(G_{p,\Delta}) = \left(\frac{2\Delta-4}{2\Delta-3}\right)n_{p,\Delta}$.

next a family of subcubic graphs G of arbitrarily large orders n that are open twin-free and contain no 4-cycles satisfying $\gamma^{IO}(G) = \frac{5}{6}n$. Let $C : u_1u_2 \dots u_pu_1$ be a cycle of length p where $p \geq 3$ and $p \neq 4$. For each vertex u_i on the cycle, we add a vertex disjoint copy, T_i say, of a reduced subdivided star T_3^* and identify one of the leaves at distance 2 from the central vertex of the reduced subdivided star with the vertex u_i for all $i \in [p]$. Let $V(T_i) = \{u_i, v_i, w_i, x_i, y_i, z_i\}$ where $u_iv_iw_ix_iy_i$ is a path and where z_i is the leaf neighbor of w_i in T_i . The resulting subcubic graph G_p is illustrated in Fig. 9.

Proposition 13. *Let $p \geq 3$ and $p \neq 4$. Moreover, let G_p be a subcubic graph of order n . Then G_p is open twin-free, contains no 4-cycles and satisfies $\gamma^{IO}(G_p) = \frac{5}{6}n$.*

Proof. Let S be an arbitrary IO-code in G_p . We show that $|S \cap V(T_i)| \geq 5$ for all $i \in [p]$. Since S is a TD-set, the set S contains the support vertices w_i and x_i . In order to identify x_i and z_i , the vertex y_i belongs to the set S . In order to identify v_i and z_i , the vertex u_i belongs to the set S . In order to identify w_i and y_i , the set S contains at least one of v_i and z_i . Hence, $|S \cap V(T_i)| \geq 5$ for all $i \in [p]$. Since S is an arbitrary IO-code in G_p and $|S| \geq \frac{5}{6}n$, this implies that $\gamma^{IO}(G_p) \geq \frac{5}{6}n$. Since the set $S^* = V(G_p) \setminus \{z_1, z_2, \dots, z_p\}$ is an IO-code in G_p , we have $\gamma^{IO}(G_p) \leq \frac{5}{6}n$. Consequently, $\gamma^{IO}(G_p) = \frac{5}{6}n$. \square

We remark that deleting the cycle edge u_1u_p in the construction of the subcubic graph G_p yields a tree T of order n that is open twin-free and satisfies $\gamma^{IO}(T) = \frac{5}{6}n$. By Proposition 13, the upper bound in Theorem 1 is tight for $\Delta = 3$ in the strong sense that there exist connected subcubic graphs of arbitrary large order n that contain no 4-cycles, are open twin-free, and that achieve the upper bound $\gamma^{IO}(G) = \left(\frac{2\Delta-1}{2\Delta}\right)n$.

For a fixed $\Delta \geq 4$ however, we believe that there are no tight examples of the upper bound in Theorem 1. Nevertheless, as we shall show below, the said upper bound in Theorem 1 is still nearly tight. To demonstrate this fact, we construct a family of graphs, denoted $G_{p,\Delta}$ where $p \geq 3$ with $p \neq 4$ and $\Delta \geq 4$, of arbitrarily large orders n and with maximum degree Δ that are open twin-free, contain no 4-cycles and whose IO-code numbers attain the value $\left(\frac{2\Delta-4}{2\Delta-3}\right)n$. As we show below, this implies that for this family of graphs $G_{p,\Delta}$, the parameter $\gamma^{IO}(G_{p,\Delta})$ falls short of the upper bound in Theorem 1 by $\frac{3p}{2\Delta}$, which for fixed p asymptotically tends to zero with large Δ .

Let $\Delta \geq 4$ be a fixed integer and let $C : u_1u_2 \dots u_pu_1$ be a cycle of length p , where $p \geq 3$ and $p \neq 4$. For each vertex u_i on the cycle, we introduce a vertex-disjoint copy, T_i say, of a subdivided star $T_{\Delta-2}$. Let $V(T_i) = \{w_i, v_i^1, x_i^1, v_i^2, x_i^2, v_i^3, x_i^3, \dots, v_i^{\Delta-2}, x_i^{\Delta-2}\}$, where w_i is the central vertex of T_i , each v_i^j for $1 \leq j \leq \Delta - 2$ is a support vertex of T_i at distance 1 from w_i , and each x_i^j is a leaf of T_i , adjacent to v_i^j , and at distance 2 from w_i . Let the graph constructed be called $G_{p,\Delta}$. An example of such a graph is illustrated in Fig. 10. Notice that $n_{p,\Delta} = n(G_{p,\Delta}) = p(2\Delta - 3)$. Also notice that the vertices u_i for $i \in [p]$ are of degree Δ each.

Proposition 14. *For $p \geq 3$ with $p \neq 4$ and $\Delta \geq 4$, if the graph $G_{p,\Delta}$ has order $n_{p,\Delta}$, then $G_{p,\Delta}$ is open twin-free, contains no 4-cycles and satisfies $\gamma^{IO}(G_{p,\Delta}) = \left(\frac{2\Delta-4}{2\Delta-3}\right)n_{p,\Delta}$.*

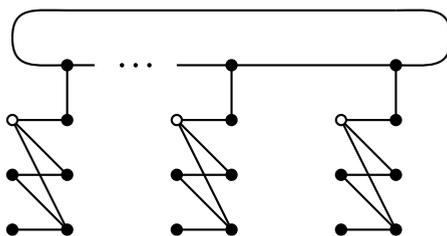


Fig. 11. A subcubic graph G of order n satisfying $\gamma^{\text{IOC}}(G) = \frac{6}{7}n$. Dark vertices belong to the optimal code.

Proof. Notice that the edges $u_i u_{i+1}$, for $i \in [p - 1]$, of the cycle C do not transform any of the leaves or support vertices of each T_i into non-leaves or non-support vertices, respectively. Therefore by the exact same proof as for Proposition 4(a), any IO-code of $G_{p,\Delta}$ must contain at least all vertices but one from each T_i . Thus, we must have $\gamma^{\text{IOC}}(G_{p,\Delta}) \geq \sum_{1 \leq i \leq p} (n(T_i) - 1) = \sum_{1 \leq i \leq p} \left(\frac{2(\Delta-2)}{2(\Delta-2)+1} \right) n(T_i) = \left(\frac{2\Delta-4}{2\Delta-3} \right) n_{p,\Delta}$. Moreover, it can be checked that the set $S = \cup_{1 \leq i \leq p} (V(T_i) \setminus \{x_i^1\})$ is an IO-code of $G_{p,\Delta}$. Thus, we also have $\gamma^{\text{IOC}}(G_{p,\Delta}) \leq |S| = p(2\Delta - 4) = \left(\frac{2\Delta-4}{2\Delta-3} \right) n_{p,\Delta}$. This proves the result. \square

Clearly, the parameter $\gamma^{\text{IOC}}(G_{p,\Delta})$ shown in Proposition 14 is smaller than the upper bound in Theorem 1 for the graphs $G_{p,\Delta}$. However, to compare this difference, we see that

$$\left(\frac{2\Delta - 1}{2\Delta} \right) n_{p,\Delta} - \left(\frac{2\Delta - 4}{2\Delta - 3} \right) n_{p,\Delta} = \frac{3}{2\Delta(2\Delta - 3)} n_{p,\Delta} = \frac{3}{2\Delta(2\Delta - 3)} p(2\Delta - 3) = \frac{3p}{2\Delta}.$$

In other words, irrespective of the length p of the cycle C (and hence, the order of the graph $G_{p,\Delta}$), we can approach the upper bound in Theorem 1 as close as necessary by taking large enough $\Delta \geq 4$.

Remark 15. The above construction can be slightly improved by adding more levels, in the following way. We start with a cycle C_p , where p is a multiple of 3. To every other vertex of the cycle, we add $\Delta - 2$ new neighbors, for each new neighbor, we further add $\Delta - 1$ neighbors, and to each such leaf we attach a copy of $T_{\Delta-1}$. One can check that the IO-code number of the resulting graph is $\left(\frac{2\Delta-4}{2\Delta-3} + \frac{2}{f(\Delta)} \right) n$, where $f(\Delta) = 3(\Delta - 1)^2(2\Delta - 3)$.

As f is a degree 3 polynomial, this is only a small improvement but significantly complicates the construction, and hence we do not give details. We did not manage to create examples with a higher ratio, such as for example $\left(\frac{2\Delta-3}{2\Delta-2} \right) n$, and leave this as an open problem.

5. Concluding remarks

Our main result, namely Theorem 1, shows that for $\Delta \geq 3$ a fixed integer, if G is a connected graph of order $n \geq 5$ that is open twin-free and satisfies $\Delta(G) \leq \Delta$, then $\gamma^{\text{IOC}}(G) \leq \left(\frac{2\Delta-1}{2\Delta} \right) n$, except in one exceptional case when G is a subdivided star of maximum degree Δ , that is, if $G \cong T_\Delta$. As shown in Proposition 4, if $G \cong T_\Delta^*$ is the reduced subdivided star of order n , then $\gamma^{\text{IOC}}(G) = \left(\frac{2\Delta-1}{2\Delta} \right) n$.

As remarked earlier, the upper bound in Theorem 1 is best possible when $\Delta = 3$, for arbitrarily large connected graphs. For $\Delta \geq 4$ however, it remains an open problem to determine if the upper bound in Theorem 1 is tight for arbitrarily large connected graphs. Nevertheless, for such values of $\Delta \geq 4$, we have presented examples of arbitrarily large open twin-free graphs of maximum degree at most Δ and without 4-cycles whose IO-code numbers almost reach the upper bound in Theorem 1. Yet, for $\Delta \geq 4$ there is room for small improvements, for example by constructing graphs whose IO-code numbers are even closer to the bound, such as $\left(\frac{2\Delta-3}{2\Delta-2} \right) n$ for example.

When we consider general graphs and thus allow 4-cycles, the bound of Theorem 1 does not hold. In fact, as shown in [10], the infinite family of half-graphs provides infinitely many connected bipartite graphs G of order n with $\gamma^{\text{IOC}}(G) = n$. By using these graphs as building blocks, one can build, for every fixed value of Δ , arbitrarily large connected graphs G of order n with $\gamma^{\text{IOC}}(G) = \left(\frac{2\Delta}{2\Delta+1} \right) n$ (similar to the graphs constructed in Proposition 13). Start with a cycle C_p of order p ; for each vertex v of C_p , consider a copy of the half-graph H_Δ and add an edge between v and a vertex of degree 1 of H_Δ . (Actually, v can be made adjacent to any vertex of the half-graph that has degree strictly less than Δ .) An example for $\Delta = 3$ is provided in Fig. 11. Thus, one may ask what is the best possible bound of this form in the general case (when one excludes half-graphs): perhaps it is indeed $\left(\frac{2\Delta}{2\Delta+1} \right) n$?

It is also interesting to notice that the above examples like half-graphs and the other graphs built using half-graphs, which serve as counterexamples to the bound in Theorem 1 for general graphs, contain induced 4-cycles. Hence, even before proving a bound like the one in Theorem 1 in the most general case (where graphs contain 4-cycles), one could also investigate if the same bound holds for graphs without induced 4-cycles. On the other hand, it would also be interesting

to find graphs (if they exist) with non-induced 4-cycles (that is, 4-cycles with chords) which are counterexamples to the bound in [Theorem 1](#).

As a remark related to the application of the bound in [Theorem 1](#) to other code numbers, it can be verified that any upper bound for IO-code numbers also holds for locating-total domination numbers of graphs. However, for the latter codes, there already exists a much stronger conjecture [12] in the case of twin-free graphs which states that the locating-total domination number of a twin-free graph on n vertices is at most $\frac{2}{3}n$. Moreover, the best known approximation to the upper bound in the said conjecture is $\frac{3}{4}n$ [12]. On the other hand, if one were to also allow twins in a graph G of maximum degree at most Δ , since any locating-total dominating set requires at least all-but-one twins in it, the $\frac{3}{4}n$ -bound on the graph G with its twins “trimmed off” and the inclusion of all-but-one twins in a locating-total dominating code roughly gives a $(\frac{\Delta-1}{\Delta}n)$ -upper bound to a locating-total domination number of G . Notice that the latter is a better bound on a location-total domination number than a direct application of the bound in [Theorem 1](#).

We also recall here two questions by Henning and Yeo from [22] about regular graphs. First, they conjectured that for every connected open twin-free cubic graph G , the upper bound $\gamma^{\text{loc}}(G) \leq \frac{3}{5}n$ holds, except for a finite number of graphs. Second, they asked whether $\gamma^{\text{loc}}(G) \leq (\frac{\Delta}{\Delta+1})n$ holds for every connected open twin-free Δ -regular graph G of order n .

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Data availability

No data was used for the research described in the article.

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