Journal of Graph Theory

WILEY

ARTICLE

Identifying Codes in Triangle-Free Graphs of Bounded Maximum Degree

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Received: 25 July 2024 | **Revised:** 19 August 2025 | **Accepted:** 9 March 2026

Funding: French government IDEX-ISITE initiative CAP 20-25, Grant/Award Number: ANR-16-IDEX-0001; International Research Center "Innovation Transportation and Production Systems" of the I-SITE CAP; ANR project GRALMECO, Grant/Award Number: ANR-21-CE48-0004; President's Intramural Research Fund, Grant/Award Number: PIRF0056; South African National Research Foundation, Grant/Award Number: 129265; University of Johannesburg; Business Finland project, Grant/Award Number: 10769/31/2022 6GTNF; Academy of Finland, Grant/Award Numbers: 338797, 358718

ABSTRACT

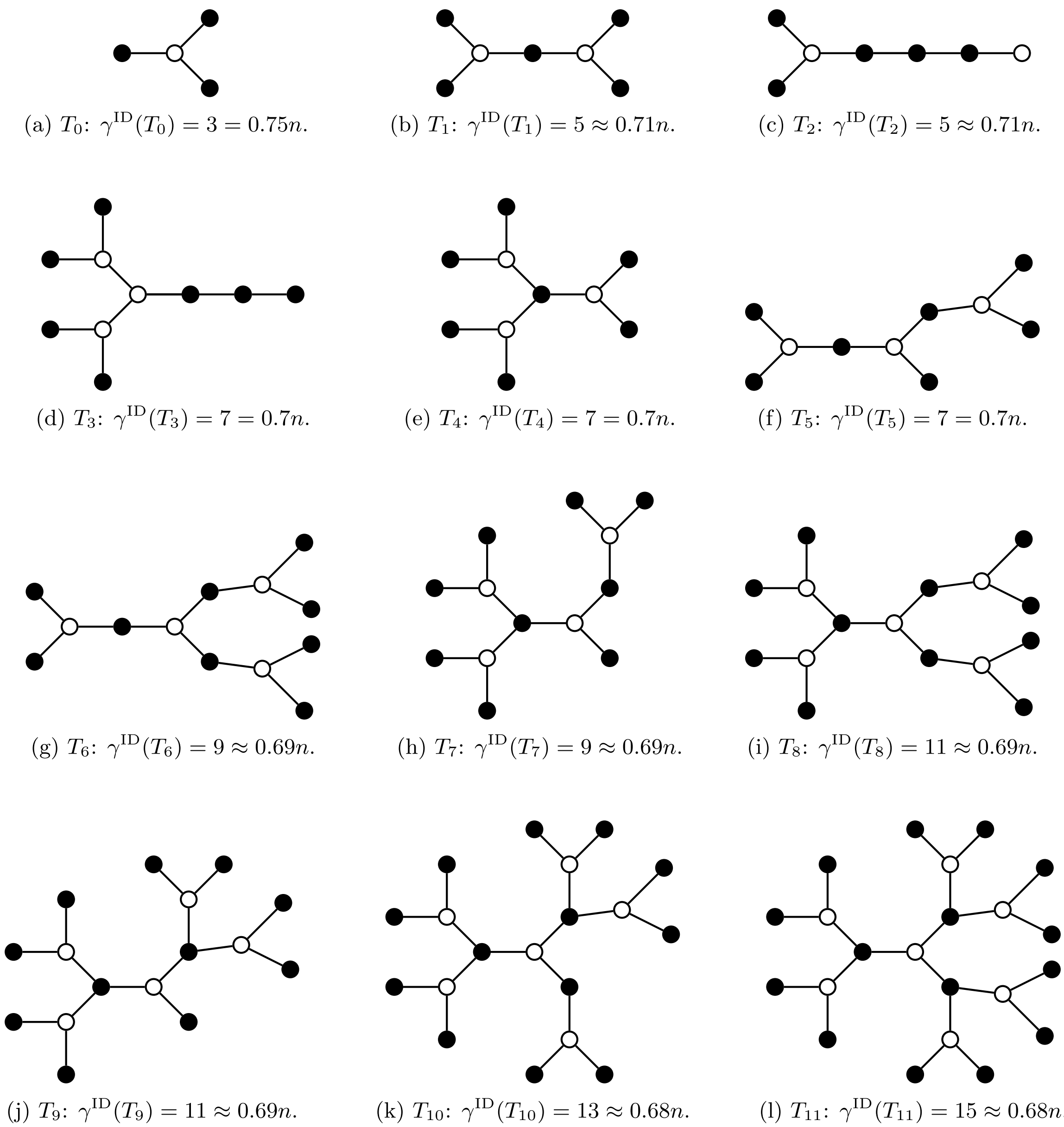
An *identifying code* of a closed-twin-free graph G is a set S of vertices of G such that any two vertices in G have a distinct, nonempty intersection between their closed neighborhood and S. It was conjectured that there exists a constant c such that for every connected closed-twin-free graph G of order n and maximum degree Δ, the graph G admits an identifying code of size at most $\left(\frac{\Delta-1}{\Delta}\right)n + c$. In [D. Chakraborty, F. Foucaud, M. A. Henning, and T. Lehtilä. Identifying codes in graphs of given maximum degree: Characterizing trees. *Discrete Mathematics* 349(2), 114826, 2026], we proved the conjecture for all trees. In this article, we show that the conjecture holds for all triangle-free graphs, with the same list of exceptional graphs needing $c > 0$ as for trees: for $\Delta \geq 3$, $c = 1/3$ suffices and there is only a set of 12 trees requiring $c > 0$ for $\Delta = 3$, and when $\Delta \geq 4$ this set is reduced to the Δ-star only. Our proof is by induction, whose starting point is the above result for trees. Along the way, we prove a generalized version of Bondy's theorem on induced subsets [J. A. Bondy. Induced subsets. Journal of Combinatorial Theory, Series B, 1972] that we use as a tool in our proofs. We also use our main result for triangle-free graphs to prove the upper bound $\left(\frac{\Delta-1}{\Delta}\right)n + 1/\Delta + 4t$ for graphs that can be made triangle-free by the removal of t edges.

1 | Introduction

In this article, we consider simple undirected loopless graphs. The *open neighborhood* of vertex u in graph G, $N_G(u)$, contains every vertex adjacent to vertex u, while the *closed neighborhood* also contains vertex u itself. A set of vertices $D \subseteq V(G)$ is a *dominating set* if every vertex outside of D is adjacent to a vertex in D. A set $S \subseteq V(G)$ is a *separating set* if every vertex $u \in V(G)$ has a unique closed neighborhood in set S, that is, the intersection $N_G[u] \cap S$ is unique for every vertex. Furthermore, a set $C \subseteq V(G)$ is an *identifying code* if it is a dominating set and a separating set [1]. See Figure 1 for examples of identifying codes. From the intuitive perspective, identifying codes allows us to locate or identify any vertex if we know which vertices in the set S are in its closed neighborhood. It is natural to ask what the minimum number of vertices in an identifying code of graph G is. This value is called the *identification number* of graph G, and it is denoted by $\gamma^{\mathrm{ID}}(G)$.

This paper concentrates on proving Conjecture 1 (stated below) on upper bounds for the minimum size of identifying codes of given maximum degree, for all triangle-free graphs. Previously, in [2], we have proved the conjecture for trees.

A thorough treatise on domination in graphs can be found in [3–5]. Bounds on domination numbers for graphs with restrictions on their degree parameters are a natural and important line of research; see, for example, the influential result by Reed [6] that if G is a connected cubic graph of order n, then its domination number is at most $\frac{3}{8}n$. A detailed discussion on upper bounds on the domination number in terms of its order and degree parameters, as well as bounds with specific structural restrictions imposed, can be found in [5], Chapters 6, 7, and 10.



(a) T_0: $\gamma^{\mathrm{ID}}(T_0) = 3 = 0.75n$. (b) T_1: $\gamma^{\mathrm{ID}}(T_1) = 5 \approx 0.71n$. (c) T_2: $\gamma^{\mathrm{ID}}(T_2) = 5 \approx 0.71n$.

(d) T_3: $\gamma^{\mathrm{ID}}(T_3) = 7 = 0.7n$. (e) T_4: $\gamma^{\mathrm{ID}}(T_4) = 7 = 0.7n$. (f) T_5: $\gamma^{\mathrm{ID}}(T_5) = 7 = 0.7n$.

(g) T_6: $\gamma^{\mathrm{ID}}(T_6) = 9 \approx 0.69n$. (h) T_7: $\gamma^{\mathrm{ID}}(T_7) = 9 \approx 0.69n$. (i) T_8: $\gamma^{\mathrm{ID}}(T_8) = 11 \approx 0.69n$.

(j) T_9: $\gamma^{\mathrm{ID}}(T_9) = 11 \approx 0.69n$. (k) T_{10}: $\gamma^{\mathrm{ID}}(T_{10}) = 13 \approx 0.68n$. (l) T_{11}: $\gamma^{\mathrm{ID}}(T_{11}) = 15 \approx 0.68n$.

FIGURE 1 | The family $\mathcal{T}_3$ of trees of maximum degree 3 requiring $c > 0$ in Conjecture 1. The set of black vertices in each figure constitutes an identifying code of the tree.

Originally, identifying codes were motivated by fault-detection in multiprocessor networks [1]. Numerous other applications have been discovered, such as threat location in facilities using sensor networks [7], logical definability of graphs [8], and canonical labeling of graphs for the graph isomorphism problem [9]. Moreover, multiple related concepts have been introduced since the 1960s, such as *separating systems*, *test covers,* and *locating-dominating sets*, which have been independently discovered and studied, forming the general area of identification problems in graphs and other discrete structures. See, for example, [10–13]. An extensive internet bibliography containing over 500 articles around these topics can be found at [14], while more information specifically about identifying codes can be found in the book chapter [15].

In this paper, we concentrate on connected, identifiable triangle-free graphs. A graph is *identifiable* if it does not contain any *closed twins*, that is, vertices with the same closed neighborhoods. Moreover, a graph is *triangle-free* if no three vertices in it form a cycle. Similarly to closed twins, we define *open twins* as vertices with the same open neighborhoods. A graph without closed or open twins is called *twin-free*. We also define similarly *open-twin-free* and *closed-twin-free* graphs. Twins are significant for separating sets and identifying codes; if two vertices are closed twins, then the graph does not admit any separating set and hence no identifying code. Moreover, if t vertices have the same open neighborhood, then any separating set and hence also any identifying code, contains at least $t - 1$ of them. Note that every connected triangle-free graph on at least three vertices is identifiable.

As our main result, we prove the following conjecture for all identifiable triangle-free graphs.

Conjecture 1 ([16], Conjecture 1). *There exists a constant c such that for every connected identifiable graph G on n vertices and of maximum degree $\Delta \geq 2$,*

$$\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta - 1}{\Delta}\right)n + c.$$

In [2], we have proved Conjecture 1 for trees and have determined the exact set of trees requiring a positive constant c together with an exact value of c. In this paper, we use our previous results for trees from [2] as a starting point for our proof of Conjecture 1 for all triangle-free graphs.

It is known that if true, the conjecture would be tight, that is, some graphs of maximum degree Δ have identification number at least $\frac{\Delta-1}{\Delta}n$. For $\Delta = 2$, the conjecture is tight for both paths and cycles with $c \leq 3/2$ (see corollary 7). For $\Delta = 3$, the conjecture is tight, for example, for trees presented in Figure 1 and for a path whose every vertex we join to a 2-path by a single edge (see [2]). For any $\Delta > 3$, the complete bipartite graph $K_{\Delta,\Delta}$ satisfies $\gamma^{\mathrm{ID}}(K_{\Delta,\Delta}) = 2\Delta - 2 = \left(\frac{\Delta-1}{\Delta}\right)n$ and, hence, gives tight examples with $c = 0$. Furthermore, for any $\Delta > 3$ and an unbounded value of n, there are trees with identification number $\gamma^{\mathrm{ID}}(T) > \frac{(\Delta-1)n}{\Delta} - \frac{n}{\Delta^2}$ [2]. Furthermore, there exist connected graphs of any maximum degree $\Delta \geq 3$ and an arbitrarily large number n of vertices, with identification number $\left(\frac{\Delta-1}{\Delta}\right)n$; see [17, 18].

The condition on the maximum degree Δ is a necessary part of the conjecture. Without it, there are graphs on n vertices with identification number $n - 1$ [19, 20].

The best known general upper bounds for connected graphs with a maximum degree Δ and number of vertices n, when n is large enough, are of the form $n - \frac{n}{\Theta(\Delta^5)}$ [19], which has been improved to $n - \frac{n}{103(\Delta+1)^3}$ in Ref.[17] (for the sake of comparison, the conjectured bound can be rewritten as $n - \frac{n}{\Delta} + c$). When we consider graph classes instead of general graphs, some improvements on these bounds are known. If every pair of closed neighborhoods in a graph differs by at least two vertices, then the general bound has been improved to $n - \frac{n}{103\Delta}$ and to $n - \frac{n}{f(k)\Delta}$ for graphs of clique number k [17]. For bipartite graphs, an upper bound of $n - \frac{n}{\Delta+9}$ has been proved [16]. In a short conference proceedings paper [21], we sketched a proof for Conjecture 1 for bipartite graphs without twins of degree 2 or greater (we did not include that proof in the current article, since the triangle-free result we present here is stronger). Moreover, in Ref. [2], we gave the proof for all trees. Conjecture 1 also holds for line graphs of graphs of average degree at least 5 [[22], Corollary 21] as well as graphs which have girth at least 5, minimum degree 2, and maximum degree at least 4 [23]. Furthermore, the conjecture holds in many cases for some graph products such as Cartesian and direct products [24–26]. See also the book chapter [15], where Conjecture 1 is presented.

Conjecture 1 has been considered also for triangle-free graphs previously. In Ref. [16], an upper bound of type $n - \frac{n}{\Delta+o(\Delta)}$ was presented for triangle-free graphs. When the triangle-free graph is also twin-free, this upper bound has been improved to $n - \frac{n}{3\Delta/(\ln\Delta - 1)}$. Note that the latter result implies that Conjecture 1 holds for triangle-free graphs without any open twins, whenever $\Delta \geq 55$ (because then, $3\Delta/(\ln\Delta - 1) \leq \Delta$). Note that the graphs containing open twins seem to be the toughest cases regarding Conjecture 1 (among triangle-free graphs). Indeed, we will see that every triangle-free graph requiring a positive constant c with $\Delta \geq 3$ contains open twins. Furthermore, in Ref. [27], it has been shown that every *twin-free* bipartite graph G on $n \geq 5$ vertices satisfies the upper bound $\gamma^{\mathrm{ID}}(G) \leq \frac{2n}{3}$, while if we allow open twins, then there exist trees T with arbitrarily large numbers of vertices such that $\gamma^{\mathrm{ID}}(T) > \left(\frac{\Delta-1}{\Delta}\right)n - \frac{n}{\Delta^2}$ [2].

Our main result is to prove Conjecture 1 (in a strong form) for all triangle-free graphs. To state it, we define, for every integer $\Delta \geq 3$, a set $\mathcal{F}_\Delta$ of exceptional graphs of maximum degree at most Δ (see Section 2.3 for greater detail). For $\Delta = 3$, this set contains twelve trees (see Figure 1), the cycles on 4 and 7 vertices, and the path on 4 vertices. For every integer $\Delta > 3$, it contains exactly the Δ-star $K_{1,\Delta}$.

Theorem 2. *Let $\Delta \geq 3$ be an integer, and let G be a connected triangle-free graph of order $n \geq 3$. If $G \in \mathcal{F}_\Delta$, then*

$$\gamma^{\mathrm{ID}}(G) = \left(\frac{\Delta - 1}{\Delta}\right)n + \frac{1}{\Delta}.$$

On the other hand, if $G \notin \mathcal{F}_\Delta$ has maximum degree Δ, then

$$\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta - 1}{\Delta}\right)n.$$

Note that graphs in $\mathcal{F}_\Delta$ for $\Delta \geq 4$ have maximum degree Δ, and the graphs in $\mathcal{F}_3$ have maximum degree either two or three.

In Ref. [2], we have shown that Conjecture 1 holds for trees (see Theorem 12). It turns out that, for triangle-free graphs with maximum degree at least 3, the set of graphs requiring a positive constant c is exactly the same as the set of trees with maximum degree at least 3 needing a positive constant. Our proof uses the result for trees from Ref. [2] as a starting point for an induction. After that, we assume that a triangle-free graph contains at least one cycle containing some edge e. We remove that edge to construct a graph G' which, by induction, satisfies the conjecture. Hence, G' contains a *small* identifying code which we can use to construct another identifying code of the same size for G. One difficulty for proving the conjecture is the existence of the set of graphs requiring $c > 0$. Since $\mathcal{F}_3$ is the largest among the sets $\mathcal{F}_\Delta$, the case $\Delta = 3$ requires a lot of special argumentation.

1.1 | Structure of the Paper

First, in Section 2, we introduce some useful definitions and lemmas. In Subsection 2.1, we introduce terminology and results about a generalization of identifying codes, which are later used in the proof of our main result. We continue in Subsection 2.2, where we discuss Conjecture 1 when $\Delta = 2$. In Subsection 2.3, we introduce every connected triangle-free graph, which requires a positive constant c for Conjecture 1. Understanding these graphs is crucial for the proof of the main theorem. In Section 3, we give the proof of Theorem 2. In Section 4, we use our bound to prove a weaker bound for graphs that have triangles, but can be made triangle-free by removing t edges. Finally, we conclude in Section 5.

2 | Preliminaries and Known Results

In the following, we go through the notation we use throughout this article. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of graph $G = (V(G), E(G))$. We usually denote $n = |V(G)|$. For a set of vertices S we denote $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. We denote by $\deg_G(v) = |N_G(v)|$ the *degree* of the vertex v in graph G. A *leaf* is a vertex with degree one and its only neighbor is called *support vertex*. In the literature, a leaf is also known as a *pendant vertex*. Naturally, any vertex of a graph G that is not a leaf of G is referred to as a *non-leaf* vertex of G. We denote the complement of a graph G by $\overline{G}$. We sometimes denote the maximum degree of graph G by $\Delta(G)$, its number of vertices by $n(G)$, and its number of edges by $m(G)$. The *girth* of the graph G refers to the number of vertices in the shortest cycle in graph G.

We say that vertex u (or vertex subset C) *separates* vertices w and v if vertex u (or some vertex $z \in C$) is in exactly one of the sets $N[w]$ and $N[v]$. Given a vertex subset C (often an identifying code), with *code neighborhood* of vertex $u \in V(G)$, we refer to the set $C \cap N[u]$.

On many occasions throughout this article, we look at a subgraph of a graph G obtained by deleting some vertices or edges. To that end, given a graph G and a set S containing some vertices and edges of G, we define $G - S$ as the subgraph of G obtained by deleting from G all edges and vertices (and edges incident with the vertices) of S.

We use the following lemma multiple times to argue that some vertices are identified.

Lemma 3. *Let G be a triangle-free graph and S, a subset of vertices of G. If three vertices u, v, w inducing a P_3 are in S, then each of them has a unique closed neighborhood within S.*

Proof. Let $u, v, w \in S \subseteq V(G)$ induce a P_3 in G where v is the middle vertex of the path. Suppose first, on the contrary, that $N[u] \cap S = N[x] \cap S$ for some $x \in V(G)\setminus\{u\}$. However, now vertices x, u, and v form a triangle, or if $x = v$, then x, u, and w form a triangle. The case with w is symmetric. Suppose then that $N[v] \cap S = N[x] \cap S$ for some $x \in V(G)\setminus\{v\}$. Now, v, w, and x form a triangle, or if $x = w$, then u, v, and w form the triangle. Hence, the claim follows since G is triangle-free. □

2.1 | (X, Y)-Separating Codes and (X, Y)-Identifying Codes

We now introduce a generalization of identifying codes, and an upper bound for them that generalizes Bondy's theorem on induced subsets [10], which will be used several times in our proofs and, we believe, can be useful in many other settings as well.

Let $G = (V, E)$ be a graph and X and Y be two (not necessarily disjoint) vertex subsets of G. Then, Y induces a partition on X by the equivalence relation $\sim$ defined by $u \sim v$ if and only if $N_G[u] \cap Y = N_G[v] \cap Y$ for any $(u, v) \in X \times X$. If the partition on X induced by Y is such that each part is a singleton set, that is, for each pair $u, v \in X$ there exists a vertex $w \in Y$ that separates u, v, then Y is called an *(X, Y)-separating set* in G, the set X is called Y-separable and the graph G is called (X, Y)-separable. In addition, any subset $C \subseteq Y$ that is an (X, C)-separating set is also referred to as an (X, Y)-separating set. Moreover, we call any vertex subset C of G an *(X, Y)-identifying code* of G if 1) C is an (X, Y)-separating set in G, and 2) C is a dominating set of X. If such an (X, Y)-identifying code of G exists, then the set X is called *Y-identifiable*. Notice that, if X is Y-identifiable and C is an (X, Y)-identifying code of G, then Y itself is an (X, Y)-identifying code of G. Furthermore, set X is also C-identifiable and C is an (X, C)-identifying code of G. In particular, if G is an identifiable graph and C is an identifying code of G, then the vertex set V is V-identifiable and C-identifiable and C is a (V, V)-identifying code and a (V, C)-identifying code of G. When X and Y are disjoint and induce a bipartite graph, an (X, Y)-identifying code has been called a *discriminating code* in the literature [28].

Lemma 4. *Let G be a graph with vertex subsets X and Y such that X is Y-identifiable. Then, there is an (X, Y)-separating set in G of size at most $|X| - 1$, and an (X, Y)-identifying code of size at most $|X|$.*

Proof. Assume that G is (X, Y)-separable. If $|X| = 1$, there is nothing to do. Otherwise, we inductively construct an (X, Y)-separating code C of G such that $|C| \leq |X| - 1$. To begin with, let u, v be an arbitrary pair of distinct vertices of X and let $c \in Y$ such that c separates the pair: c exists since G is (X, Y)-separable. Then, we let $C = \{c\}$. Let $\mathcal{P}_C$ be the partition induced by C on X, where two vertices of X are in the same part if and only if their closed neighborhood in G intersects the same subset of C. Then, for as long as there exists a part P of $\mathcal{P}_C$ such that $u', v' \in P$ for two distinct vertices u', v' of X, the construction of C follows inductively by selecting an element $c' \in Y$ such that c' separates u', v', and letting $c' \in C$. At each step, since G is (X, Y)-separable, such c' exists. Moreover, we notice that at each inductive step, we have $|C| \leq |\mathcal{P}_C| - 1$, since at each step, we increase the number of parts by at least 1, and the size of C by exactly 1. This implies that we must have $|C| \leq |X| - 1$, since $|\mathcal{P}_C| \leq |X|$, affirming the first claim.

Now, if moreover G is (X, Y)-identifiable, we proceed as above to first build the (X, Y)-separating set C of G of size at most $|X| - 1$. Now, any two vertices of X are separated by C. Furthermore, there exists at most one vertex of X that is not dominated by C; for otherwise, if there exist two distinct vertices $x, x' \in X$ not dominated by C, it would imply that $N_G[x] \cap C = N_G[x'] \cap C = \emptyset$ and so, C would not be an (X, Y)-separating set of X, a contradiction. Therefore, let $x \in X$ be not dominated by C (if such an x exists). Since Y dominates X, there exists $c'' \in N_G[x] \cap Y$. Then, we let $c'' \in C$, thus making C an (X, Y)-identifying code of G with $|C| \leq |X|$. This completes the proof. □

Lemma 4 generalizes Bondy's celebrated theorem on "induced subsets" [10]. Indeed, in Bondy's theorem, one is given a set X of elements and a collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of subsets of X; it is proved by Bondy that there is a subset of at most $|X| - 1$ subsets of $\mathcal{A}$ that form an $(X, \mathcal{A})$-separating set, when viewing X and $\mathcal{A}$ as the two partite sets of a bipartite graph (the incidence bipartite graph of the hypergraph $(X, \mathcal{A})$), provided this graph is $\mathcal{A}$-identifiable. Bondy's original proof uses an elegant graph-theoretic argument [10] (several proofs of algebraic nature have also been provided, see e.g., [29]). A similar statement, formulated in the language of graphs, is also proved by Gutin, Ramanujan, Reidl, and Wahlström in Ref [[30], Lemma 8], by an inductive argument similar to the one presented here. These prior results, however, are only concerned with separating sets (thus, in their setting, one vertex may remain undominated), and with the special case where X and Y are disjoint. Hence, our result both generalizes and strengthens these setups.

2.2 | Paths and Cycles

Our main result requires a precise understanding of graphs with $\Delta = 2$ and triangle-free graphs, which need a positive constant c for Conjecture 1. Hence, in this subsection, we recall results on all connected graphs with $\Delta = 2$, that is, on paths and cycles. A path (cycle) on n vertices is denoted by P_n (C_n).

The identification number of all identifiable paths (i.e., of all paths except P_2) was determined by Bertrand et al. [31]. Moreover, using an upper bound from Ref. [31] on even cycles of order at least 6, Gravier et al. [32] provided the exact values of the identification numbers of all identifiable cycles (i.e., cycles of length at least 4). We summarize these results in the following theorems.

Theorem 5 ([31], Theorem 3). *If P_n is a path on n vertices, then we have*

$$\gamma^{\mathrm{ID}}(P_n) = \begin{cases} \dfrac{n}{2} + \dfrac{1}{2}, & \text{if } n \geq 1 \text{ is odd}, \\ \dfrac{n}{2} + 1, & \text{if } n \geq 4 \text{ is even}. \end{cases}$$

Theorem 6 ([32], Theorems 2 and 4). *If C_n is a cycle on n vertices, then we have*

$$\gamma^{\mathrm{ID}}(C_n) = \begin{cases} 3, & \text{if } n = 4, 5, \\ \dfrac{n}{2}, & \text{if } n \geq 6 \text{ is even}, \\ \dfrac{n}{2} + \dfrac{3}{2}, & \text{if } n \geq 7 \text{ is odd}. \end{cases}$$

Using Theorems 5 and 6, therefore, one has the following corollary.

Corollary 7. *The following holds.*

a. *If G is a path, then $\gamma^{\mathrm{ID}}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.*
b. *If $G = C_4$ or $G = C_5$, then $\gamma^{\mathrm{ID}}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.*
c. *If $n \geq 6$ is even, then $\gamma^{\mathrm{ID}}(C_n) = \frac{n}{2}$.*
d. *If $n \geq 7$ is odd, then $\gamma^{\mathrm{ID}}(C_n) = \frac{n}{2} + \frac{3}{2}$.*
e. *If $n = 4$, then $\gamma^{\mathrm{ID}}(P_n) = \frac{3}{4}n$, and if $n \geq 3$ and $n \neq 4$, then $\gamma^{\mathrm{ID}}(P_n) \leq \frac{2}{3}n$.*
f. *If $n \in \{4, 7\}$, then $\frac{2}{3}n < \gamma^{\mathrm{ID}}(C_n) \leq \frac{3}{4}n$, and if $n \geq 3$ and $n \notin \{4, 7\}$, then $\gamma^{\mathrm{ID}}(C_n) \leq \frac{2}{3}n$.*

We shall need the following elementary property of odd cycles with one edge added.

Observation 8 (Proposition 4.1 of [33]). *If $n = 2k + 1 \geq 7$ is odd, $G = C_n$ and if $e \in E(\overline{G})$, then $\gamma^{\mathrm{ID}}(G + e) \leq k + 1 \leq \frac{2}{3}n$.*

2.3 | Extremal Triangle-Free Graphs

In this section, we define and discuss the exceptional triangle-free graphs of the statement of Theorem 2, that is, those in the set $\mathcal{F}_\Delta$ that require $c > 0$ in the bound of Conjecture 1. The notation for set $\mathcal{T}_3$ (already defined in Ref.[2]) will be useful in our proof for the main theorem.

Definition 9. For $\Delta = 3$, we define $\mathcal{T}_3 = \{T_0, T_1, T_2, \ldots, T_{11}\}$ to be the collection of 12 trees of maximum degree 3 as in Figure 1, and for $\Delta \geq 4$, we let $\mathcal{T}_\Delta = \{K_{1,\Delta}\}$.

For $\Delta = 3$, we let $\mathcal{F}_3 = \mathcal{T}_3 \cup \{P_4, C_4, C_7\}$ and for $\Delta \geq 4$, we let $\mathcal{F}_\Delta = \mathcal{T}_\Delta = \{K_{1,\Delta}\}$.

We note that $\gamma^{\text{ID}}(T_0) = 3, \gamma^{\text{ID}}(T_1) = \gamma^{\text{ID}}(T_2) = 5, \gamma^{\text{ID}}(T_3) = \gamma^{\text{ID}}(T_4) = \gamma^{\text{ID}}(T_5) = 7, \gamma^{\text{ID}}(T_6) = \gamma^{\text{ID}}(T_7) = 9, \gamma^{\text{ID}}(T_8) = \gamma^{\text{ID}}(T_9) = 11, \gamma^{\text{ID}}(T_{10}) = 13$, and $\gamma^{\text{ID}}(T_{11}) = 15$. Generally, we have the following.

Proposition 10 [2]. *If $\Delta \geq 3$ is an integer and T is a tree of order n in $\mathcal{T}_\Delta$, then $\gamma^{\text{ID}}(T) = \left(\frac{\Delta-1}{\Delta}\right)n + \frac{1}{\Delta}$.*

By Corollary 7, if G has maximum degree $\Delta = 2$, then $\gamma^{\text{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n + \frac{3}{2}$. When $\Delta \geq 3$, we have the following.

Proposition 11. *If $\Delta \geq 3$ is an integer and G is a graph of order n and maximum degree at most Δ in $\mathcal{F}_\Delta$ (possibly, if $\Delta = 3$, the maximum degree of G is 2), then $\gamma^{\text{ID}}(G) = \left(\frac{\Delta-1}{\Delta}\right)n + \frac{1}{\Delta}$.*

Proof. If G is a tree in $\mathcal{T}_\Delta$, then this follows from Proposition 10. Otherwise, $\Delta = 3$ and $G \in \{P_4, C_4, C_7\}$. If $G \in \{P_4, C_4\}$, by Corollary 7, $\gamma^{\text{ID}}(G) = 3 = \frac{2}{3}n + \frac{1}{3}$ and if $G = C_7, \gamma^{\text{ID}}(G) = 5 = \frac{2}{3}n + \frac{1}{3}$. □

We have shown that trees satisfy Conjecture 1 in Ref. [2], as follows.

Theorem 12 [2]. *If $T \notin \mathcal{T}_\Delta$ is a tree of order $n \geq 3$ with maximum degree $\Delta \geq 3$, then*

$$\gamma^{\text{ID}}(T) \leq \left(\frac{\Delta - 1}{\Delta}\right)n.$$

In the following proposition, we present some useful properties for the structure of identifying codes in trees in set $\mathcal{T}_3$. The proof is provided in Ref. [2].

Proposition 13 [2]. *If T is a tree in $\mathcal{T}_3$, then the following properties hold.*

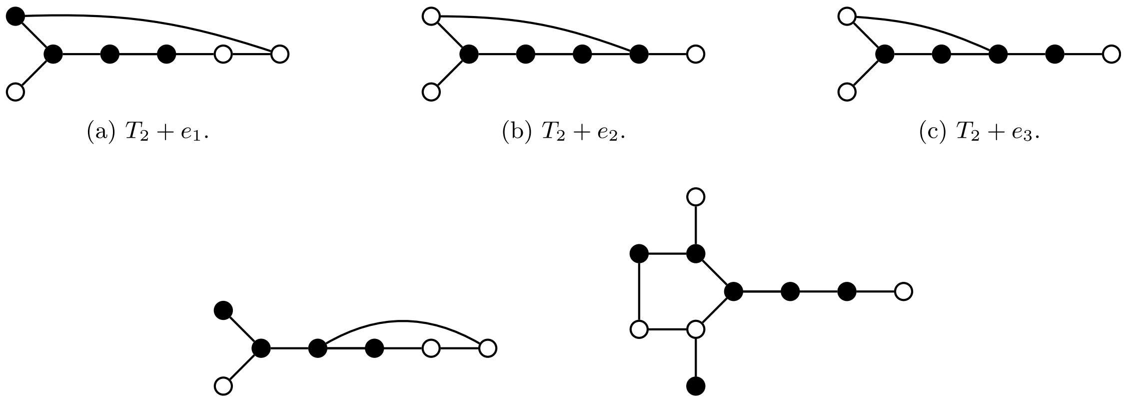
i. *If $T \neq T_2$, then T has an optimal identifying code $C(T)$ which includes all vertices of degree at most 2.*
ii. *If $T \notin \{T_2, T_3\}$, then $C(T)$ can be chosen as an independent set. When we delete any code vertex v from T, set $C(T)\backslash\{v\}$ forms an optimal identifying code of the forest $T - v$.*

We shall also need the following property of trees in family $\mathcal{T}_3$.

Lemma 14. *Let $T \in \mathcal{T}_3$ be a tree of order n. If $e \in E(\overline{T})$ is such that $T + e$ is triangle-free and $\Delta(T + e) = 3$, then $\gamma^{\text{ID}}(T + e) < \frac{2}{3}n$.*

Proof. Assume first that $T \notin \{T_2, T_3\}$. Note that since $T + e$ is triangle-free, $T \neq T_0$. Let $C(T)$ be an optimal identifying code in T which includes every vertex of degree at most 2 such that $C(T)$ is also an independent set. Such an identifying code exists by Proposition 13 and is presented in Figure 1. Observe that $C(T)$ 3-dominates every vertex in $V(T)\backslash C(T)$. Let us denote the endpoints of edge e by u and v. Since $\Delta(T + e) = 3$, both u and v have degree at most 2 in T and hence, $u, v \in C(T)$. Furthermore, since $T + e$ is triangle-free, at least one of u, v is a leaf (since in all trees in $\mathcal{T}_3\backslash\{T_2, T_3\}$, all degree 2 vertices are at distance 2 apart). Let us assume without loss of generality that u is a leaf and u_s is the adjacent support vertex. Furthermore, let $u' \in C(T)$ be the vertex at distance 2 from u with $d_T(u, v) = d_T(u', v)$ (vertex u' exists due to the structure of trees in $\mathcal{T}_3\backslash\{T_2, T_3\}$ and the assumption that $C(T)$ 3-dominates every vertex in $V(T)\backslash C(T)$).

Consider the set $C = (C(T)\backslash\{v, u'\}) \cup \{u_s\}$. Since $|C| = |C(T)| - 1$, we have $|C| < \frac{2}{3}n$. We claim that C is an identifying code in $T + e$. Since each vertex in $V(T)\backslash C(T)$ is 3-dominated by $C(T)$, each vertex in $V(T)\backslash C(T)$ is still dominated by C.



(a) $T_2 + e_1$. (b) $T_2 + e_2$. (c) $T_2 + e_3$. (d) $T_2 + e_4$. (e) $T_3 + e$.

FIGURE 2 | Some triangle-free graphs of types $T_2 + e$ and $T_3 + e$.

TABLE 1 | Known graphs requiring a positive constant c for Conjecture 1.

Graph class	Δ	c	Reference
Odd paths	2	$1/\Delta = 1/2$	[31] (Theorem 5)
Even paths	2	$2/\Delta = 1$	[31] (Theorem 5)
Odd cycles C_n for $n \geq 7$	2	$3/\Delta = 3/2$	[32] (Theorem 6)
C_4	2	$2/\Delta = 1$	[32] (Theorem 6)
C_5	2	$1/\Delta = 1/2$	[32] (Theorem 6)
$\mathcal{T}_3$	3	$1/\Delta = 1/3$	[2]
$K_{1,\Delta}$	$\Delta \geq 3$	$1/\Delta$	[2]
Complements of half-graphs and their complete joins	even $\Delta \geq 2$	$2/\Delta$	[19]
Complements of half-graphs and their complete joins plus one universal vertex	odd $\Delta \geq 3$	$1/\Delta$	[19]

Furthermore, u' is dominated by u_s, and v is dominated by u. Thus, C is a dominating set in $T + e$. We now check that C is separating. Since only u' and v have been removed from $C(T)$ when constructing C, it suffices to show that all vertices in $N[u'] \cup N[v]$ are separated from all other vertices of $T + e$, as well as u and v are pairwise separated, since we added the edge e. (Other pairs of vertices, such as the neighbors of u_s, may have been affected, but since we only added code vertices to their closed neighborhood, they remain separated by C.) We note that there exists a vertex $u_c \in C \cap N(u_s)\backslash\{u\}$ since u_s is 3-dominated by $C(T)$. Hence, u, u_s, and u_c form a 3-path in C, and they have a unique neighborhood in C by Lemma 3. Furthermore, vertex v is the only vertex with exactly u in its code neighborhood, while any neighbor of v is in $V(T)\backslash C$ and remains at 2-dominated or 3-dominated. Hence, vertices in $N(v)$ have unique code neighborhoods. Vertex u' is the only vertex with exactly u_s in its code neighborhood, and any vertex in $N(u')\backslash C$ has a unique code neighborhood since it is at least 2-dominated. Therefore, C is an identifying code in $T + e$ satisfying the claim. Hence, we are only left with considering trees $\mathcal{T}_2$ and $\mathcal{T}_3$.

For the case $T = \mathcal{T}_2$, we have presented each possible triangle-free graph $T + e$ of maximum degree 3 in Figures 2a–d, together with identifying codes satisfying the claim. For $T = \mathcal{T}_3$, notice that we may obtain $\mathcal{T}_3$ from $\mathcal{T}_2$ by attaching a P_3 from the middle vertex to a vertex of $\mathcal{T}_2$. Hence, when the new edge e is not adjacent to one of the vertices in the P_3, we may use the identifying code in $\mathcal{T}_2 + e$ together with the two new leaves in the newly attached P_3. When we have the edge e adjacent to the vertices in the newly attached P_3, we may use the identifying code from Figure 2e. □

We have listed in Table 1 the graphs which are known (to us) to require a positive constant for Conjecture 1. Among these are the extremal graphs discovered in [19] (those graphs of order n with identifying code number $n - 1$). Those are either stars, or can be built from any number of complements of half-graphs[1] by taking their complete join, and optionally, adding

[1] A *half-graph* is a special bipartite graph with both parts of the same size, where each part can be ordered so that the open neighborhoods of consecutive vertices differ by exactly one vertex [34]. Their complements can also be described as powers of paths of the form P_{2k}^{k-1} [19].

a single universal vertex. Note that the latter examples have large cliques and thus are far from triangle-free. They have a maximum degree $n - 1$ or $n - 2$, and so, they need $c = 1/\Delta$ or $c = 2/\Delta$ in the bounds of Conjecture 1.

It is an interesting open question whether there exist any other such graphs and if the constant $c = \frac{3}{2}$ is enough for all graphs. Notice that by Theorem 2, if there exists a graph not listed in Table 1 that requires a positive constant c, then it must contain triangles.

3 | Proof of the Main Result

In this section, we shall prove our main result, namely Theorem 2. Recall its statement.

Theorem 2. *Let $\Delta \geq 3$ be an integer, and let G be a connected triangle-free graph of order $n \geq 3$. If $G \in \mathcal{F}_\Delta$, then*

$$\gamma^{\mathrm{ID}}(G) = \left(\frac{\Delta - 1}{\Delta}\right)n + \frac{1}{\Delta}.$$

On the other hand, if $G \notin \mathcal{F}_\Delta$ has maximum degree Δ, then

$$\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta - 1}{\Delta}\right)n.$$

Proof of Theorem 2. The first part of the statement is held by Proposition 11.

For the second part, let G be a connected triangle-free graph of order n and size m with maximum degree $\Delta \geq 3$ such that $G \notin \mathcal{F}_\Delta$. Thus, $n \geq 5$. We proceed by induction on $n + m$ to show that $\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$. Since G is connected, we note that $m \geq n - 1$, and so $n + m \geq 2n - 1 \geq 9$. If $n + m = 9$, then G is formed from a star by joining a pendant leaf to another leaf. By Theorem 12, we have $\gamma^{\mathrm{ID}}(G) \leq \frac{2}{3}n$, in this case. Furthermore, if $n = m = 5$, then G is formed from a four-cycle joined by a leaf. Observe that such a graph has $\gamma^{\mathrm{ID}}(G) = 3 < \frac{2}{3}n$. This establishes the base cases. Let $n + m \geq 11$, where $n \geq 5$.

For the inductive hypothesis, assume that if G' is a connected triangle-free graph of order $n' \geq 3$ and size m' with $n' + m' < n + m$ and with maximum degree $\Delta' = \Delta(G') \geq 3$ such that $G' \notin \mathcal{F}_{\Delta'}$, then $\gamma^{\mathrm{ID}}(G') \leq \left(\frac{\Delta'-1}{\Delta'}\right)n'$.

If $m = n - 1$, then G is a tree. Since $G \notin \mathcal{F}_\Delta$, by Theorem 12, we have $\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$. Hence, we may assume that $m \geq n$, for otherwise the desired upper bound follows. Thus, the graph G contains a cycle edge, that is, an edge that belongs to a cycle in G. Moreover, since $\Delta \geq 3$, the graph G is not a cycle.

Among all cycle edges in G, let $e = uv$ be chosen so that the sum of the degrees of its ends is as large as possible, that is, $\deg_G(u) + \deg_G(v)$ is maximal. Since $\Delta(G) \geq 3$, we have for the edge e that

$$\deg_G(u) + \deg_G(v) \geq 5.$$

Let

$$G' = G - e.$$

Since e is a cycle edge of G, the graph G' is a connected triangle-free graph of order n. Let $\Delta(G') = \Delta'$, and so $\Delta' \geq \Delta - 1$. We note that $n(G') = n$ and $m(G') = m - 1$.

Suppose that $\Delta' = 2$. In this case, $\Delta = 3$ and G' is either a path or a cycle. Suppose firstly that G' is a cycle. By the triangle-free condition and our choice of the edge e, we infer that G is obtained from a cycle C_n where $n \geq 6$ by adding a chord between two non-consecutive vertices on the cycle in such a way as to create two cycles that both contain the edge e and both have length at least 4. Assume first that $n \neq 7$. By Corollary 7f, $\gamma^{\mathrm{ID}}(G') \leq \frac{2}{3}n$. By Observation 8, $\gamma^{\mathrm{ID}}(G) \leq \gamma^{\mathrm{ID}}(G') \leq \frac{2}{3}n$ if n is odd. Hence, we may consider the case with even $n \geq 6$. Let G' be the cycle $v_1 v_2 \cdots v_n v_1$. Observe that both the set V_{even} of vertices with even subscript and the set V_{odd} of vertices with odd subscript are identifying codes of G' of size $n/2$. If the chord in G is between two vertices with even subscripts, then the set V_{odd} is an identifying code in G, and vice versa. Moreover, if the chord is between a vertex with even subscript and a vertex with odd subscript, then both sets V_{even} and V_{odd} are identifying codes of G. Hence, $\gamma^{\text{ID}}(G) \leq \frac{1}{2}n$. If $n = 7$, then $\gamma^{\text{ID}}(G) = \gamma^{\text{ID}}(G') - 1 = 4 < \frac{2}{3}n$ by Observation 8. Therefore, if G' is a cycle, then $\gamma^{\text{ID}}(G) \leq \frac{2}{3}n$, as desired. Suppose secondly that G' is a path. In this case, $n \geq 5$. For small values of n, namely $n \in \{5, 6, 7, 8, 10\}$, it can readily be checked that $\gamma^{\text{ID}}(G) \leq \gamma^{\text{ID}}(G') = \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \frac{2}{3}n$. Hence, we may assume that $n \geq 9$ is odd or $n \geq 12$ is even. Any identifying code in the path G' can be extended to an identifying code in $G = G' + e$ by adding at most one vertex, and so, by Corollary 7a, we have $\gamma^{\text{ID}}(G) \leq \gamma^{\text{ID}}(G') + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq \frac{2}{3}n$. Hence, we may assume that $\Delta' \geq 3$, for otherwise the desired bound holds.

Since G is triangle-free, we note that $G' \neq K_{1,\Delta'}$ (since otherwise adding back the deleted edge e would create a triangle in G). In particular, if $\Delta' \geq 4$, then $G' \notin \mathcal{F}_{\Delta'}$. If $\Delta' = 3$ and $G' \in \mathcal{F}_{\Delta'}$, then by Observation 8 and Lemma 14 we infer that $\gamma^{\text{ID}}(G) < \frac{2}{3}n \leq \left(\frac{\Delta - 1}{\Delta}\right)n$. Hence, we may assume that $G' \notin \mathcal{F}_{\Delta}$, for otherwise the desired bound holds. Applying the inductive hypothesis to the graph G', we have $\gamma^{\text{ID}}(G') \leq \left(\frac{\Delta' - 1}{\Delta'}\right)n \leq \left(\frac{\Delta - 1}{\Delta}\right)n$.

For notational convenience, let $N_u = N_G(u) \backslash \{v\}$ and let $N_v = N_G(v) \backslash \{u\}$. Since G is triangle-free and $uv \in E(G)$, we note that $N_u \cap N_v = \emptyset$. Let A be the boundary of the set $\{u, v\}$, that is, $A = N_u \cup N_v$ is the set of vertices different from u and v that are adjacent to u or v. Further, let

$$A_{uv} = A \cup \{u, v\} = N_G[u] \cup N_G[v] \quad \text{and} \quad \overline{A}_{uv} = V(G) \backslash A_{uv}.$$

See Figure 3 for an illustration.

Let C' be an optimal identifying code in G', and so C' is an identifying code in G' and $|C'| = \gamma^{\text{ID}}(G')$. If C' is an identifying code in G, then $\gamma^{\text{ID}}(G) \leq |C'| = \gamma^{\text{ID}}(G') \leq \left(\frac{\Delta - 1}{\Delta}\right)n$. Hence, we may assume that C' is not an identifying code in G, for otherwise the desired bound holds.

We will next proceed with proving a series of claims.

Claim A. If $V(G) = A_{uv}$, then

$$\gamma^{\text{ID}}(G) \leq \left(\frac{\Delta - 1}{\Delta}\right)n.$$

Proof. Suppose that $V(G) = A_{uv}$. Recall that by our choice of the edge e, we have $\deg_G(u) \geq 2$, $\deg_G(v) \geq 2$, and $5 \leq \deg_G(u) + \deg_G(v) \leq 2\Delta$. In this case since $V(G) = A_{uv}$, we have $n = \deg_G(u) + \deg_G(v) \leq 2\Delta$. Let u' be an

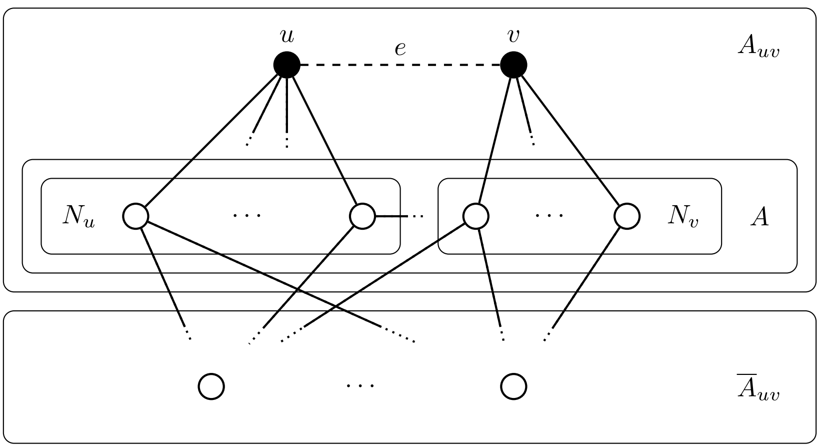


FIGURE 3 | The general setting of the proof of Theorem 2. Since G is triangle-free, N_u and N_v are independent sets, but there can be edges across them.

arbitrary neighbor of u in G different from v, and let v' be an arbitrary neighbor of v in G different from u. The code $C = V(G)\setminus\{u', v'\}$ is an identifying code in G, implying that

$$\begin{aligned}\gamma^{\mathrm{ID}}(G) \le |C| &= \deg_G(u) + \deg_G(v) - 2\\ &= \left(\frac{\deg_G(u) + \deg_G(v) - 2}{\deg_G(u) + \deg_G(v)}\right)n\\ &\le \left(\frac{2\Delta - 2}{2\Delta}\right)n\\ &= \left(\frac{\Delta - 1}{\Delta}\right)n,\end{aligned}$$

yielding the desired upper bound. □

By Claim A, we may assume that $V(G) \neq A_{uv}$, for otherwise the desired result follows. Let

$$G_{uv} = G - (N_G[u] \cup N_G[v]),$$

and so $V(G_{uv}) = \overline{A}_{uv} = V(G)\setminus(A \cup \{u, v\})$. Since C' is an identifying code in G' but not in G, a pair of vertices in G is not identified by the code C' when adding back the deleted edge e to the graph G' to reconstruct G. Since G is triangle-free, the only possible pairs of vertices not identified by the code C' in G are the pairs $\{u, v\}$ or $\{u', v\}$ or $\{u, v'\}$ where u' is some neighbor of u different from v, and v' is some neighbor of v different from u.

Claim B. If for every optimal identifying code C' in G' the pair $\{u, v\}$ is the only pair not identified by the code C' in G, then

$$\gamma^{\mathrm{ID}}(G) \le \left(\frac{\Delta - 1}{\Delta}\right)n.$$

Proof. Let C' be an optimal identifying code in G', and suppose the pair $\{u, v\}$ is the only pair not identified by the code C' in G. Necessarily, $\{u, v\} \subseteq C'$, no neighbor of u different from v belongs to C' and no neighbor of v different from u belongs to C'; that is, $C' \cap A = \emptyset$, and so no neighbor of u in G' and no neighbor of v in G' belongs to C'. Equivalently, $C'\setminus\{u, v\} = C' \cap \overline{A}_{uv}$. See Figure 3, where the black vertices belong to C', and the other ones do not. We proceed further with a series of subclaims that we will need when proving Claim B. Since every neighbor of u (respectively, v) is identified by the code C' in the graph G', we infer the following claim.

Claim B.1. Every neighbor of u (respectively, v) in G' has at least one neighbor that belongs to the set $C'\setminus\{u, v\}$.

We shall frequently use the following claim when obtaining structural properties of the graph G.

Claim B.2. If there exists a vertex $w \in C'\setminus\{u, v\}$ and a vertex $z \in A$ such that $(C'\setminus\{w\}) \cup \{z\}$ is an identifying code in the graph G, then $\gamma^{\mathrm{ID}}(G) \le \left(\frac{\Delta-1}{\Delta}\right)n$.

Proof. Consider the set $(C'\setminus\{w\}) \cup \{z\}$ as defined in the statement of the claim. If this set is an identifying code in the graph G, then $\gamma^{\mathrm{ID}}(G) \le |C'| \le \left(\frac{\Delta-1}{\Delta}\right)n$. □

Thus, Claim B.2 proves the result when the graph G_{uv} contains a component of order 1. By Claim B.2, we may assume that there does not exist a vertex $w \in C'\setminus\{u, v\}$ and a vertex $z \in A$ such that $(C'\setminus\{w\}) \cup \{z\}$ is an identifying code in the graph G, for otherwise the desired bound holds. Recall that $C' \cap A = \emptyset$. In the following claim, we show that there are no P_2-components in G_{uv}.

Claim B.3. No component in G_{uv} has order 2.

Proof. Suppose that the graph G_{uv} contains a component F of order 2, and so F is isomorphic to P_2. As observed earlier, in the graph G', we have $C' \cap N(u) = C' \cap N(v) = \emptyset$. In this case, the two vertices in the component F are not separated by the code C', a contradiction. □

In the following subclaims, we consider the case with $\Delta = 3$ separately. This is due to the more complex structure of set $\mathcal{F}_3$ compared to sets $\mathcal{F}_i$ for $i \geq 4$.

Claim B.4. Let $\Delta = 3$. If C_7 is a component in G_{uv}, then $\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$.

Proof. Let $F = C_7$ be a cycle component in G_{uv}. Since C' is an optimal identifying code in G' and since C' contains no vertices in the boundary A, the set $V(F) \cap C'$ is an identifying code in F, that is, $\gamma^{\mathrm{ID}}(F) \leq |V(F) \cap C'|$. By Theorem 6, we have $\gamma^{\mathrm{ID}}(C_7) = 5$. Let $V(F) = \{w_1, w_2, \ldots, w_7\}$ and $E(F) = \{w_7w_1\} \cup \{w_iw_{i+1} | 1 \leq i \leq 6\}$. Observe that each vertex in F can be adjacent to at most one vertex in A since $\Delta = 3$. Note that, since $\Delta = 3$, we have $1 \leq |N_u| \leq 2$ and $1 \leq |N_v| \leq 2$, and so $|A| \leq 4$. We denote $N_u = \{u_1, u_2\}$ and $N_v = \{v_1, v_2\}$ (if these vertices exist). We further assume that $|N(F) \cap N_u| \geq |N(F) \cap N_v|$. Moreover, let w_2 be adjacent to $u_1 \in N_u$. Let us consider vertex sets

$$C_1 = (C' \backslash V(F)) \cup \{u_1, w_1, w_4, w_5, w_6\}$$

and

$$C_2 = (C' \backslash V(F)) \cup \{u_1, w_3, w_5, w_6, w_7\}.$$

In the following, we show that at least one of these two sets is an identifying code in G. Notice that $|C_1| = |C_2| \leq |C'|$.

If u_1 is the only vertex of A adjacent to a vertex in F, then C_1 is an identifying code in G. In particular, we may use Lemma 3 to see that u, v, and u_1 have unique neighborhoods in C_1, while $C_1 \cap V(F)$ forms an identifying code for $F \backslash \{w_2\}$, and w_2 is identified by vertices u_1 and w_1. The remaining vertices are identified by the vertices in C'.

Assume then that there are two vertices of A (u_1 and, say, x) adjacent to vertices in F. If x is in N_u ($x = u_2$), then C_1 is an identifying code in G, even if x is not dominated by a vertex in $C_1 \cap V(F)$. Indeed, we have $u \in N[u_2] \cap C_1$ but $v, u_1 \notin N[u_2] \cap C_1$. Thus, u_2 is separated from all other vertices. If $x \in N_v$, then C_1 or C_2 is an identifying code in G. In this case, we choose such a set C_i ($i \in \{1, 2\}$) so that $N(x) \cap V(F) \cap C_i \neq \emptyset$.

Assume next that there are three vertices of A adjacent to the vertices in F. In this case, due to our assumption that $|N_u \cap N(F)| \geq |N_v \cap N(F)|$, we have $|N_u \cap N(F)| = 2$ and hence, we may assume that $A \cap N(F) = \{u_1, u_2, v_1\}$. Again we choose a set C_i ($i \in \{1, 2\}$), such that $N(v_1) \cap V(F) \cap C_i \neq \emptyset$, as our identifying code. Note that, as in the previous case, we do not need to dominate u_2 from F.

Finally, when (all) four vertices of A are adjacent to vertices in F, we again choose a set C_i ($i \in \{1, 2\}$) such that $N(v_1) \cap V(F) \cap C_i \neq \emptyset$ as our identifying code. As in the previous cases, we do not need to dominate vertices u_2 and v_2 from F. The argument for u_2 is similar as in the previous cases. Moreover, by Lemma 3, vertices v and u have unique neighborhoods in C_i, vertex v separates v_2 from vertices other than u, v, and v_1, while v_1 is separated from v_2 by a vertex in F.

The claim follows in these cases since $|C_1| = |C_2| \leq |C'|$ and since C_1 or C_2 is an identifying code in G. ☐

Claim B.5. Let $\Delta = 3$. If C_4 is a component in G_{uv}, then $\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$.

Proof. Let $F = C_4$ be a cycle component in G_{uv}. Since C' is an optimal identifying code in G' and since C' contains no vertices in the boundary A, the set $V(F) \cap C'$ is an identifying code in F, that is (by Theorem 6), $\gamma^{\mathrm{ID}}(F) = 3 \leq |V(F) \cap C'|$. Let $V(F) = \{w_1, w_2, w_3, w_4\}$ and $E(F) = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1\}$. Observe that each vertex in F can be adjacent to at most one vertex in A since $\Delta = 3$. Recall that by Claim B.1 every vertex in A is dominated by two vertices in C'.

Let us first assume that $V(F) \subseteq C'$. Then, we can apply Claim B.2 to any vertex in F and its neighbor in A, and the claim follows. Thus, we can now assume that $|V(F) \cap C'| = 3$.

Let us first assume that there are one or three vertices in A adjacent to vertices in F. Note that each vertex in F can be adjacent to at most one vertex in A. If there is one vertex in A adjacent to a vertex in F, then we assume that the adjacent vertex in F is $w_2 \in F$. If there are three vertices in A adjacent to vertices in F, then we assume that they are adjacent to vertices w_1, w_2, and w_3 in F. Furthermore, we assume that $z \in A$ is adjacent to w_2. We will consider vertex set $C = (C' \backslash V(F)) \cup \{w_1, w_3, z\}$. This is an identifying code in G since all vertices in A_{uv} are separated from other vertices by u and v, and vertices u, v, and z have unique code neighborhoods in C by Lemma 3. Moreover, vertices in F are separated from each other by z, w_1, and w_3, and finally, vertices in $A \backslash \{z\}$ are separated from each other by the same vertices as in C'.

Assume then that there are four vertices in A adjacent to vertices in F. Let us assume without loss of generality that $\{w_1, w_2, w_3\} \subset C'$ and that $z \in A$ is adjacent to w_2. As in the previous case, now $C = (C' \backslash \{w_2\}) \cup \{z\}$ is an identifying code in G.

Finally, we have the case where we have exactly two vertices in A adjacent to vertices in F. Assume first that these vertices in A are adjacent to two adjacent vertices of F, say to w_1 and w_2. Let $z \in A$ be a vertex adjacent to w_2. As in the previous cases, we let $C = (C' \backslash V(F)) \cup \{w_1, z, w_3\}$. With the same arguments as in the previous cases, C is an identifying code in G.

Let us next consider the case where there are two vertices in A adjacent to only non-adjacent vertices in F, say, vertices w_1 and w_3. Observe that now w_2 and w_4 are open twins in G' and we have exactly two edges between A and F. Let $z \in A$ be a neighbor of w_1. Assume that $|N_u| \geq |N_v|$. Let us denote the neighbors of u in A by u_1 and u_2, and neighbors of v in A by v_1 and v_2 (if v_2 exists). Notice that if both edges from F to A are between F and N_u or F and N_v, then $C = \{z, w_2, w_4\} \cup (C' \backslash V(F))$ is an identifying code of G. Thus, we may assume without loss of generality that there is an edge from w_1 to u_1 and from w_3 to v_1. Assume next that for u_2 or v_2 there exists vertex $c \in C'$ such that $c \in N(v_2) \backslash N(v_1)$ or $c \in N(u_2) \backslash N(u_1)$. If the former holds, then $C_u = \{u_1, w_2, w_4\} \cup (C' \backslash V(F))$ is an identifying code of G, and if the latter holds, then $C_v = \{v_1, w_2, w_4\} \cup (C' \backslash V(F))$ is an identifying code of G. Hence, we next assume that c does not exist. Moreover, if v_2 does not exist, then again C_u is an identifying code of G, and we are done. Hence, we assume that v_2 exists. Recall that each vertex in A is dominated by at least two vertices of C' (since otherwise, a vertex of A dominated only by one vertex of C' would not be separated from either u or v in G'). Hence, u_2 has a neighbor in $C' \cap \overline{A}_{uv}$, and since every such vertex is also a neighbor of u_1 (by the above argument that c does not exist), there exists a vertex $c_u \in C' \cap (N(u_1) \cap N(u_2)) \backslash \{u\}$. By the same argument applied to v_2, there is also a vertex $c_v \in C' \cap (N(v_1) \cap N(v_2)) \backslash \{v\}$. Observe that if there exists a vertex w_u such that $N[w_u] \cap C' = \{c_u\}$ or a vertex w_v such that $N[w_v] \cap C' = \{c_v\}$, then $w_u = c_u$ and $w_v = c_v$. Indeed, otherwise we would have $N[w_u] \cap C' = N[c_u] \cap C'$ or $N[w_v] \cap C' = N[c_v] \cap C'$. We may also observe that we have either $\{w_1, w_2, w_3\} \subseteq C'$ or $\{w_1, w_4, w_3\} \subseteq C'$. Let us assume, without loss of generality, the former. However, now $C = \{u_1\} \cup (C' \backslash \{u\})$ is an identifying code in G. Indeed, only u, u_1 and u_2 have lost a codevertex from their neighborhoods compared to C'. Moreover, u is the only vertex adjacent to both v and u_1, u_1 is the only vertex adjacent to w_1 and c_u and finally, $C \cap N[u_2] = \{c_u\}$ but now $C \cap N[c_u] \supseteq \{c_u, u_1\}$. Thus, also u_2 has a unique code neighborhood. Now, the claim follows. ☐

Claim B.6. Let $\Delta = 3$. If P_4 is a component in G_{uv}, then $\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta - 1}{\Delta}\right)n$.

Proof. Let $F = P_4$ be a path component in G_{uv}. Since C' is an optimal identifying code in G' and since C' contains no vertices in the boundary A, the set $V(F) \cap C'$ is an identifying code in F, that is, by Theorem 5, $\gamma^{\mathrm{ID}}(F) = 3 \leq |V(F) \cap C'|$. Let $V(F) = \{w_1, w_2, w_3, w_4\}$ and $E(F) = \{w_1w_2, w_2w_3, w_3w_4\}$.

Case 1. *There is an edge between w_1 or w_4 and A*. Without loss of generality, let it be w_1. Consider now the graph $G_1 = G - \{w_2, w_3, w_4\}$. First observe that G_1 is connected. Second, if $G_1 \notin \mathcal{F}_3$, then by induction we have an identifying code $C_1 \subseteq V(G_1)$ of cardinality at most $\frac{2}{3}n(G_1)$. Moreover, if no neighbor (in G) of w_2, w_3 or w_4 is in C_1, then $C_1 \cup \{w_2, w_4\}$ is an identifying code of claimed cardinality in G. If on the other hand $C_1 \cap (N_G(w_2) \cup N_G(w_3)) \neq \varnothing$, then at least one of the sets $C_1 \cup \{w_2, w_4\}$ and $C_1 \cup \{w_3, w_4\}$ is an identifying code in G. The case with $C_1 \cap (N_G(w_4) \cup N(w_3)) \neq \varnothing$ is analogous. Moreover, these identifying codes have the claimed cardinality.

Let us next consider the case with $G_1 \in \mathcal{F}_3$. Furthermore, there is a vertex with degree 3, and there are at least six vertices in G_1 since $\deg(u) + \deg(v) \geq 5$ by our assumption. Thus, $G_1 \in \mathcal{F}_3 \backslash \{P_4, C_4, C_7, K_{1,3}\} = \mathcal{T}_3 \backslash \{K_{1,3}\}$. Let us assume first that $G_1 = T_2$. In this case, we have $n = 10$. Notice that $\deg(u) + \deg(v) = 5$ and either u or v is the support vertex of degree 3 in T_2, and the other one is the adjacent vertex of degree 2. However, since w_1 is a leaf in T_2, this means that it must have a distance of either 1 or 3 to set $\{u, v\}$, a contradiction since that distance is actually 2.

Let us next assume that $G_1 = T_3$. Observe that in this case, $\deg(u) + \deg(v) = 6$. Since w_1 has distance 2 to set $\{u, v\}$, vertex w_1 has to be one of the leaves adjacent to one of the support vertices of degree 3 of T_3. But then, the leaf of T_3 adjacent to the unique support vertex of degree 2 in T_3 as well as that support vertex form a component of order 2 in G_{uv}, contradicting Claim B.3.

Assume next that $G_1 \in \mathcal{T}_3 \setminus \{K_{1,3}, T_2, T_3\}$. Let C_1 be an optimal identifying code of G_1 such that every vertex with degree at most 2 in G_1 is in C_1 (such a code exists by Proposition 13, see Figure 1). Since G is not a tree, there are at least two edges between $A \cup \{w_1\}$ and $\{w_2, w_3, w_4\}$. Hence, there is an edge between w_1 and w_2 and between $G_1' = G_1 - \{w_1\}$ and $\{w_2, w_3, w_4\}$.

Let us consider the case where there are no edges between A and w_2 or w_3 but there is an edge to w_4. In this case, we may consider the identifying code $C = C_1 \cup \{w_3\}$ which has the claimed cardinality. Notice that this is indeed an identifying code since $w_1 \in C_1$ and w_4 is dominated by some vertex in C_1.

Therefore, we may assume from now on that there is an edge between A and w_2 or w_3. Let us assume that the edge is from A to w_2 (the case with an edge to w_3 is similar). Consider now graph $G_1'' = G - \{w_1, w_3, w_4\}$ together with an optimal identifying code C_1''. Observe that G_1'' is a tree since G_1 is a tree and w_2 has only one edge to A. Moreover, we also notice that there are no edges between vertices in A since G_1 is a tree. Thus, if a 4-cycle contains exactly two vertices in F, then those vertices are w_1 and w_4.

If $G_1'' \notin \mathcal{F}_3$, then we have two cases based on whether $w_2 \in C_1''$.

If $w_2 \in C_1''$, then $C_1'' \cup \{w_3, w_4\}$ is an identifying code in G. Indeed, by Lemma 3 vertices w_2, w_3, and w_4 have unique code neighborhoods. Moreover, also w_1 has a unique neighborhood in $C_1'' \cup \{w_2\}$ since any vertex of A adjacent to w_2 is also either in C_1'' or has another neighbor outside of A in C_1''.

If $w_2 \notin C_1''$, then we use set $C_{23} = C_1'' \cup \{w_2, w_3\}$. Since C_1'' is an identifying code in G_1'', the vertex adjacent to w_2 (call it z) in A is in C_1'', and z has another code neighbor. Hence, by Lemma 3, vertices z, w_2, and w_3 have unique code neighborhoods. Moreover, since w_1 is adjacent to w_2 and all other vertices in $N[w_2]$ have unique code neighborhoods, also w_1 has a unique neighborhood in C_{23}. Furthermore, also w_4 has a unique neighborhood in C_{23} since the only other vertex x adjacent to w_3 which might not be separated from w_4 is in A and is in C_1'' or has another vertex in C_1'' adjacent to it. Since w_3 and w_4 cannot belong to the same 4-cycle in G and there are no triangles, vertices w_4 and x are separated. All the other vertices are pairwise separated by the set C_1''. Hence, C_{23} is an identifying code in G.

Thus, we may assume that $G_1'' \in \mathcal{F}_3$. Observe that we may construct G_1'' from G_1 by removing a leaf and then adding a new leaf, and vice versa. Hence, G_1, G_1'' are two trees of $\mathcal{F}_3$ (and thus, $\mathcal{T}_3$) with the same order of at least 6. Observe that for every i such that $i \in \{0, 1\}$ or $i \geq 4$, there are no support vertices of degree 2 in T_i ($T_i \in \mathcal{T}_3$). However, when we remove a leaf from any such tree of $\mathcal{T}_3$ and add a different leaf to a vertex of degree at most 2, there necessarily exists a support vertex of degree 2 in the resulting tree. Thus, at least one of G_1 and G_1'' must be in $\{T_2, T_3\}$. Note that we cannot obtain an isomorphic copy of T_3 by this operation when starting from T_3, so at most one of G_1 and G_1'' is T_3. Thus, $\{G_1, G_1''\} = \{T_1, T_2\}$, $\{G_1, G_1''\} = \{T_2, T_2\}$ or $\{G_1, G_1''\} = \{T_3, T_4\}$. Let us first assume that $G_1, G_1'' = \{T_3, T_4\}$. Consider the leaf adjacent to the support vertex of degree 2 in T_3. Notice that the leaf has to belong to F. Moreover, its distance from set $\{u, v\}$ is exactly 2. However, now when we form T_4 by removing this leaf and attaching the new leaf, the new leaf is attached adjacent to u or v, and hence, $A \cap V(F) \neq \emptyset$, a contradiction. Thus, $G_1, G_1'' \in \{T_1, T_2\}$.

We thus have $\{G_1, G_1''\} = \{T_1, T_2\}$ or $\{G_1, G_1''\} = \{T_2, T_2\}$. In both cases, we have $\deg(u) + \deg(v) = 5$. Hence, one of u or v is the degree 3 support vertex in T_2, and the other one is the adjacent degree 2 vertex. Moreover, the leaf adjacent to the support vertex of degree 2 in T_2 is in F. But this vertex should have distance 2 to set $\{u, v\}$, a contradiction. This finishes the proof of Case 1.

Case 2. *There are no edges from w_1 or w_4 to A.* Then, there is an edge from A to w_2 or w_3; without loss of generality, assume there is an edge from A to w_2 and denote $G_2 = G - \{w_1, w_3, w_4\}$. Observe that if $G_2 \notin \mathcal{F}_3$, then by induction, there exists an identifying code C_2 in G_2 with $|C_2| \leq \frac{2}{3}n(G_2)$ and $|C_2| + 2 \leq \frac{2}{3}n$. Moreover, either $C_2 \cup \{w_1, w_3\}$ or $C_2 \cup \{w_2, w_3\}$ is an identifying code in G, depending on whether $w_2 \in C_2$ or not. Hence, we may assume that $G_2 \in \mathcal{F}_3$ and in fact $G_2 \in \mathcal{T}_3$ since G_2 has a vertex of degree 3. If we do not have an edge from w_3 to A, then G would be a tree, a contradiction. Hence, we may assume that there exists an edge from w_3 to A. Let us assume that $G_2 \neq T_2$ and C_2 contains all vertices of degree at most 2 in G_2. This is possible by Proposition 13. Observe that in this case, $C_2 \cup \{w_3\}$ is an identifying code of claimed size in G. Hence, we may assume that $G_2 = T_2$. Since $\deg(u) + \deg(v) \geq 5$, the only leaf of T_2 not in A is the leaf adjacent to the degree 2 support vertex. However, this vertex has a distance 3 to set $\{u, v\}$. Thus, $G_2 \neq T_2$, and the claim follows. □

Claim B.7. Let $\Delta = 3$. If $T \in \mathcal{T}_3$ is a component in G_{uv}, then $\gamma^{\text{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$.

Proof. Let $F = T \in \mathcal{T}_3$ be a tree component in G_{uv}. Since C' is an optimal identifying code in G' and since C' contains no vertices in the boundary A, the set $V(F) \cap C'$ is an identifying code in F, that is, $\gamma^{\text{ID}}(F) \leq |V(F) \cap C'|$. Let us call by T^2 the set of vertices of degree at most 2 in $G[F]$. Note that only vertices of T^2 can have a neighbor in A.

Let us first assume that $F \in \mathcal{T}_3 \backslash \{T_2, T_3\}$. By Proposition 13 we can choose an optimal identifying code C_T for T such that $T^2 \subseteq C_T$ (see Figure 1). Moreover, by Proposition 13(ii) $C_T \backslash \{t\}$ is an optimal identifying code in $T - \{t\}$ for any $t \in T^2$. Notice that $C'_T = (C' \backslash V(F)) \cup C_T$ is an identifying code in G' and $|C'| = |C'_T|$. Notice that Claim B.2 holds also for C'_T. Assume first that $z \in A$ is adjacent to $t \in T^2$. If $C = \{z\} \cup C'_T \backslash \{t\}$ is an identifying code in G, then the claim follows from Claim B.2. If C is not an identifying code, then we will show next that vertex t must be adjacent to a vertex $u_1 \in N_u$ and $v_1 \in N_v$ and t is the only vertex in C'_T separating u_1 from another vertex $u_2 \in N_u$, and similarly, vertex $v_2 \in N_v$ from $v_1 \in N_v$. Indeed, since $u, v \in C$, all vertices of A are separated from the vertices in F. Moreover, since $C_T \backslash \{t\}$ is an identifying code in $F - \{t\}$, set $C \cap V(F)$ is an identifying code in $F - \{t\}$. Furthermore, vertex t is the only vertex that has exactly z in its code neighborhood. Consequently, z, u, and v are identified by Lemma 3. Therefore, the only vertices that might not be separated belong to A. Since C'_T separated all the vertices in A, we require t to separate some vertices that are not separated by z. Moreover, t can have at most two neighbors in A. If t is adjacent to only one vertex in A, then that vertex is z and $z \in C$ separates itself from other vertices in A. Thus, t is adjacent to two vertices in A. If both of these vertices are in N_u (or N_v), then u and v separate them from other vertices in A. Moreover, $z \in C$ separates itself from other neighbors of t. Hence, t is adjacent to a vertex $u_1 \in N_u$ and $v_1 \in N_v$. Since $u, v \in C$, the only vertex u_1 that can have the same code neighborhood with respect to C is vertex $u_2 \in N_u$. A similar statement holds for v_1 and $v_2 \in N_v$. Assume first that $C \backslash \{z\}$ separates v_1 and v_2 in G. In this case $(C \cup \{u_1\}) \backslash \{z\}$ is an identifying code in G. A similar argument holds for $C \backslash \{z\}$ separating u_1 and u_2 in G. Hence, we may assume that $C \backslash \{z\}$ does not separate pairs u_1, u_2, and v_1, v_2. Thus, we require t to separate these two pairs in C'_T, as claimed.

Since v_2 and u_2 are dominated by a vertex in $C'_T \backslash \{u, v\}$ (Claim B.1), there is a code vertex $w_u \in (C'_T \backslash \{u, v\}) \cap N(u_1) \cap N(u_2)$ and a code vertex $w_v \in (C'_T \backslash \{u, v\}) \cap N(v_1) \cap N(v_2)$. In this case, we may consider graph G' and modify the identifying code C'_T into $C'' = \{u_1, v_1\} \cup C'_T \backslash \{u, v\}$, which is an identifying code in G'. This is a contradiction because C'' is an optimal identifying code of G' such that u, v are identified by C'' in G, contradicting the hypothesis of Claim B.

Assume next that $F = T_2$. Denote its unique support vertex of degree 3 by s. Let the leaves adjacent to s be denoted by l_1 and l_2, and the third one be l_3. Finally, let f_1, f_2, f_3 be the three vertices on the path from s to l_3 (in that order). Assume first that there are no edges between A and f_2, f_3, or l_3. Now consider the graph $G_f = G - \{f_2, f_3, l_3\}$. Assume first that $G_f \notin \mathcal{F}_3$ and let C_f be an optimal identifying code in G_f. In this case, if $f_1 \in C_f$, then $C_f \cup \{f_2, f_3\}$ is an identifying code of claimed cardinality in G. If $f_1 \notin C_f$, then $C_f \cup \{f_2, l_3\}$ is an identifying code in G of claimed cardinality. Assume then that $G_f \in \mathcal{F}_3$. Since there is a vertex of degree 3 in G_f, we have $G_f \in \mathcal{T}_3$. However, since there are no edges from A to $\{f_2, f_3, l_3\}$, graph G is a tree, a contradiction. Hence, we may assume that there is an edge from A to $\{f_2, f_3, l_3\}$.

Let us next consider graph $G_s = G - \{l_1, l_2, s\}$ and its optimal identifying code by C_s. Notice that G_s is connected. Assume first that $G_s \notin \mathcal{F}_3$. If one of the three vertices in $\{l_1, l_2, s\}$ is dominated by a vertex in C_s in G, then the set C_s together with two adjacent vertices from $\{l_1, l_2, s\}$ such that at least one of them is dominated by a vertex from C_s is an identifying code of claimed cardinality. If no vertex in $\{l_1, l_2, s\}$ is dominated by a vertex from C_s, then we consider $C_s \cup \{l_1, l_2\}$. This is an identifying code, since s is separated from all other vertices. Indeed, if some vertex in $V(G) \backslash V(F)$ is adjacent to l_1 and l_2, then it is also adjacent to some vertex in $C_s \cap (V(G) \backslash V(F))$, while vertex s cannot be (since it is not dominated by C_s). Hence, we may assume that $G_s \in \mathcal{F}_3$ and more specifically $G_s \in \mathcal{T}_3$ since there is a vertex of degree 3. Hence, there is a single edge between $\{f_1, f_2, f_3, l_3\}$ and A. Furthermore, if there is an edge from A to f_2 or f_3, then $G_s \notin \mathcal{T}_3$, and if the edge is from A to f_1 or l_3, then G_s has to be T_2. However, there are at least five vertices in A_{uv} and hence $G_s \neq T_2$. Thus $G_s \notin \mathcal{F}_3$.

Let us finally consider the case where $F = T_3$. Denote by s_1 and s_2 its two support vertices of degree 3, and by f_2 the support vertex of degree 2. Let leaves l_1 and l_2 be adjacent to s_1, and leaves l_3 and l_4 be adjacent to s_2. Furthermore, let the leaf adjacent to f_2 be l_5, and the other vertex adjacent to f_2 be f_1. Further denote the vertex of degree 3 adjacent to f_1 by

f_s. Assume first that there are no edges from A to $\{f_1, f_2, l_5\}$. Thus, $G_f = G - \{f_1, f_2, l_5\}$ is a connected graph, and let C_f be an optimal identifying code in G_f. Notice that if $G_f \in \mathcal{F}_3$, then $G_f \in \mathcal{T}_3$, and then G is a tree, a contradiction. Thus, $G_f \notin \mathcal{F}_3$ and C_f contains at most two-thirds of the vertices of G_f. In this case, if $f_s \in C_f$, then the set $C = C_f \cup \{f_1, f_2\}$ is an identifying code in G, and if $f_s \notin C_f$, then the set $C = C_f \cup \{f_1, l_5\}$ is an identifying code in G. Moreover, both of these sets contain at most two-thirds of the vertices in G, and we are done. Hence, we may assume from now on that there is an edge from A to $\{f_1, f_2, l_5\}$. Assume then that there are no edges from A to $\{s_1, l_1, l_2\}$. Now, graph $G_s = G - \{s_1, l_1, l_2\}$ is connected, has a cycle and hence, by induction, also an optimal identifying code C_s satisfying the two-thirds upper bound. Moreover, set $C_s \cup \{l_1, l_2\}$ is an identifying code in G. Hence, there is an edge from A to $\{s_1, l_1, l_2\}$. By symmetry, a similar argument holds for $\{s_2, l_3, l_4\}$, so there is an edge from A to $\{s_2, l_3, l_4\}$. Hence, we can assume next that there is an edge from A to sets $\{s_1, l_1, l_2\}, \{s_2, l_3, l_4\}$, and $\{l_5, f_1, f_2\}$. Let us consider graph $G'_f = G - \{s_2, l_3, l_4\}$ together with an optimal identifying code C'_f of G'_f. Notice that G'_f is connected, has a cycle, and maximum degree 3. Hence, $G'_f \notin \mathcal{F}_3$ and by induction, $|C'_f| \leq \frac{2}{3}n(G'_f)$. Consider set $C'_f \cup \{l_3, l_4\}$ if no vertex in C'_f dominates a vertex in $\{l_3, l_4\}$, and otherwise, set C'_f together with s_2 and a vertex in $\{l_3, l_4\}$ that is dominated by C'_f. Notice that in each case, the corresponding set is an identifying code of claimed cardinality for G. □

By the above claims, we may assume that if $\Delta = 3$, then for any component F in G_{uv} we have $\gamma^{\mathrm{ID}}(F) \leq \frac{2}{3}n(F)$.

Claim B.8. Let $\Delta \geq 4$. If T is a Δ-star component in G_{uv}, then $\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$.

Proof. Let T be a Δ-star component in G_{uv}. Let $V(T) = \{w, w_1, ..., w_\Delta\}$ and w be the center vertex of T. Observe that w is adjacent only to vertices in T. Let w_1 be adjacent to a vertex $z \in A$. Let $T' = T - \{w_1\}$ and let $G_T = G - T'$. Observe that G_T is a connected graph, and at least one of u or v has degree at least 3 in G_T. Hence, there are at least six vertices in G_T. Let us assume first that $G_T \in \mathcal{F}_3$; then, $G_T \in \mathcal{T}_3$. In this case we have $\gamma^{\mathrm{ID}}(G_T) \leq \frac{3}{4}n(G_T)$ and $\gamma^{\mathrm{ID}}(T') \leq \left(\frac{\Delta-1}{\Delta}\right)n(T')$. Moreover, there are at least three leaves in T'. Let C_T be an optimal identifying code in G_T. Observe that $C = C_T \cup \{w, w_2, ..., w_{\Delta-1}\}$ is an identifying code in G of cardinality at most $|C| \leq \frac{3}{4}n(G_T) + \left(\frac{\Delta-1}{\Delta}\right)n(T') \leq \left(\frac{\Delta-1}{\Delta}\right)n$. Furthermore, if $G_T \notin \mathcal{F}_3$, since also $G_T \notin \mathcal{F}_{\Delta'}$, where $\Delta' = \Delta(G_T)$, there exists an identifying code C_T of G_T such that $|C_T| = \gamma^{\mathrm{ID}}(G_T) \leq \left(\frac{\Delta'-1}{\Delta'}\right)n(G_T) \leq \left(\frac{\Delta-1}{\Delta}\right)n(G_T)$ and hence, we may again consider $C = C_T \cup \{w, w_2, ..., w_{\Delta-1}\}$ as our identifying code for G and $|C| \leq \left(\frac{\Delta-1}{\Delta}\right)n$ which completes the proof of the claim. □

Let us denote the set of isolated vertices in G_{uv} by $\mathcal{I}$ and let us further denote $G_{AI} = G[A_{uv} \cup \mathcal{I}]$.

Claim B.9. Set A_{uv} is an identifying code in G_{AI}.

Proof. Observe that $\mathcal{I} \subseteq C'$ (otherwise some vertex of $\mathcal{I}$ would not be dominated by C' in G'). If set A_{uv} is not an identifying code in G_{AI}, then there are open twins w_1, w_2 in $\mathcal{I}$. Let $z \in N(w_1)$. Notice that $z \in A$ and hence, $z \notin C'$. Since vertices in $\{w_1, w_2\}$ are open twins, we may consider set $C = \{z\} \cup C' \setminus \{w_1\}$. Observe that C is an identifying code of G since w_2 separates all the same vertices as vertex w_1. Hence, we obtain a contradiction from Claim B.2. Thus, there are no twins in $\mathcal{I}$, and set A_{uv} is an identifying code in G_{AI}. □

Let us next consider the minimum cardinality of an identifying code in G_{AI}. Assume that $|N_u| \geq |N_v|$. Since $|N_v| \leq |N_u| \leq \Delta - 1$, we have $|A_{uv}| \leq 2\Delta$. Thus, if $|A_{uv}| > \left(\frac{\Delta-1}{\Delta}\right)n(G_{AI}) = \left(\frac{\Delta-1}{\Delta}\right)(|A_{uv}| + |\mathcal{I}|)$, then $|A_{uv}| > (\Delta - 1)|\mathcal{I}|$. Hence, this implies that $|\mathcal{I}| \leq 2$. By Claim B.9, the set $\mathcal{I}$ is A-identifiable. Hence, by Lemma 4, if $|\mathcal{I}| \leq 2$, then we require at most two vertices from A to separate and dominate the vertices in $\mathcal{I}$. Moreover, we have $|A| \geq 3$ due to the assumption $\deg(u) + \deg(v) \geq 5$. Notice that set $A_{uv} \setminus \{z_1, z_2\}$ remains an identifying code of G_{AI} when $z_1 \in N_u$ and $z_2 \in N_v$ and they are not required for dominating or separating vertices in $\mathcal{I}$. Indeed, these vertices will be the only ones that are adjacent to exactly u or v in the identifying code.

Hence, if $|\mathcal{I}| = 2$ and $|A_{uv}| = 2\Delta - 2 - a \leq 2\Delta - 2$ for some $a \geq 0$, then A_{uv} is an identifying code of G_{AI} that satisfies the conjectured bound for G_{AI}. Indeed,

$$|A_{uv}| = 2\Delta - 2 - a = \left(\frac{\Delta - 1}{\Delta}\right)((2\Delta - 2) + 2) - a \leq \left(\frac{\Delta - 1}{\Delta}\right)n(G_{AI}).$$

If $|\mathcal{I}| = 2$ and $|A_{uv}| \geq 2\Delta - 1$, then we may remove one well-chosen vertex from A since we require at most two vertices of A to identify vertices in $\mathcal{I}$ while $|A| \geq 3$, and the resulting set remains an identifying code (as described above) while satisfying the conjectured bound in G_{AI}. Indeed, $2\Delta - 2 < \left(\frac{\Delta-1}{\Delta}\right)((2\Delta - 2) + 3)$ and $2\Delta - 1 < \left(\frac{\Delta-1}{\Delta}\right)((2\Delta - 1) + 3)$. If $|\mathcal{I}| = 1$ and $|A_{uv}| \leq 2\Delta - 1$, then we may remove one well-chosen vertex from N_u. If $|\mathcal{I}| = 1$ and $|A_{uv}| = 2\Delta$, then we may remove one well-chosen vertex from N_u and N_v. Finally, if $\mathcal{I} = \emptyset$, then we may just remove any single vertex from both N_u and N_v to obtain an identifying code satisfying the conjectured bound.

We are now ready to complete the proof of Claim B. Observe that we have obtained above an identifying code, which contains u and v, satisfying the conjectured bound in G_{AI}. Denote by $\mathcal{F}_{uv}$ the components in G_{uv}. Furthermore, observe that if $F \in \mathcal{F}_{uv}$ has size at least 2 in G_{uv}, then F is also a component in $G - G_{AI}$. Furthermore, by Claims B.3, B.4, B.5, B.6, B.7 and B.8, every component F of $\mathcal{F}_{uv}$ admits an identifying code, and we have $F \notin \mathcal{F}_\Delta$, and so, by the induction hypothesis, $\gamma^{\mathrm{ID}}(F) \leq \left(\frac{\Delta-1}{\Delta}\right)n(F)$. Let us denote by C_F an optimal identifying code in component F and by C_{AI} an identifying code in G_{AI} which contains u and v, and has size at most $\left(\frac{\Delta-1}{\Delta}\right)n(G_{AI})$, as constructed above. Since there are no edges between components of $\mathcal{F}_{uv}$ and each edge from such a component F is to a vertex in A, which are separated from vertices in F by u and v, we have an identifying code

$$C = C_{AI} \cup \bigcup_{F \in \mathcal{F}_{uv}, n(F) \geq 3} C_F$$

and

$$\begin{aligned} |C| &\leq \left(\frac{\Delta - 1}{\Delta}\right)n(G_{AI}) + \sum_{F \in \mathcal{F}_{uv}, n(F) \geq 3} \left(\frac{\Delta - 1}{\Delta}\right)n(F) \\ &= \left(\frac{\Delta - 1}{\Delta}\right)n. \end{aligned}$$

This completes the proof of Claim B. □

By Claim B, we may assume that there exists an optimal identifying code C' in G' such that $\{u', v\}$ or $\{u, v'\}$ is not identified by C' in G, where u' is some neighbor of u different from v and v' is some neighbor of v different from u. Necessarily, in the case where $\{u', v\}$ is not identified, $u \in C'$ and $v \notin C'$, and if $X = N_{G'}(v) \cap C'$, then $X \neq \emptyset$ (in order for C' to dominate v in G') and $N_{G'}(u') \cap C' = X \cup \{u\}$. (We have the symmetric facts for the case where $\{u, v'\}$ is not identified.) Let v' be an arbitrary vertex in X, and let Q_{uv} be the cycle $uu'v'vu$ in G. Recall that $G_{uv} = G - N_G[u] - N_G[v]$. Further recall that $N_u = N_G(u)\backslash\{v\}$, $N_v = N_G(v)\backslash\{u\}$, and $A = N_u \cup N_v$. See Figure 4 for an illustration.

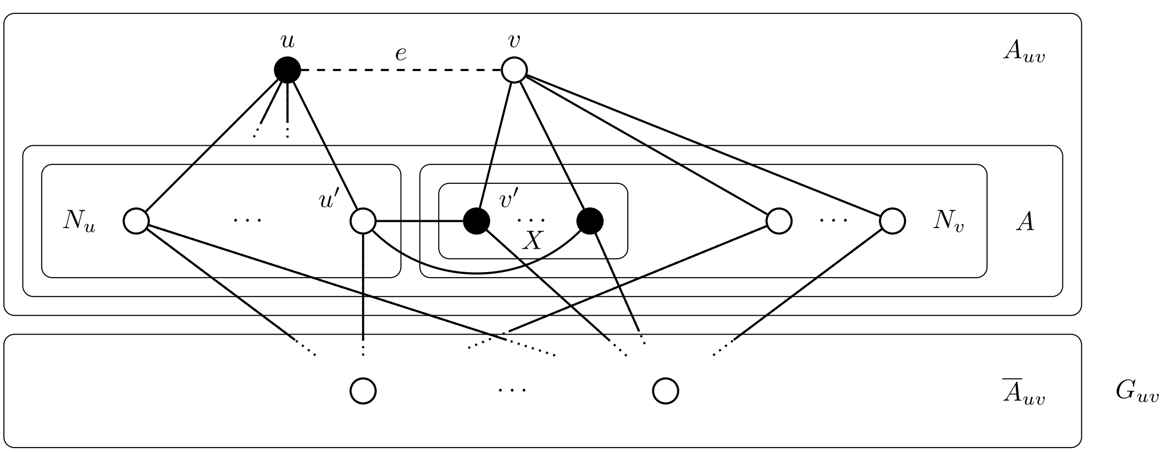


FIGURE 4 | The setting of the proof of Theorem 2 after applying Claim B. Here, $\{u', v\}$ is not identified by C' (black vertices).

Claim C. If F is a component in G_{uv} and $F \in \mathcal{F}_\Delta$, then

$$\gamma^{\mathrm{ID}}(G) \le \left(\frac{\Delta - 1}{\Delta}\right)n.$$

Proof. Suppose that F is a component in G_{uv} and $F \in \mathcal{F}_\Delta$. Suppose, firstly, that $F = K_{1,\Delta}$, where $\Delta \ge 3$. Let $V(F) = \{x, x_1, x_2, \ldots, x_\Delta\}$ where x is the central vertex of F with leaf neighbors $x_1, x_2, \ldots, x_\Delta$. Since G is connected, there is an edge f joining a vertex $z \in A$ and a vertex in $V(F)$. Renaming vertices if necessary, we may assume that $f = zx_\Delta$. Let $G^* = G - (V(F)\backslash\{x_\Delta\})$. Let G^* have order n^*, and so $n^* = n - \Delta$. Further, let $\Delta(G^*) = \Delta^*$. Since at least one of u and v has degree at least 3 in G, and since the degrees of u and v are the same in G and in G^*, we note that $\Delta^* \ge 3$. Thus, G^* is a connected triangle-free graph and $\Delta \ge \Delta^* \ge 3$. Moreover, G^* contains the cycle Q_{uv} and has order $n^* \ge 6$. These properties of G^* imply that $G^* \notin \mathcal{F}_{\Delta^*}$. Applying the inductive hypothesis to G^*, we have

$$\gamma^{\mathrm{ID}}(G^*) \le \left(\frac{\Delta^* - 1}{\Delta^*}\right)n^* \le \left(\frac{\Delta - 1}{\Delta}\right)n^* = \left(\frac{\Delta - 1}{\Delta}\right)n - (\Delta - 1).$$

Let C^* be an optimal identifying code in G^*, and so $|C^*| = \gamma^{\mathrm{ID}}(G^*)$. The code $C^* \cup \{x, x_1, \ldots, x_{\Delta-2}\}$ is an identifying code in G if $\Delta \ge 4$, or $\Delta = 3$ and if $N_G(x_1) \cap C^* \ne \emptyset$. Similarly, if $N_G(x_2) \cap C^* \ne \emptyset$ and $\Delta = 3$, then $C^* \cup \{x, x_2\}$ is an identifying code in G. Finally, if $\Delta = 3$ and $N_G(x_1) \cap C^* = N_G(x_2) \cap C^* = \emptyset$, then $C^* \cup \{x_1, x_2\}$ is an identifying code in G. Hence, we have

$$\begin{aligned}\gamma^{\mathrm{ID}}(G) &\le |C^*| + \Delta - 1\\ &= \gamma^{\mathrm{ID}}(G^*) + \Delta - 1\\ &\le \left(\left(\frac{\Delta - 1}{\Delta}\right)n - \Delta + 1\right) + \Delta - 1\\ &= \left(\frac{\Delta - 1}{\Delta}\right)n,\end{aligned}$$

which yields the desired upper bound in the case when $F = K_{1,\Delta}$, where $\Delta \ge 3$. Hence, we may assume that $\Delta = 3$ and $F \in \mathcal{F}_\Delta\backslash\{K_{1,3}\}$. Thus, $F \in \{P_4, C_4, C_7\} \cup (\mathcal{T}_3\backslash\{K_{1,3}\})$.

We next distinguish two cases.

Case 1. $F \ne P_4$. In this case, since the graph G is connected and $\Delta = 3$, one can check that there exists an induced path $P : x_1x_2x_3$ in F such that $G'' = G - V(P)$ is a connected graph. Let $n'' = n(G'')$, and so $n'' = n - 3$. Let $\Delta(G'') = \Delta''$. We note that G'' is a connected triangle-free graph and $\Delta \ge \Delta'' \ge 3$. Thus, $\Delta'' = 3$. Moreover, G'' contains the cycle Q_{uv} and has order $n'' \ge 6$. In particular, $G'' \notin \mathcal{F}_3$. Applying the inductive hypothesis to G'', we have

$$\gamma^{\mathrm{ID}}(G'') \le \frac{2}{3}n'' = \frac{2}{3}n - 2.$$

Let C'' be an optimal identifying code in G'', and so, $|C''| = \gamma^{\mathrm{ID}}(G'')$. If $N_G[x_1] \cap C'' = N_G[x_3] \cap C'' = \emptyset$, then we consider $C'' \cup \{x_1, x_3\}$ as a potential identifying code in G. It suffices to check that x_1, x_2, x_3 are now identified by the new code. Note that with this code, each of x_1 and x_3 is only dominated by itself, and no other vertex is dominated only by x_1 or only by x_3. If there exists some vertex y not separated from x_2 by $C'' \cup \{x_1, x_3\}$, then y must be adjacent to both x_1, x_3. Since G is triangle-free, y is not adjacent to x_2. Thus, $y \notin C''$ (otherwise y, x_2 would be separated). Thus, y is dominated by its third neighbor z, which is in C'', and hence, z is adjacent to x_2. Since $\Delta'' = \Delta = 3$, vertex y has no other neighbors in G. Thus, $y \notin A$, as in that case, y should be adjacent to u or v, but $z \notin \{u, v\}$ since z is adjacent to x_2. Hence, $y \in F$ and F is C_4. Thus, if $F \ne C_4$, we are done. If $F = C_4$, we show that $C'' \cup \{x_1, x_2\}$ is an identifying code in G. Indeed, z, x_1, x_2 are code vertices inducing a P_3, and we may apply Lemma 3 on them. Furthermore, x_3 is separated from all other neighbors of x_2 since they are in the identifying code. Furthermore, y is adjacent to z and x_1. If this is true also for some vertex w, then $w \in A$ and z is adjacent to w, y, x_2, which is not possible since $\Delta = 3$ and z is adjacent to u or v. Hence, $C'' \cup \{x_1, x_2\}$ is an identifying code in G.

Assume next that for an index $j \in \{1, 3\}$, say $j = 1$, we have $N_G[x_1] \cap C'' \neq \emptyset$. Consider the set $C'' \cup \{x_1, x_2\}$. If it is an identifying code of G, then we are done. Thus, assume it is not the case. Vertices x_1, x_2, and the neighbor of x_1 in C'' induce a P_3 and thus by Lemma 3, they are uniquely identified. Hence, x_3 is not separated from some other vertex w. Thus, w is a neighbor of x_2, and $w \notin C''$. As w is dominated by C'', say by w', vertices w and x_3 have $w' \in C''$ as a second common neighbor. Notice that in this case, vertices x_3, x_2, w, w' form a 4-cycle and $N(x_2) = \{x_1, x_3, w\}$. However, now set $C'' \cup \{x_2, x_3\}$ is an identifying code in G. Indeed, x_2, x_3, and w' are separated from other vertices by Lemma 3. Assume first that w' is adjacent to x_1. Then, since G'' is connected, $n'' \geq 6$ and $\Delta = 3$, vertex w has a neighbor other than w' and x_2, say w''. Since w and w' are separated by C'', vertex w'' belongs to C''. Since x_3 and w were not separated by $C'' \cup \{x_1, x_2\}, w''$ is adjacent to x_3. Again, since $n'' \geq 6$, there is no edge between x_1 and w'' (otherwise all considered vertices have degree 3 and $n'' = 3$). Hence, x_1 is identified by $C'' \cup \{x_2, x_3\}$. On the other hand, if w' is not adjacent to x_1, then vertex w' separates x_1 from the two other neighbors of x_2. In both cases, $C'' \cup \{x_2, x_3\}$ is an identifying code of G. Therefore,

$$\begin{aligned}\gamma^{\mathrm{ID}}(G) &\leq |C''| + 2 \\ &= \gamma^{\mathrm{ID}}(G'') + 2 \\ &\leq \left(\frac{2}{3}n - 2\right) + 2 \\ &= \frac{2}{3}n,\end{aligned}$$

which yields the desired upper bound.

Case 2. $F = P_4$. Let F be the path $x_1x_2x_3x_4$. If x_4 is adjacent to a vertex in the set A, then, as before, there exists an induced path $P : x_1x_2x_3$ in F such that $G'' = G - V(P)$ is a connected graph of order at least 6 not in $\mathcal{F}_3$. Then, the same arguments as in Case 1 apply, and we obtain an identifying code of the desired size.

Hence, we may assume that x_4 has degree 1 in G (with x_3 as its unique neighbor in G). Analogously, we may assume that x_1 has degree 1 in G (with x_2 as its unique neighbor in G). Since G is connected, we may assume, renaming vertices if necessary, that x_2 is adjacent to a vertex in the set A.

We now consider the graph $G_F = G - \{x_3, x_4\}$. Let $n_F = n(G_F)$, and so $n_F = n - 2$. Let $\Delta(G_F) = \Delta_F$. We note that G_F is a connected triangle-free graph and $\Delta \geq \Delta_F \geq 3$. Thus, $\Delta_F = 3$. Moreover, G_F contains the cycle Q_{uv} and has order $n_F \geq 7$. In particular, $G_F \notin \mathcal{F}_3$. Applying the inductive hypothesis to G_F, we have

$$\gamma^{\mathrm{ID}}(G_F) \leq \frac{2}{3}n_F = \frac{2}{3}(n - 2).$$

Let C_F be an optimal identifying code in G_F, and so $|C_F| = \gamma^{\mathrm{ID}}(G_F)$. If $x_2 \in C_F$, then in order to identify the vertices x_1 and x_2, the code C_F contains at least one neighbor of x_2 in G_F. In this case, we let $C = C_F \cup \{x_3\}$. If $x_2 \notin C_F$, then $x_1 \in C_F$ and at least one neighbor of x_2 in the set A belongs to C_F. In this case, we let $C = (C_F \setminus \{x_1\}) \cup \{x_2, x_3\}$. In both cases (using Lemma 3 in the second case), C is an identifying code of G and $|C| = |C_F| + 1$, implying that

$$\begin{aligned}\gamma^{\mathrm{ID}}(G) \leq |C| = \gamma^{\mathrm{ID}}(G_F) + 1 \\ &\leq \frac{2}{3}(n - 2) + 1 \\ &< \frac{2}{3}n.\end{aligned}$$

This completes the proof of Claim C. □

By Claim C, we will from now on assume that if F is a component in G_{uv} of order at least 3, then $F \notin \mathcal{F}_\Delta$.

Let B be the set of all vertices that belong to a P_1-component or to a P_2-component in G_{uv}. Furthermore, let us denote $G^* = G[A \cup B \cup \{u, v\}]$. Note that it is possible that $G^* = G$ and hence we do not apply the induction hypothesis directly to G^*. Recall that $A = N_u \cup N_v$ and let $|A| = a$. Further, let $|B| = b$. Thus, $V(G^*) = A \cup B \cup \{u, v\}$ and $n^* = a + b + 2$. Next, in Claim D, we will show that G^* admits an identifying code of cardinality at most $\left(\frac{\Delta-1}{\Delta}\right)n^*$ that contains both vertices u and v. Notice that $G^* \notin \mathcal{F}_\Delta$ since G^* contains a 4-cycle and a vertex of degree at least 3.

Claim D. Graph G^* admits an identifying code containing vertices u and v of cardinality at most

$$\left(\frac{\Delta - 1}{\Delta}\right)n^*.$$

Proof. Since G is a connected graph, every vertex in a P_1-component of $G[B]$ is adjacent in G^* to at least one vertex in A. Moreover, by the connectivity of G^*, at least one vertex from every P_2-component of $G[B]$ is adjacent in G^* to at least one vertex in A. However, it is possible that a P_2-component of $G[B]$ contains exactly one vertex that has degree 1 in G^* (and is therefore not adjacent in G^* to a vertex from the set A). Let B_A be the set of all vertices in B that have at least one neighbor in A in the graph G^*, and let $B_L = B \backslash B_A$. Thus, each vertex in B_L is a vertex of degree 1 in G^* that belongs to a P_2-component of $G[B]$ and is adjacent to no vertex of A in G^*. Possibly, $B_L = \varnothing$.

Let $B = (B_1, ..., B_k)$ be a partition of the set B such that the following properties hold.

a. All vertices in the set $B_i \cap B_A$ have the same neighborhood in the set A for all $i \in [k]$, that is, if $x, y \in B_i \cap B_A$, then $N_{G^*}(x) \cap A = N_{G^*}(y) \cap A \neq \varnothing$.
b. Each vertex in the set $B_i \cap B_L$ has its unique neighbor (in G^*) in the set $B_i \cap B_A$ for all $i \in [k]$.
c. Vertices in distinct sets $B_i \cap B_A$ and $B_j \cap B_A$ have different neighborhoods in A, that is, if $x \in B_i \cap B_A$ and $y \in B_j \cap B_A$ where $1 \leq i < j \leq k$, then $N_{G^*}(x) \cap A \neq N_{G^*}(y) \cap A$.

An illustration is given in Figure 5. We note that $|B_i| = |B_i \cap B_A| + |B_i \cap B_L|$ for all $i \in [k]$. Moreover, $|B_i \cap B_L| \leq |B_i \cap B_A|$ and $|B_i \cap B_A| \geq 1$ for all $i \in [k]$. If $|B_i \cap B_L| = |B_i \cap B_A|$, then we observe that the subgraph $G[B_i]$ induced by the set B_i consists of vertex-disjoint copies of K_2 (and each such copy of K_2 contains exactly one vertex of degree 1 in G). Since each vertex in A is adjacent to either the vertex u or v, each vertex in A has at most $\Delta - 1$ neighbors in the set B, implying that $|B_i \cap B_A| \leq \Delta - 1$ for all $i \in [k]$. Let

$$|B_i| = b_i$$

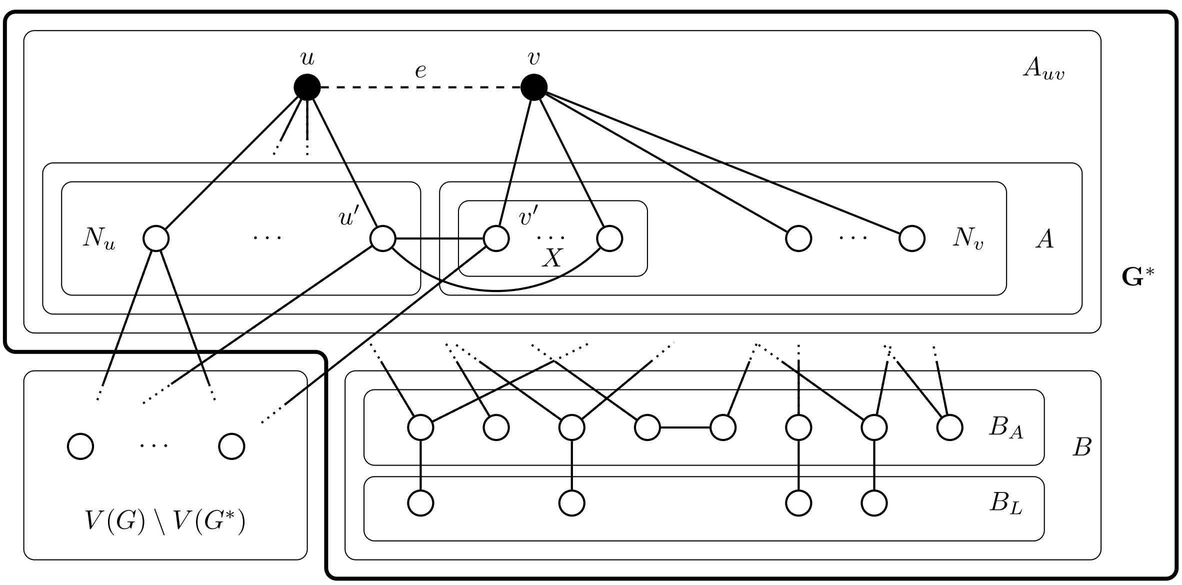


FIGURE 5 | The setting of Claim D in the proof of Theorem 2. The aim is to construct a small identifying code containing both u and v.

for all $i \in [k]$. Thus, if $B_i \cap B_L = \emptyset$, then $b_i = |B_i \cap B_A| \leq \Delta - 1$, while if $B_i \cap B_L \neq \emptyset$, then $b_i \leq 2|B_i \cap B_A| \leq 2(\Delta - 1)$ for all $i \in [k]$. Recall that

$$b = |B| = \sum_{i=1}^{k} b_i.$$

Since G is triangle-free, we note that $B_i \cap B_A$ is an independent set in G^* for all $i \in [k]$. Thus, if two vertices w and z belong to the same P_2-component in $G[B]$, then either $w \in B_i$ and $z \in B_j$ where $1 \leq i, j \leq k$ and $i \neq j$, or one of w and z belongs to $B_i \cap B_A$ and the other to $B_i \cap B_L$ for some $i \in [k]$. For $i \in [k]$, we now define the set B_i^* as follows. If $|B_i \cap B_L| = |B_i \cap B_A|$, then we define

$$B_i^* = B_i \cap B_L,$$

while if $|B_i \cap B_L| < |B_i \cap B_A|$, then we let w_i be an arbitrary vertex in the set $B_i \cap B_A$ that is isolated in the subgraph $G[B_i]$ induced by B_i (and so, w_i has no neighbor in B_i), and we define

$$B_i^* = (B_i \cap B_L) \cup \{w_i\}.$$

We now define

$$B^* = \bigcup_{i=1}^{k} B_i^*.$$

Moreover, let $|B_i^*| = b_i^*$ for all $i \in [k]$ and

$$b^* = \sum_{i=1}^{k} b_i^*.$$

An illustration of the set B^* is given in Figure 6.

Claim D.1. The following hold.

a. $b_i^* \geq \frac{b_i}{\Delta - 1}$ for all $i \in [k]$.
b. $b \leq (\Delta - 1)b^*$.

Proof. By our earlier observations, if $B_i \cap B_L = \emptyset$, then $b_i \leq \Delta - 1$ where $i \in [k]$. Moreover, $b_i \leq 2(\Delta - 1)$ for all $i \in [k]$. If $b_i \leq \Delta - 1$ for some $i \in [k]$, then

$$\frac{b_i - b_i^*}{b_i} \leq \frac{b_i - 1}{b_i} \leq \frac{\Delta - 2}{\Delta - 1}.$$

We note that if $b_i \geq \Delta$ for some $i \in [k]$, then $b_i^* \geq 2$, and so in this case

$$\frac{b_i - b_i^*}{b_i} \leq \frac{b_i - 2}{b_i} \leq \frac{2(\Delta - 1) - 2}{2(\Delta - 1)} = \frac{\Delta - 2}{\Delta - 1}.$$

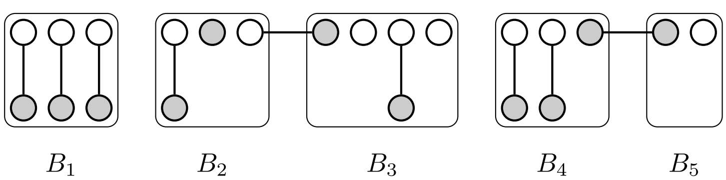


FIGURE 6 | The construction of the set B^* in the proof of Claim D. The vertices in B^* are shaded.

Thus in both cases,

$$\frac{b_i - b_i^*}{b_i} \leq \frac{\Delta - 2}{\Delta - 1}.$$

Rearranging terms in the above inequality yields the inequality

$$b_i^* \geq \frac{b_i}{\Delta - 1}$$

for all $i \in [k]$. This proves property (a) in the statement of the claim. Hence,

$$b = \sum_{i=1}^{k} b_i \leq (\Delta - 1) \sum_{i=1}^{k} b_i^* = (\Delta - 1)b^*,$$

and so property (b) in the statement of the claim holds. □

Let z_i be an arbitrary vertex in $B_i \cap B_A$, and so $z_i \in B_i$ and z_i has a neighbor in the set A for all $i \in [k]$. Let $Z = \{z_1, z_2, \ldots, z_k\}$. By construction, the set A identifies the set Z since every pair of vertices in Z have distinct neighborhoods in A. Equivalently, the set Z is (Z, A)-identifiable by the set A. Let A^* be a subset of A of minimum cardinality that A-identifies Z. By Lemma 4, $1 \leq |A^*| \leq k$. Since $b_i^* \geq 1$ for all $i \in [k]$, we note that $b^* \geq k$, and so $|A^*| \leq b^*$. Recall that $Q_{uv} : uu'v'vu$ is a 4-cycle in G and hence in G^*, where u' is a neighbor of u in G different from v, and v' is a neighbor of v in G different from u.

Claim D.2. If $A = A^*$, then there exists an identifying code in G^* containing vertices u and v with cardinality at most

$$\left(\frac{\Delta - 1}{\Delta}\right)n^*.$$

Proof. Suppose that $A = A^*$. By the minimality of the set A^*, every vertex in A^* has a neighbor in B. Recall that $V(G^*) = A \cup B \cup \{u, v\}$ and $n^* = a + b + 2$. We now let

$$C^* = A \cup \{u, v\} \cup (B \backslash B^*).$$

The set C^* is an identifying code of G^*. Indeed, C^* is a dominating set which is connected in G^*. Thus, by Lemma 3, it separates vertices in C^*. Furthermore, each vertex in B^* has a unique neighborhood in $A \cup (B \backslash B^*)$.

Claim D.2(i). If $b^* \geq k + 2$, then $|C^*| \leq \left(\frac{\Delta - 1}{\Delta}\right)n^*$.

Proof. Suppose that $b^* \geq k + 2$. Thus, $a = |A^*| \leq k \leq b^* - 2$. By Claim D.1(b) and our earlier observations, we have

$$\begin{aligned}
|C^*| &= a + 2 + b - b^* \\
&= \left(\frac{\Delta - 1}{\Delta}\right)(a + 2 + b) + \frac{1}{\Delta}(a + 2 + b - \Delta b^*) \\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* + \frac{1}{\Delta}((b^* - 2) + 2 + (\Delta - 1)b^* - \Delta b^*) \\
&= \left(\frac{\Delta - 1}{\Delta}\right)n^*,
\end{aligned}$$

yielding the desired upper bound. □

By Claim D.2, we may assume that $b^* \le k+1$, implying that $b^* \in \{k, k+1\}$. With this assumption, $b_i^* = 1$ for all $i \in [k]$, except for possibly exactly one value of i satisfying $b_i^* = 2$.

Recall that every vertex in A^* has a neighbor in B. Thus, since $A = A^*$, we note in particular that both vertices u' and v' have a neighbor in B. Since G^* is triangle-free, u' and v' do not have a common neighbor. Renaming the partitions in $B = (B_1, \dots, B_k)$ if necessary, we may assume that u' is adjacent to a vertex in B_1 and v' is adjacent to a vertex in B_2. Since u' is adjacent to both u and v', it has at most $\Delta - 2$ additional neighbors, implying that $|B_1 \cap B_A| \le \Delta - 2$. Analogously, $|B_2 \cap B_A| \le \Delta - 2$. Therefore,

$$1 \le b_1^* \le b_1 \le 2(\Delta - 2) \quad \text{and} \quad 1 \le b_2^* \le b_2 \le 2(\Delta - 2).$$

Let

$$B_{3,k} = B \backslash (B_1 \cup B_2) \quad \text{and} \quad B_{3,k}^* = B^* \backslash \left(B_1^* \cup B_2^*\right).$$

Further, let $b_{3,k} = |B_{3,k}|$ and $b_{3,k}^* = |B_{3,k}^*|$, and so $b = b_1 + b_2 + b_{3,k}$ and $b^* = b_1^* + b_2^* + b_{3,k}^*$. We note that

$$b_{3,k} = \sum_{i=3}^{k} b_i \quad \text{and} \quad b_{3,k}^* = \sum_{i=3}^{k} b_i^* \ge \sum_{i=3}^{k} 1 = k - 2.$$

We now define a partition (V_1, V_2, V_3) of $V(G^*)$ as follows. We let

$$\begin{aligned} V_1 &= B_1 \cup \{u, u'\}, \\ V_2 &= B_2 \cup \{v, v'\}, \\ V_3 &= B_{3,k} \cup (A \backslash \{u', v'\}). \end{aligned}$$

Further we define $n_i = |V_i|$ for $i \in [3]$, and so $n^* = n_1 + n_2 + n_3$. We note that $n_1 = b_1 + 2, n_2 = b_2 + 2$, and $n_3 = b_{3,k} + a - 2$. We again consider the identifying code C^* of G^* and we let $C_i^* = C^* \cap V_i$ for $i \in [3]$. Thus,

$$|C^*| = \sum_{i=1}^{3} |C_i^*|.$$

Claim D.2(ii). $|C_3^*| \le \left(\frac{\Delta - 1}{\Delta}\right) n_3$.

Proof. By Claim D.1(a), we have

$$b_{3,k}^* = \sum_{i=3}^{k} b_i^* \ge \sum_{i=3}^{k} \frac{b_i}{\Delta - 1} = \frac{b_{3,k}}{\Delta - 1},$$

and so $(\Delta - 1) b_{3,k}^* \ge b_{3,k}$, or, equivalently,

$$b_{3,k} - \Delta b_{3,k}^* \le -b_{3,k}^* \le -k + 2. \tag{1}$$

Recall that $n_3 = a - 2 + b_{3,k}$ and that $a \le k$. Hence, by Inequality (1), we have

$$\begin{aligned}\left|C_3^*\right| &= |A\setminus\{u',v'\}| + b_{3,k} - b_{3,k}^* \\ &= a - 2 + b_{3,k} - b_{3,k}^* \\ &= \left(\frac{\Delta-1}{\Delta}\right)(a-2+b_{3,k}) + \frac{1}{\Delta}\left(a-2+b_{3,k}-\Delta b_{3,k}^*\right) \\ &\le \left(\frac{\Delta-1}{\Delta}\right)n_3 + \frac{1}{\Delta}(k-2-k+2) \\ &= \left(\frac{\Delta-1}{\Delta}\right)n_3,\end{aligned}$$

yielding the desired upper bound. □

Claim D.2(iii). If $b_i^* \ge \frac{b_i}{\Delta-2}$, then $|C_i^*| \le \left(\frac{\Delta-1}{\Delta}\right)n_i$ for $i \in [2]$.

Proof. Let $i \in [2]$. Recall that $b_i^* \ge 1$ and $n_i = b_i + 2$. Suppose that $b_i^* \ge \frac{b_i}{\Delta-2}$. Thus, $(\Delta-2)b_i^* \ge b_i$, or, equivalently, $b_i - \Delta b_i^* \le -2b_i^* \le -2$ noting that $b_i^* \ge 1$. By definition, we have $C_1^* = \{u, u'\} \cup (B_1\setminus B_1^*)$ and $C_2^* = \{v, v'\} \cup (B_2\setminus B_2^*)$. Hence,

$$\begin{aligned}|C_i^*| &= 2 + b_i - b_i^* \\ &= \left(\frac{\Delta-1}{\Delta}\right)(2+b_i) + \frac{1}{\Delta}\left(2+b_i-\Delta b_i^*\right) \\ &\le \left(\frac{\Delta-1}{\Delta}\right)n_i + \frac{1}{\Delta}(2-2) \\ &= \left(\frac{\Delta-1}{\Delta}\right)n_i,\end{aligned}$$

yielding the desired upper bound. □

If $b_i^* \ge \frac{b_i}{\Delta-2}$ for $i \in [2]$, then by Claims D.2(ii) and D.2(iii), we have

$$|C^*| = \sum_{i=1}^{3}\left|C_i^*\right| \le \sum_{i=1}^{3}\left(\frac{\Delta-1}{\Delta}\right)n_i = \left(\frac{\Delta-1}{\Delta}\right)n^*,$$

yielding the desired upper bound. Hence, we may assume that $b_i^* < \frac{b_i}{\Delta-2}$ for some $i \in [2]$. By symmetry, we may assume that $b_1^* < \frac{b_1}{\Delta-2}$. Let

$$|B_1 \cap B_A| = t_1 + t_2 \quad \text{and} \quad |B_1 \cap B_L| = t_2.$$

We note that $1 \le t_1 + t_2 \le \Delta - 2$.

If $t_1 \ge 1$ and $t_2 \ge 1$, then $b_1 = t_1 + 2t_2$ and $b_1^* = 1 + t_2 \ge 2$, implying that

$$\frac{b_1}{\Delta-2} = \frac{t_2+(t_1+t_2)}{\Delta-2} \le \frac{t_2+(\Delta-2)}{\Delta-2} = \frac{t_2}{\Delta-2} + 1 \le \frac{\Delta-3}{\Delta-2} + 1 < 2 \le b_1^*.$$

If $t_1 \ge 1$ and $t_2 = 0$, then $b_1 = t_1 \le \Delta - 2$ and $b_1^* = 1$, implying that

$$\frac{b_1}{\Delta-2} \le \frac{\Delta-2}{\Delta-2} = 1 = b_1^*.$$

If $t_1 = 0$ and $t_2 \geq 2$, then $b_1 = 2t_2$ and $b_1^* = t_2 \geq 2$. Moreover, $t_2 \leq \Delta - 2$. Thus,

$$\frac{b_1}{\Delta - 2} = \frac{2t_2}{\Delta - 2} \leq \frac{2(\Delta - 2)}{\Delta - 2} = 2 \leq b_1^*.$$

If $t_1 = 0$, $t_2 = 1$ and $\Delta \geq 4$, then $b_1 = 2$, $b_1^* = 1$ and

$$\frac{b_1}{\Delta - 2} = \frac{2}{\Delta - 2} \leq \frac{2}{2} = 1 = b_1^*.$$

In all the above four cases, we contradict our assumption that $b_1^* < \frac{b_1}{\Delta - 2}$. Hence, $t_1 = 0$, $t_2 = 1$, and $\Delta = 3$. In this case, B_1 induces a P_2-component in G^*. Let $B_1 \cap B_A = \{u_1\}$ and let $B_1 \cap B_L = \{u_2\}$, and so $uu'u_1u_2$ is a path in G^*. We note that u_2 is a vertex of degree 1 in G^*. Since $\Delta = 3$ and v' is adjacent to v and u', we note that either $B_2 = P_1$ or $B_2 = P_2$. We denote the vertex in $B_2 \cap N(v')$ by v_1, and if v_1 has another adjacent vertex outside of A, we denote it by v_2. By our choice of u and v (maximizing $\deg_G(u) + \deg_G(v)$ for all adjacent u, v with uv a cycle edge), we note that both u and v have degree 3 in G^* (otherwise, $\deg_G(u') + \deg_G(v') > \deg_G(u) + \deg_G(v)$). We denote the third neighbor of u and v by u'' and v'', respectively.

Let us assume next that $|N(u_1) \cap A| = 2$. Since G is triangle-free, the other vertex in $N(u_1) \cap A$ is not v'. Notice that in this case, we may remove u' from C_1^*, resulting in C_1^{**}, and the set $C^{**} = C_1^{**} \cup C_2^* \cup C_3^*$ is an identifying code in G^*. Indeed, C^* was an identifying code in G^*. Moreover, every neighbor of u' is in C^{**}, and vertices in C^{**} induce a single component in $G[C^{**}]$. Hence, every vertex in C^{**} is separated by Lemma 3 while u' is the unique vertex not in C^{**} adjacent to vertices u and u_1. Notice that $|C^{**}| \leq \frac{2}{3}n^*$. Indeed, we have $n_1 = 4$, $|C_1^{**}| = 2$, $(n_2, |C_2^*|) \in \{(3, 2), (4, 3)\}$ and $|C_3^*| \leq \frac{2}{3}n_3$ by Claim D.2(ii). Since $\frac{|C_1^{**}| + |C_2^*|}{n_1 + n_2} \leq \frac{5}{8} < \frac{2}{3}$, we have $|C^{**}| \leq \frac{2}{3}n^*$. Therefore, we may assume that one of B_1 or B_2 is a P_2-component such that the vertex in B_A is adjacent to exactly one vertex in A. We may assume from now on without loss of generality, that $N(u_1) \cap A = \{u'\}$.

Recall that in graph G', any minimum-size identifying code C' is such that it does not separate either vertices $\{u, v'\}$ or $\{v, u'\}$ in G. Thus, $C' \cap \{u, v, v', u'\}$ is either $\{u, v'\}$ or $\{v, u'\}$. Furthermore, since u_2 is a leaf and u_1 is adjacent only to u' in A, the only vertex that can separate u_1 and u_2 is u'. Therefore, $u' \in C'$. Thus, $C' \cap \{u, v, v', u'\} = \{u', v\}$ and $\{u, v'\}$ are not separated in G. Furthermore, we have $u'' \notin C'$ (otherwise $\{u, v'\}$ would be separated by C' in G). Notice that since C' separates u' and u in G', we have u_1 in C' and since C' separates u' and u_1, we have $u_2 \in C'$. However, now the set $C = \{u\} \cup (C' \backslash \{u_1\})$ is an identifying code of claimed cardinality in G. Indeed, vertices $u', u, v \in C$ are adjacent and are thus separated by Lemma 3. Moreover, vertices u_1 and u_2 also have unique code neighborhoods. This contradicts the properties of G and completes the proof of Claim D.2. □

Recall that

$$1 \leq |A^*| \leq k \leq b^* \leq b.$$

By Claim D.2, we may assume that $A^* \subset A$, and so $1 \leq |A^*| < |A| = a \leq 2(\Delta - 1)$. Let $\overline{A^*} = A \backslash A^*$. Thus, $|\overline{A^*}| = |A| - |A^*| \geq 1$. We note that either $|\overline{A^*}| \leq \Delta - 2$ or $|\overline{A^*}| \geq \Delta - 1$. Further, either $\overline{A^*}$ contains a neighbor of u and a neighbor of v or $\overline{A^*} \subset N_{G^*}(u)$ or $\overline{A^*} \subset N_{G^*}(v)$. We proceed further with three claims.

Claim D.3. If $|\overline{A^*}| \leq \Delta - 2$, then there exists an identifying code in G^* containing vertices u and v with cardinality at most

$$\left(\frac{\Delta - 1}{\Delta}\right) n^*.$$

Proof. Suppose that $|\overline{A^*}| \leq \Delta - 2$. As observed earlier, $|A^*| \leq b^*$. Thus in this case, $a = |A| = |A^*| + |\overline{A^*}| \leq \Delta + b^* - 2$. By Claim D.1(b), we have $b - (\Delta - 1)b^* \leq 0$. Let x be an arbitrary vertex in $\overline{A^*}$ and let

$$C^* = V(G^*) \backslash (B^* \cup \{x\}).$$

The code C^* is an identifying code of G^*. Indeed, C^* is a dominating set in G^*. Furthermore, it is also connected and hence, by Lemma 3, separates every vertex in C^*. Furthermore, x is the only vertex in A that does not belong to set C^*, and it is separated by $\{u, v\}$ from the vertices in B. Finally, vertices in B^* are separated by $A^* \cup \{B \backslash B^*\}$. Next, we consider the cardinality of C^*:

$$\begin{aligned}
|C^*| &= (a + b + 2) - (b^* + 1) \\
&= \left(\frac{\Delta - 1}{\Delta}\right)(a + b + 2) + \frac{1}{\Delta}(a + b - (\Delta - 2) - \Delta b^*) \\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* + \frac{1}{\Delta}((\Delta + b^* - 2) + b - \Delta + 2 - \Delta b^*) \\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* + \frac{1}{\Delta}(b - (\Delta - 1)b^*) \\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^*,
\end{aligned}$$

yielding the desired upper bound. □

Claim D.4. If $\overline{A^*}$ contains a neighbor of u and a neighbor of v, then there exists an identifying code in G^* containing vertices u and v with cardinality at most

$$\left(\frac{\Delta - 1}{\Delta}\right)n^*.$$

Proof. Suppose that $\overline{A^*}$ contains a neighbor, u_1, of u and a neighbor, v_1, of v. By our earlier observations, $|\overline{A^*}| = a - |A^*| \leq 2(\Delta - 1) - 1 = 2\Delta - 3$. Hence, $a = |A^*| + |\overline{A^*}| \leq b^* + 2\Delta - 3$. By Claim D.1(b), we have $b - (\Delta - 1)b^* \leq 0$. We now let

$$C^* = V(G^*) \backslash (B^* \cup \{u_1, v_1\}).$$

The set C^* is an identifying code of G^*. Indeed, C^* is a dominating set in G^*. Furthermore, it is also connected and hence, by Lemma 3, separates every vertex in C^*. Furthermore, u_1 and v_1 are the only vertices in A that do not belong to set C^* and are separated by $\{u, v\}$ from vertices in B and from each other. Finally, vertices in B^* are separated by $A^* \cup \{B \backslash B^*\}$. Next, we consider the cardinality of C^*:

$$\begin{aligned}
|C^*| &= (a + b + 2) - (b^* + 2) \\
&= \left(\frac{\Delta - 1}{\Delta}\right)(a + b + 2) + \frac{1}{\Delta}(a + b - 2(\Delta - 1) - \Delta b^*) \\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* + \frac{1}{\Delta}((b^* + 2\Delta - 3) + b - 2(\Delta - 1) - \Delta b^*) \\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* + \frac{1}{\Delta}(b - (\Delta - 1)b^* - 1) \\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* - \frac{1}{\Delta} \\
&< \left(\frac{\Delta - 1}{\Delta}\right)n^*,
\end{aligned}$$

yielding the desired upper bound. □

By Claim D.3, we may assume that $|\overline{A^*}| \geq \Delta - 1$, for otherwise there exists an identifying code of size at most $\left(\frac{\Delta-1}{\Delta}\right)n^*$ in G^* containing vertices u and v.

Claim D.5. If $\overline{A^*} \subseteq N_u$ or $\overline{A^*} \subseteq N_v$, then there exists an identifying code in G^* containing vertices u and v with cardinality at most

$$\left(\frac{\Delta - 1}{\Delta}\right)n^*.$$

Proof. Suppose that either $\overline{A^*} \subseteq N_u$ or $\overline{A^*} \subseteq N_v$. Renaming u and v if necessary, we may assume that $\overline{A^*} \subseteq N_v$, and so $\overline{A^*}$ contains no neighbor of u. Since $\overline{A^*} \subseteq N_v$, we have $|\overline{A^*}| \leq |N_{G^*}(v)| - 1 \leq \Delta - 1$. However, by our earlier assumption due to Claim D.3, $|\overline{A^*}| \geq \Delta - 1$. Therefore, $|\overline{A^*}| = \Delta - 1$, that is, $\deg_{G^*}(v) = \Delta$ and $\overline{A^*} = N_{G^*}(v)\backslash\{u\}$. Thus,

$$a = |A| = |A^*| + |\overline{A^*}| \leq b^* + \Delta - 1.$$

Recall that $u'v'$ is an edge in G^*, where u' is a neighbor of u in G^* different from v and v' is a neighbor of v in G^* different from u. Since $u' \in A^*$, the vertex u' is adjacent to at least one vertex in B, and therefore u' is adjacent to at most $\Delta - 2$ vertices in $\overline{A^*}$. Let v'' be a vertex in $\overline{A^*}$ that is not adjacent to the vertex u'. By Claim D.1(b), we have $b - (\Delta - 1)b^* \leq 0$. We now let

$$C^* = V(G^*)\backslash(B^* \cup \{v', v''\}).$$

The set C^* is an identifying code of G^*. Indeed, C^* is a dominating set in G^*. Furthermore, it is also connected and hence, by Lemma 3, separates every vertex in C^*. Furthermore, v' and v'' are the only vertices in A that do not belong to set C^* and are separated by v from vertices in B and by u' from each other. Finally, vertices in B^* are separated by $A^* \cup \{B\backslash B^*\}$. Next, we consider the cardinality of C^*:

$$\begin{aligned}
|C^*| &= (a + b + 2) - (b^* + 2)\\
&= \left(\frac{\Delta - 1}{\Delta}\right)(a + b + 2) + \frac{1}{\Delta}(a + b - 2(\Delta - 1) - \Delta b^*)\\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* + \frac{1}{\Delta}((b^* + \Delta - 1) + b - 2(\Delta - 1) - \Delta b^*)\\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* + \frac{1}{\Delta}(b - (\Delta - 1)b^* - (\Delta - 1))\\
&\leq \left(\frac{\Delta - 1}{\Delta}\right)n^* - \left(\frac{\Delta - 1}{\Delta}\right)\\
&< \left(\frac{\Delta - 1}{\Delta}\right)n^*,
\end{aligned}$$

yielding the desired upper bound. □

By Claims D.2, D.3, D.4 and D.5, we have shown that graph G^* admits an identifying code of cardinality at most $\left(\frac{\Delta-1}{\Delta}\right)n^*$ containing vertices u and v. Hence, Claim D follows. □

If $G^* = G$, then the theorem statement follows from Claim D. If $G^* \neq G$, then graph G_{uv} contains a component of cardinality at least 3. Furthermore, by Claim C, graph G_{uv} does not contain any components that belong to the set $\mathcal{F}_\Delta$. Hence, we may assume that G_{uv} contains a component of order at least 3. Next, we finalize the proof by showing that also in this case, $\gamma^{\text{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$.

Let us denote by $\mathcal{K}$ the set of components of order at least 3 in G_{uv}. By Claim C, $K \notin \mathcal{F}_\Delta$ for any $K \in \mathcal{K}$. Applying the inductive hypothesis to a component $K \in \mathcal{K}$ of maximum degree Δ_K, component K satisfies

$$\gamma^{\mathrm{ID}}(K) \leq \left(\frac{\Delta_K - 1}{\Delta_K}\right) n(K) \leq \left(\frac{\Delta - 1}{\Delta}\right) n(K). \tag{2}$$

Let C_K be an identifying code in K with minimum cardinality.

Observe that $G^* = G - \bigcup_{K \in \mathcal{K}} V(K)$. Let C^* be an identifying code of G^* with cardinality at most $\left(\frac{\Delta-1}{\Delta}\right)n^*$ containing vertices u and v, that exists by Claim D.

Let us next consider set $C = C^* \cup (\bigcup_{K \in \mathcal{K}} C_K)$. Set C is an identifying code in G since set C^* contains vertices u and v. Indeed, since C is a union of multiple identifying codes, each identifying code dominates and pairwise separates the vertices within the corresponding component. Thus, the only problems might be between the vertices of different components. In particular, in the case the component K contains vertex $z \in C_K$ such that $N[z] \cap C_K = \{z\}$, component G^* contains vertex $y \in C^*$ such that $N[y] \cap C^* = \{y\}$, and y and z are adjacent in G. However, this is not possible since $y \in A$ and thus, $N[y] \cap C^* \cap \{u, v\} \neq \varnothing$.

Notice that we have

$$\gamma^{\mathrm{ID}}(G) \leq |C| = |C^*| + \sum_{K \in \mathcal{K}} |C_K| \leq \left(\frac{\Delta - 1}{\Delta}\right) n^* + \sum_{K \in \mathcal{K}} \left(\frac{\Delta - 1}{\Delta}\right) n(K) = \left(\frac{\Delta - 1}{\Delta}\right) n.$$

This completes the proof of Theorem 2. □

As an immediate consequence of Theorem 2, we have the following corollary.

Corollary 15. *If G is a connected, identifiable, triangle-free graph of order n, then the following holds.*

a. *If G is cubic, then $\gamma^{\mathrm{ID}}(G) \leq \frac{2}{3}n$.*
b. *If G is subcubic and $n \geq 23$, then $\gamma^{\mathrm{ID}}(G) \leq \frac{2}{3}n$.*
c. *If G has maximum degree Δ where $\Delta \geq 4$ is fixed and $n \geq \Delta + 2$, then $\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$.*

4 | Beyond Triangle-Free Graphs

We now apply Theorem 2 to graphs having triangles and obtain a bound weaker than the conjectured one, as follows.

Corollary 16. *If G is a connected identifiable graph of order $n \geq 3$ with maximum degree $\Delta \geq 3$ such that G can be made triangle-free by deleting t edges, then*

$$\gamma^{\mathrm{ID}}(G) \leq \left(\frac{\Delta - 1}{\Delta}\right) n + 4t + \frac{1}{\Delta}.$$

Proof. We prove the claim by induction on t. Let us assume that $E_t \subseteq E(G)$, with $|E_t| = t$, is the smallest set of edges that we can remove from G so that $G_t = G - E_t$ is triangle-free. Observe that G_t is connected and identifiable, since the only connected triangle-free graph that is not identifiable is K_2, but we assume here that $n \geq 3$. By Table 1 and Theorem 2, the claim holds when G is triangle-free, that is, for $t = 0$. Assume next that the claim holds for every $t \leq t'$ and let $t = t' + 1$. Assume that G_t has a maximum degree $\Delta_t \leq \Delta$. We have $\Delta_t \geq 2$ since G_t is connected and $n \geq 3$.

Let C_t be an optimal identifying code in G_t. By Theorem 2, if $\Delta_t \geq 3$, we have

$$|C_t| \leq \left(\frac{\Delta_t - 1}{\Delta_t}\right) n + \frac{1}{\Delta_t} \leq \left(\frac{\Delta - 1}{\Delta}\right) n + \frac{1}{\Delta}.$$

If $\Delta_t = 2$, by Corollary 7(e)-(f) and $n \notin \{4, 7\}$, then $|C_t| \leq \frac{2}{3}n \leq \left(\frac{\Delta-1}{\Delta}\right)n$ since $\Delta \geq 3$. If $n = 4$, $|C_t| = 3 \leq \left(\frac{\Delta-1}{\Delta}\right)n + \frac{1}{\Delta}$ and if $n = 7$, $|C_t| \leq 5 \leq \left(\frac{\Delta-1}{\Delta}\right)n + \frac{1}{\Delta}$ (again since $\Delta \geq 3$). Let edge $uv \in E_t$ and let us consider graph G_t together with edge uv, denoted by G'_t. Observe that if C_t is not an identifying code in G'_t, then the addition of edge uv modified some code-neighborhoods (this implies that u or v is in C_t). Therefore, there are at most four vertices that are no longer separated (possibly, vertex u together with some vertex u', and possibly, vertex v with some vertex v'). As we add these edges back one at a time, each time we create at most four new vertices among which some vertex-pairs are not separated by C_t. Thus, in the graph G, there is a set S of at most $4t$ vertices in which some vertex-pairs are not separated by C_t. However, since G is identifiable, S is $V(G)$-identifiable and thus, by Lemma 4, we can find an $(S, V(G))$-identifying code C_S of size at most $|S| \leq 4t$. The set $C_t \cup C_S$ is an identifying code of G of the desired size, proving the claim. □

5 | Conclusion

We proved Conjecture 1 for all triangle-free graphs. In fact, we proved (see Theorem 2) the following stronger result. Let G be a connected, identifiable, triangle-free graph of order $n \geq 3$ with maximum degree Δ. If $\Delta \geq 4$, then we proved that $\gamma^{\text{ID}}(G) \leq \left(\frac{\Delta-1}{\Delta}\right)n$, except for one exceptional graph, namely the star $K_{1,\Delta}$. Moreover, if $\Delta \leq 3$, then $\gamma^{\text{ID}}(G) \leq \frac{2}{3}n$, unless G belongs to a forbidden family that contains fifteen graphs (all of order at most 22): P_4, C_4, C_7, and the twelve trees of maximum degree 3 from $\mathcal{T}_3$.

In the special case when G is a triangle-free cubic graph, this implies that $\gamma^{\text{ID}}(G) \leq \frac{2}{3}n$ always holds. This establishes a best possible upper bound for triangle-free cubic graphs, since $\gamma^{\text{ID}}(K_{3,3}) = 4$ (we do not know if other cubic graphs for which this bound is tight exist). The previously best known upper bounds (prior to this paper) when G is triangle-free, subcubic, and cubic were, $\gamma^{\text{ID}}(G) \leq \frac{8}{9}n$ and $\gamma^{\text{ID}}(G) \leq \frac{5}{6}n$, respectively (see [[35], Corollary 4.46]; the proof used the technique developed in Ref. [16]).

Towards a positive resolution of Conjecture 1, it would be interesting to prove it for all cubic graphs.

When it comes to general graphs, the list of exceptional graphs is larger than $\mathcal{F}_\Delta$. Indeed, as mentioned in Table 1, the complements of half-graphs and related constructions defined in Ref.[19] do require $c > 0$. Nevertheless, those constructions have maximum degree Δ very close to the number n of vertices ($n - 1$ or $n - 2$), and, for any given Δ, there is only a finite number of such examples. Thus, it is possible that even for the general case, if the conjecture is true, the list of graphs requiring $c > 0$ is also finite for every fixed value of Δ.

As seen in Table 1, all graphs known to us that require $c > 0$ have $c \leq 3/2$ (which is reached only by odd cycles), and when $\Delta \geq 3$, in fact $c \leq 1/3$. Are there graphs that require higher values of c? Another way to formulate these constants is in terms of the maximum degree. Do there exist any graphs that require the constant c to be larger than $3/\Delta$? By our results, such graphs would necessarily contain triangles. Note that it seems necessary to understand those graphs needing $c > 0$, in order to prove the conjecture.

Acknowledgments

The first two authors were supported by the French government IDEX-ISITE initiative CAP 20-25 (ANR-16-IDEX-0001), the International Research Center "Innovation Transportation and Production Systems" of the I-SITE CAP 20-25, and the ANR project GRALMECO (ANR-21-CE48-0004). The research of Dipayan Chakraborty was supported in part by the Lebanese American University under the President's Intramural Research Fund PIRF0056. The research of Michael Henning was supported in part by the South African National Research Foundation (grant number 129265) and the University of Johannesburg. The Research of Tuomo Lehtilä was partially supported by Business Finland project 10769/31/2022 6GTNF and Academy of Finland grants 338797 and 358718.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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