

TIGHT (DOUBLE) EXPONENTIAL BOUNDS FOR IDENTIFICATION PROBLEMS: LOCATING-DOMINATING SET AND TEST COVER*

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Abstract. We investigate fine-grained algorithmic aspects for classical identification problems in graphs, namely, LOCATING-DOMINATING SET, and in set systems, namely, TEST COVER. In the first problem, an input is a graph G on n vertices and an integer k , and the objective is to decide whether there is a subset S of k vertices such that any two distinct vertices not in S are dominated by distinct subsets of S . In the second problem, an input is a set U of items, a collection $\mathcal{F}$ of subsets of U called *tests*, and an integer k , and the objective is to select a set S of at most k tests such that any two distinct items are contained in a distinct subset of tests of S . For our first result, we LOCATING-DOMINATING SET (respectively, TEST COVER) parameterized by the treewidth of the input graph (respectively, the incidence graph) does not admit an algorithm running in time $2^{2^{o(\text{tw})}} \cdot \text{poly}(n)$ (respectively, $2^{2^{o(\text{tw})}} \cdot \text{poly}(|U| + |\mathcal{F}|)$), unless the exponential-time hypothesis (ETH) fails. Next, we prove that unless the ETH fails, LOCATING-DOMINATING SET does not admit an algorithm running in time $2^{o(k^2)} \cdot \text{poly}(n)$ or a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{o(k)}$ vertices, and TEST COVER does not admit an algorithm running in time $2^{2^{o(k)}} \cdot \text{poly}(|U| + |\mathcal{F}|)$ or a kernel with $2^{2^{o(k)}}$ vertices. Again, we show that these lower bounds are tight by designing (kernelization) algorithms with matching running times.

Key words. identification problems, locating-dominating set, test cover, double-exponential lower bound, ETH, kernelization lower bounds

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1. Introduction. The article aims to study the algorithmic properties of certain identification problems in discrete structures. In identification problems, one wishes to

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select a solution substructure of an input structure (a subset of vertices, the coloring of a graph, etc.) so that the solution substructure uniquely identifies each element. Some well-studied examples are, for example, the problems TEST COVER for set systems and METRIC DIMENSION for graphs (problems [SP6] and [GT61], respectively, in the book by Garey and Johnson [39]). This type of problem has been studied since the 1960s, both in combinatorics community (see, e.g., Rényi [64] or Bondy [8]) and in the algorithms community since the 1970s [6, 9, 25, 59]. They have multiple practical and theoretical applications, such as network monitoring [63], medical diagnosis [59], bioinformatics [6], coin-weighing problems [66], graph isomorphism [3], games [20], machine learning [19], etc. An online bibliography on the topic with over 500 entries as of 2024 is maintained at [48].

In this article, we investigate fine-grained algorithmic aspects of two identification problems. One of them is LOCATING-DOMINATING SET, which is a graph-theoretic problem, and the other one, TEST COVER, is a problem on set systems. Like most other interesting and practically motivated computational problems, identification problems also turned out to be NP-hard, even in very restricted settings. See, for example, [21] and [39], respectively. We refer the reader to “Related work” toward the end of this section for a more detailed overview of their algorithmic complexity.

To cope with this hardness, these problems have been studied through the lens of parameterized complexity. In this paradigm, we associate each instance I with a parameter ℓ and are interested to know whether the problem admits a *fixed-parameter tractable* (FPT) algorithm, i.e., an algorithm with the running time $f(\ell) \cdot |I|^{\mathcal{O}(1)}$, for some computable function f . A parameter may originate from the formulation of the problem itself or can be a property of the input graph. If a parameter originates from the formulation of the problem itself, then it is called a *natural parameter*. Otherwise, a parameter that is described by the structural property of the input graph is called a *structural parameter*. One of the most well-studied structural parameters is “treewidth” (which, informally, quantifies how close the input graph is to a tree, and is denoted by tw). We refer readers to [26, Chapter 7] for a formal definition. Courcelle’s celebrated theorem [22] states that there exists a computable function f such that, given a graph G of treewidth tw and a Monadic Second-Order Logic (MSOL) formula φ of length L , one can decide whether $G \models \varphi$ in time $f(\text{tw}, L) \cdot n$, where $n = |V(G)|$. In particular, for every fixed MSOL formula φ , the model-checking problem is FPT when parameterized by tw . Hence, a large class of problems admits an FPT algorithm when parameterized by the treewidth. Unfortunately, the function f is a tower of exponents whose height depends roughly on the size of the MSOL formula. Hence, this result serves as a starting point to obtain an (usually impractical) FPT algorithm.

Over the years, researchers have searched for more efficient problem-specific algorithms when parameterized by the treewidth. There is a rich collection of problems that admit an FPT algorithm with single- or almost-single-exponential dependency with respect to treewidth, i.e., of the form $2^{\mathcal{O}(\text{tw})} \cdot n^{\mathcal{O}(1)}$ or $2^{\mathcal{O}(\text{tw} \log(\text{tw}))} \cdot n^{\mathcal{O}(1)}$ (see, for example, [26, Chapter 7]). There are a handful of graph problems that only admit FPT algorithms with double- or triple-exponential dependence in the treewidth [7, 29, 30, 31, 32, 43, 58]. In the respective articles, the authors prove that this double- (respectively, triple-) dependence on the treewidth cannot be improved unless the exponential-time hypothesis (ETH)¹ fails.

Foucaud et al. [36] studied three NP-complete *metric-based graph problems* viz. METRIC DIMENSION, STRONG METRIC DIMENSION, and GEODETIC SET. They

¹The ETH roughly states that n -variable 3-SAT cannot be solved in time $2^{o(n)} n^{\mathcal{O}(1)}$. See [26, Chapter 14].

proved that these problems admit double-exponential lower bounds in tw (and, in fact, in the size of the minimum vertex cover size vc for the second problem) under the ETH. The first two of these three problems are identification problems.

In this article, we continue this line of research and highlight the usefulness of identification problems to prove relatively rare types of lower bounds, by investigating fine-grained algorithmic aspects of LOCATING-DOMINATING SET and TEST COVER, two classical (non-metric-based) identification problems. This also shows that this type of bound can hold for “local” (i.e., non-metric-based) problems (the problems studied in [36] were all metric-based). Apart from serving as examples for double-exponential dependence on treewidth, these problems are of interest in their own right and possess a rich literature both in the algorithms and discrete mathematics communities, as highlighted in “Related work.”

LOCATING-DOMINATING SET

Input: A graph G on n vertices and an integer k .

Question: Does there exist a locating-dominating set of size k in G , that is, a set S of $V(G)$ of size at most k such that for any two different vertices $u, v \in V(G) \setminus S$, their neighborhoods in S are different, i.e., $N(u) \cap S \neq N(v) \cap S$ and nonempty?

TEST COVER

Input: A set of items U , a collection $\mathcal{F}$ of subsets of U called *tests*, and an integer k .

Question: Does there exist a collection of at most k tests such that for each pair of items, there is a test that contains exactly one of the two items?

As TEST COVER is defined over set systems, for structural parameters, we define an *incidence graph* in the natural way: A bipartite graph G on n vertices with bipartition $\langle R, B \rangle$ of $V(G)$ such that sets R and B contain a vertex for every set in $\mathcal{F}$ and for every item in U , respectively. A vertex $r \in R$ and a vertex $b \in B$ are adjacent if and only if the set corresponding to r contains the element corresponding to b . In this incidence graph, a test cover corresponds to a subset S of R such that any two vertices of B have distinct neighborhoods within S .

The LOCATING-DOMINATING SET problem is also a *graph domination problem*. In the classical DOMINATING SET problem, an input is an undirected graph G and an integer k , and the objective is to decide whether there is a subset $S \subseteq V(G)$ of size k such that for any vertex $u \in V(G) \setminus S$, at least one of its neighbors is in S . It can also be seen as a local version of METRIC DIMENSION² in which the input is the same and the objective is to determine a set S of $V(G)$ such that for any two vertices $u, v \in V(G) \setminus S$, there exists a vertex $s \in S$ such that $\text{dist}(u, s) \neq \text{dist}(v, s)$.

We demonstrate the applicability of the techniques from [36] to the “local” problems LOCATING-DOMINATING SET and TEST COVER, showing that the metric nature of, e.g., METRIC DIMENSION was in fact not essential to obtain this type of

²Note that METRIC DIMENSION is also an identification problem, but it is inherently nonlocal in nature, and indeed was studied together with two other nonlocal problems in [36], where the similarities between these nonlocal problems was noticed.

lower bounds. To do so, we adapt the main techniques developed in [36] to our setting, namely, the *set-representation gadgets* and *bit-representation gadgets*.

We prove the following result.

THEOREM 1.1. *Unless the ETH fails, LOCATING-DOMINATING SET (respectively, TEST COVER) does not admit an algorithm running in time $2^{2^{o(tw)}} \cdot \text{poly}(n)$, where tw is the treewidth and n is the order of the graph (respectively, of the incidence graph).*

We also prove that both LOCATING-DOMINATING SET and TEST COVER admit an algorithm with matching running time (Theorem 3.3 and Theorem 5.1), by nontrivial dynamic programming schemes on tree-decompositions.

In contrast to Theorem 1.1, DOMINATING SET admits an algorithm running in time $\mathcal{O}(3^{tw} \cdot n^2)$ [56, 72].

We remark that the algorithmic lower bound of Theorem 1.1 holds true even with respect to vertex integrity [40], a parameter larger than treewidth. This also implies the same lower bounds for parameters treedepth and pathwidth, whose values both always lie between the vertex integrity and the treewidth.

Theorem 1.1 adds LOCATING-DOMINATING SET and TEST COVER to the short list of NP-complete problems that admit (tight) double-exponential lower bounds for treewidth. Using the techniques mentioned in [36], two more problems (from learning theory), viz. NONCLASHING TEACHING MAP and NONCLASHING TEACHING DIMENSION, were recently shown in [15] to admit similar lower bounds.

Next, we prove that LOCATING-DOMINATING SET and TEST COVER also admit “exotic” lower bounds, when parameterized by the solution size k . First, note that both problems are trivially FPT when parameterized by the solution size. Indeed, as any solution must have size at least logarithmic in the number of elements/vertices (assuming no redundancy in the input), the whole instance is a trivial single-exponential kernel for LOCATING-DOMINATING SET and double-exponential in the case of TEST COVER. To see this, note that in both problems, any two vertices/items must be assigned a distinct subset from the solution set. Hence, if there are more than 2^k of them, we can safely reject the instance. Thus, for LOCATING-DOMINATING SET, we can assume that the graph has at most $2^k + k$ vertices, and for TEST COVER, at most 2^k items. Moreover, for TEST COVER, one can also assume that every test is unique (otherwise, delete any redundant test), in which case there are at most 2^{2^k} tests. Hence, LOCATING-DOMINATING SET admits a kernel with size $\mathcal{O}(2^k)$ and an FPT algorithm running in time $2^{\mathcal{O}(k^2)}$ (see Proposition 4.1). We prove that both of these bounds are optimal.

THEOREM 1.2. *Unless the ETH fails, LOCATING-DOMINATING SET does not admit*

- *an algorithm running in time $2^{o(k^2)} \cdot \text{poly}(n)$, or*
- *a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{o(k)}$ vertices.*

To the best of our knowledge, LOCATING-DOMINATING SET is the first known problem to admit such an algorithmic lower bound, with a matching upper bound, when parameterized by the solution size. The only other problems known to us, admitting similar lower bounds, are for structural parameterizations like vertex cover [1, 15, 36] or pathwidth [61, 65]. The second result is also quite rare in the literature. The only results known to us about ETH-based conditional lower bounds on the number of vertices in a kernel when parameterized by the solution size are for EDGE CLIQUE

COVER [27] and BICLIQUE COVER [16].³ Theorem 1.2 also improves upon a “no $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ algorithm” bound from [4] (under $W[2] \neq \text{FPT}$) and a $2^{o(k \log k)}$ ETH-based lower bound recently proved in [11].

Now, consider the case of TEST COVER. As mentioned before, it is safe to assume that $|\mathcal{F}| \leq 2^{|U|}$ and $|U| \leq 2^k$. By Bondy’s celebrated theorem [8], which asserts that in any feasible instance of TEST COVER, there is always a solution of size at most $|U| - 1$; we can also assume that $k \leq |U| - 1$. Hence, the brute-force algorithm that enumerates all the subcollections of tests of size at most k runs in time $|\mathcal{F}|^{\mathcal{O}(|U|)} = 2^{\mathcal{O}(|U|^2)} = 2^{2^{\mathcal{O}(k)}}$. Our next result proves that this simple algorithm is again optimal.

THEOREM 1.3. *Unless the ETH fails, TEST COVER does not admit*

- *an algorithm running in time $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, or*
- *a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{2^{o(k)}}$ vertices.*

This result adds TEST COVER to the relatively rare list of NP-complete problems that admit such double-exponential lower bounds when parameterized by the solution size, and have a matching algorithm. The only other examples that we know of are EDGE CLIQUE COVER [27], DISTINCT VECTORS PROBLEM [62], and TELEPHONE BROADCAST [70]. For double-exponential algorithmic lower bounds with respect to structural parameters, please see [33, 43, 47, 50, 52, 54, 55, 57]. The second result in the theorem is a simple corollary of the first result. Assume that the problem admits a kernel with $2^{2^{o(k)}}$ vertices. Then, the brute-force algorithm enumerating all the possible solutions works in time $\binom{2^{2^{o(k)}}}{k} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, which is $2^{k \cdot 2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, which is $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, contradicting the first result. To the best of our knowledge, TEST COVER is the first problem that admits a double-exponential kernelization lower bound for the number of vertices when parameterized by solution size, or by any natural parameter.

Related work. LOCATING-DOMINATING SET was introduced by Slater in the 1980s [67, 68]. The problem is NP-complete [21], even for special graph classes such as planar unit disk graphs [60], planar bipartite subcubic graphs, chordal bipartite graphs, split graphs, and co-bipartite graphs [35], interval and permutation graphs of diameter 2 [38]. By a straightforward application of Courcelle’s theorem [22], LOCATING-DOMINATING SET is FPT for parameter treewidth and even cliquewidth [23]. Explicit polynomial-time algorithms were given for trees [67], block graphs [2], series-parallel graphs [21], and cographs [37]. Regarding the approximation complexity of LOCATING-DOMINATING SET, see [35, 41, 69].

In [10], structural parameterizations of LOCATING-DOMINATING SET were studied. It was shown that the problem admits a linear kernel for the parameter max-leaf number; however, under standard complexity assumptions, no polynomial kernel exists for the solution size, combined with either the vertex cover number or the distance to a clique. They also provide a double-exponential kernel for the parameter distance to the cluster. In [13], the authors of the present paper design an improved parameterized algorithm for LOCATING-DOMINATING SET with respect to the vertex cover number and a linear kernel for the feedback edge set number.

It was shown in [4] that LOCATING-DOMINATING SET cannot be solved in time $2^{o(n)}$ on bipartite graphs, or in time $2^{o(\sqrt{n})}$ on planar bipartite graphs, assuming

³Additionally, POINT LINE COVER does not admit a kernel with $\mathcal{O}(k^{2-\epsilon})$ vertices, for any $\epsilon > 0$, unless $\text{NP} \subseteq \text{coNP}/\text{poly}$ [53].

the ETH. Moreover, they also showed that LOCATING-DOMINATING SET cannot be solved in time $2^{O(k)}n^{O(1)}$ on bipartite graphs, unless $W[2] = \text{FPT}$. Note that the authors of [4] have designed a complex framework with the goal of studying a large class of identification problems related to LOCATING-DOMINATING SET and similar problems. In the full version [11] of [10], it is shown that LOCATING-DOMINATING SET admits neither a $2^{o(k \log k)}n^{O(1)}$ -time nor an $n^{o(k)}$ -time algorithm, assuming the ETH. In [13], the authors of the present paper showed that LOCATING-DOMINATING SET does not admit a compression algorithm returning an input with a subquadratic number of bits, under standard assumptions.

TEST COVER was shown to be NP-complete by Garey and Johnson [39, problem SP6], and it is also hard to approximate within a ratio of $(1 - \epsilon) \ln n$ [9] (an approximation algorithm with ratio $1 + \ln n$ exists by reduction to SET COVER [6]). As any solution has size at least $\log_2(n)$, the problem admits a trivial kernel of size 2^{2^k} , and thus TEST COVER is FPT parameterized by solution size k . TEST COVER was studied within the framework of above/below guarantee" parameterizations in [5, 24, 25, 42] and kernelization in [5, 24, 42]. These results have shown an intriguing behavior for TEST COVER, with some nontrivial techniques being developed to solve the problem [5, 25]. TEST COVER is FPT for parameters $n - k$, but $W[1]$ -hard for parameters $m - k$ and $k - \log_2(n)$ [25]. However, assuming standard assumptions, there is no polynomial kernel for the parameterizations by k and $n - k$ [42], although there exists a partially polynomial kernel" for parameter $n - k$ [5] (i.e., one with $O((n - k)^7)$ elements, but potentially exponentially many tests). When all the tests have a fixed upper bound r on their size, the parameterizations by k , $n - k$, and $m - k$ all become FPT with a polynomial kernel [24, 42]. In [13], the authors of the present paper showed that TEST COVER can be solved in time $2^{O(|U| \log |U|)}(|U| + |\mathcal{F}|)^{O(1)}$, but does not admit a compression algorithm returning an input with a subquadratic number of bits, under standard assumptions.

The problem DISCRIMINATING CODE [17] is very similar to TEST COVER (with the distinction that the input is presented as a bipartite graph, one part representing the elements and the other representing the tests, and that every element has to be covered by some solution test) and has been shown to be NP-complete even for planar instances [18].

Organization. We use standard notations which we specify in section 2. We use the LOCATING-DOMINATING SET problem to demonstrate key technical concepts regarding our lower bounds and algorithms. We present an overview of the arguments about LOCATING-DOMINATING SET in sections 3 and 4, respectively, for parameters treewidth and solution size. The arguments regarding TEST COVER follow the same line and are presented in section 5. We conclude with some open problems in section 6.

2. Preliminaries. For a positive integer q , we denote the set $\{1, 2, \dots, q\}$ by $[q]$. We use $\mathbb{N}$ to denote the collection of all nonnegative integers.

Graph theory. We use standard graph-theoretic notation, and we refer the reader to [28] for any undefined notation. For an undirected graph G , sets $V(G)$ and $E(G)$ denote its set of vertices and edges, respectively. We denote an edge with two endpoints u, v as uv . Unless otherwise specified, we use n to denote the number of vertices in the input graph G of the problem under consideration. Two vertices u, v in $V(G)$ are *adjacent* if there is an edge uv in G . The *open neighborhood* of a vertex v , denoted by $N_G(v)$, is the set of vertices adjacent to v . The *closed neighborhood* of a vertex v , denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. We say that a vertex u is a *pendant vertex*

if $|N_G(v)| = 1$. We omit the subscript in the notation for neighborhood if the graph under consideration is clear. For a subset S of $V(G)$, we define $N[S] = \bigcup_{v \in S} N[v]$ and $N(S) = N[S] \setminus S$. For a subset S of $V(G)$, we denote the graph obtained by deleting S from G by $G - S$. We denote the subgraph of G induced on the set S by $G[S]$.

The *vertex integrity* of a graph G is the minimum over $|S| + \max_{D \in cc(G-S)} |D|$, where S is a subset of $V(G)$ and $cc(H)$ denotes the set of connected components of H . It is known that the vertex integrity of a graph G is an upper bound for the treedepth of G , and thus, also of the pathwidth and the treewidth of G . See [40] for more details.

Locating-dominating sets. A subset of vertices S in graph G is called its *dominating set* if $N[S] = V(G)$. A dominating set S is said to be a *locating-dominating set* if for any two different vertices $u, v \in V(G) \setminus S$, we have $N(u) \cap S \neq N(v) \cap S$. In this case, we say vertices u and v are *distinguished* by the set S . We say a vertex u is *located* by set S if for any vertex $v \in V(G) \setminus \{u\}$, $N(u) \cap S \neq N(v) \cap S$. By extension, a set X is *located* by S if all vertices in X are located by S . We note the following simple observation (see also [14, Lemma 5]).

Observation 2.1. If S is a locating-dominating set of a graph G , then there exists a locating-dominating set S' of G such that $|S'| \leq |S|$ and that contains all vertices that are adjacent to pendant vertices (i.e., vertices of degree 1) in G .

Proof of Claim. Let u be a pendant vertex which is adjacent to a vertex v of G . We now look for a locating-dominating set S' of G such that $|S'| \leq |S|$ and contains the vertex v . As S is a (locating) dominating set, we have $\{u, v\} \cap S \neq \emptyset$. If $v \in S$, then take $S' = S$. Therefore, let us assume that $u \in S$ and $v \notin S$. Define $S' = (S \cup \{v\}) \setminus \{u\}$. It is easy to see that S' is a dominating set. If S' is not a locating-dominating set, then there exists w , apart from u , in the neighborhood of v such that both u and w are adjacent to *only* v in S' . As u is a pendant vertex and v its unique neighbor, w is not adjacent to u . Hence, w was not adjacent any vertex in $S' \setminus \{v\} = S \setminus \{u\}$. This, however, contradicts the fact that S is a (locating) dominating set. Hence, S' is a locating-dominating set and $|S'| = |S|$. Thus, the result follows from repeating this argument for each vertex of G adjacent to a pendant vertex. $\square$

Parameterized complexity. An instance of a parameterized problem Π consists of an input I , which is an input of the nonparameterized version of the problem, and an integer k , which is called the *parameter*. Formally, $\Pi \subseteq \Sigma^* \times \mathbb{N}$. A parameterized problem Π is FPT if given an instance (I, k) of Π , we can correctly decide whether (I, k) is a YES-instance of Π in time $f(k) \cdot |I|^{\mathcal{O}(1)}$. Here, $f: \mathbb{N} \rightarrow \mathbb{N}$ is some computable function depending only on k . A parameterized problem Π is said to admit a *kernelization* if there exists an algorithm that, given an instance (I, k) of Π , runs in time polynomial in $|I| + k$ and constructs an instance (I', k') of Π such that

- (i) $(I, k) \in \Pi$ if and only if $(I', k') \in \Pi$, and
- (ii) $|I'| + k' \leq g(k)$ for some computable function $g: \mathbb{N} \rightarrow \mathbb{N}$ depending only on k .

If $g(\cdot)$ is a polynomial function, then Π is said to admit a *polynomial kernelization*. For a detailed introduction to parameterized complexity and related terminologies, we refer the reader to the recent books by Cygan et al. [26] and Fomin et al. [34].

3. Locating-Dominating Set parameterized by treewidth. In the first subsection, we present a dynamic programming algorithm that solves the problem in $2^{2^{\mathcal{O}(\text{tw})}} \cdot n^{\mathcal{O}(1)}$. In the second subsection, we prove that this dependency on treewidth is optimal, up to the multiplicative constant factors, under the ETH.

3.1. Upper bound. In this subsection, we present a dynamic programming (DP)-based algorithm for the LOCATING-DOMINATING SET problem when parameterized by the treewidth of the input graph. For completeness, we begin with the necessary definitions.

DEFINITION 3.1 (tree-decomposition). *A tree-decomposition of an undirected graph $G = (V, E)$ is a pair $\mathcal{T} = (T, \mathcal{X} = X_t \mid t \in V(T))$, where T is a tree and $\mathcal{X}$ is a collection of subsets of $V(G)$, called bags, such that*

1. *For every vertex $u \in V(G)$, there exists $t \in V(T)$ such that $u \in X_t$.*
2. *For every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in X_t$.*
3. *For every vertex $u \in V(G)$, the set $\{t \in V(T) \mid u \in X_t\}$ induces a connected subtree of T .*

Given a tree-decomposition $\mathcal{T}$, its width is defined as $\max_{t \in V(T)} (|X_t| - 1)$. The treewidth of a graph G is the minimum width over all possible tree-decompositions of G .

To facilitate the description of our DP algorithm, we use the notion of a *nice tree-decomposition* [49].

DEFINITION 3.2 (nice tree-decomposition). *A rooted tree-decomposition $\mathcal{T} = (T, X_t \mid t \in V(T))$ is said to be nice if every node $t \in V(T)$ has at most two children and is of one of the following types:*

1. *Root node: A node r with $X_r = \emptyset$ and no parent.*
2. *Leaf node: A node t with $X_t = \emptyset$ and no children.*
3. *Introduce node: A node t with a unique child t' such that $X_t = X_{t'} \cup \{u\}$ for $u \notin X_{t'}$.*
4. *Forget node: A node t with a unique child t' such that $X_t = X_{t'} \setminus \{u\}$ for $u \in X_{t'}$.*
5. *Join node: A node t with exactly two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.*

For each node $t \in V(T)$, we consider the subtree T_t of T rooted at t . Let G_t denote the subgraph of G induced by the vertices that appear in the bags of nodes in T_t .

Without loss of generality, we assume that a nice tree-decomposition of width $\mathcal{O}(\text{tw})$ is provided. If not, one can be constructed in time $2^{\mathcal{O}(\text{tw})}n$ [49, 51].

We now state the main result of this subsection.

THEOREM 3.3. *LOCATING-DOMINATING SET admits an algorithm with running time $2^{2^{\mathcal{O}(\text{tw})}} \cdot n$, where tw is the treewidth and n is the order of the input graph.*

Consider a locating-dominating set $S \subseteq V(G)$, and let $S_t \subseteq V(G_t)$ denote the *partial solution* induced by restricting S to the subgraph G_t . Formally, we define $S_t := S \cap V(G_t)$.

We now informally describe the information needed for a DP-state for a node t of a tree-decomposition, in order to convey to the reader the main ideas needed to understand the algorithms. The formal proof follows.

Observe that every vertex in $V(G_t) \setminus X_t$ is uniquely located by its neighborhood within S_t , since such vertices will not appear in any future bag of the decomposition.

Let us now examine the behavior of vertices in the current bag X_t . See Figure 1 for an illustration. Define the following:

- $Y := S_t \cap X_t$, i.e., the subset of the current bag that is included in the partial solution.

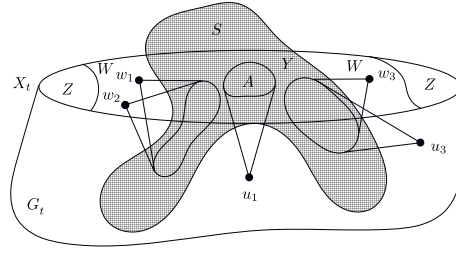


FIG. 1. The figure illustrates the definitions used in the DP algorithm. Let S be a locating-dominating set of G . Its intersections with $V(G_t)$ and X_t are denoted by S_t and Y , respectively. The region containing the vertices w_1, w_2 , and w_3 represents subset $W \subseteq X_t$. The remaining vertices of X_t form the set Z . Consider a subset $A \subseteq Y$. Observe that the vertex u_1 is adjacent only to the vertices in A within S (and hence within S_t). Therefore, the set A is not suitable for locating purposes. $\mathcal{Y}$ denote the collection of all such subsets. Note that the vertices w_1 and w_2 have identical neighborhoods in S_t . Similarly, the vertices w_3 and u_3 have identical neighborhoods in S_t . Since both w_1 and w_2 belong to X_t , we add the pair (w_1, w_2) to $\mathbb{W}$. However, as u_3 is forgotten (that is, $u_3 \notin X_t$), we add the pair $(w_3, +)$ to $\mathbb{W}$. By the definition of W_0 , all three vertices w_1, w_2 , and w_3 are in it.

- $W := (N(S_t) \cap X_t) \setminus Y$, i.e., the set of vertices in $X_t \setminus Y$ that are dominated by vertices in S_t (but may not be located). Also, define $W_0 = \{w \in W \mid N_{G_t}(w) \cap S_t \subseteq Y\}$. Note that every vertex in $W \setminus W_0$ is adjacent to some vertex in $S_t \setminus Y$.
- $Z := X_t \setminus (Y \cup W)$.

Note that the vertices in Z are not dominated by S_t .

Since S is a locating-dominating set, it must hold that no two vertices outside of S have identical neighborhoods within S . In particular, if there exists a vertex $u \in V(G_t)$ such that $N(u) \cap S \subseteq S_t$, then we must ensure that no vertex $v \in V(G) \setminus V(G_t)$ satisfies $N(v) \cap S = N(u) \cap S$. Otherwise, u and v would be indistinguishable in G , violating the location constraint. We remark that for the above case to happen, it must be that $N(u) \cap S \subseteq (S_t \cap X_t)$. To facilitate the extension of the partial solution S_t to a full solution $S \subseteq V(G)$, we track which subsets of Y are used to identify vertices. Specifically, we consider subsets $A \subseteq Y$ such that there exists $u \in V(G_t) \setminus S_t$ with $N_{G_t}(u) \cap S_t = A$. If $u \in V(G_t) \setminus X_t$, then A is *not usable for location* in future extensions, since no vertex added in later bags can change the neighborhood of u in the solution. Suppose that $\mathcal{Y}$ denotes the collection of subsets of Y that are not usable for future location.

For the reason hinted above, we need to distinguish between the vertices in W that are only adjacent to a subset of Y and those that are adjacent to some vertices in $S_t \setminus Y$ also. This is why we defined W_0 above.

Finally, we define a set $\mathbb{W}$ to keep track of pairs of vertices that currently have identical neighborhoods in S_t and must be distinguished in future extensions. Each element of $\mathbb{W}$ is of one of the following forms:

- A pair (w_1, w_2) in X_t and more specifically in W , indicating that w_1 and w_2 currently have identical neighborhoods within S_t , and some vertex in $S \setminus S_t$ must be adjacent to exactly one of them to distinguish them.
- A pair $(w_1, +)$, indicating that $w_1 \in W$ and some vertex $u \in V(G_t) \setminus X_t$ share the same neighborhood in S_t , and hence a vertex in $S \setminus S_t$ must be adjacent to w_1 to ensure distinguishability.

Note that for any $(w_1, w_2) \in \mathbb{W}$, vertex w_1 is in W_0 (respectively, $W \setminus W_0$) if and only if w_2 is in W_0 (respectively, $W \setminus W_0$).

For every node $t \in V(T)$, we define a DP-state using the notion of a *valid tuple*, which is a combination of the sets described above.

DEFINITION 3.4 (valid tuple). *Let $t \in V(T)$ be a node in the tree-decomposition. A tuple $\langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ is said to be a valid tuple at t if the following conditions hold:*

- $Y, W \subseteq X_t$ are disjoint sets such that $N(Y) \cap X_t \subseteq W$, $W_0 \subseteq W$,
- $\mathcal{Y}$ is a family of subsets of Y ,
- $\mathbb{W}$ is a collection of pairs which can be of two types: pairs (w_1, w_2) with $w_1, w_2 \in W$, and pairs of the form $(w_1, +)$, with $w_1 \in W$.

We now define how a valid tuple for node t can correspond to a potential subset S_t of a locating-dominating S of G .

DEFINITION 3.5 (candidate solution). *Consider a valid tuple $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ at node t . We say that set $S_t \subseteq V(G_t)$ is a candidate solution at t with respect to τ if it satisfies the following properties:*

1. S_t locates (in G_t) all vertices in $V(G_t) \setminus X_t$, i.e., for any distinct vertices $u, v \in V(G_t) \setminus (S_t \cup X_t)$, the sets $N_{G_t}(u) \cap S_t$ and $N_{G_t}(v) \cap S_t$ are nonempty and distinct.
2. The intersection of S_t with the current bag is exactly Y , and the vertices in $X_t \setminus Y$ that are dominated by S_t are given by W , i.e., $Y = S_t \cap X_t$, and $W = (N(S_t) \cap X_t) \setminus Y$. Moreover, no vertex in W_0 is adjacent to a vertex in $S_t \setminus Y$, and every vertex $w \in W \setminus W_0$ is adjacent to some vertex in $S_t \setminus Y$.
3. A subset A belongs to $\mathcal{Y}$ if and only if there exists a vertex $u \in V(G_t) \setminus (S_t \cup X_t)$ such that $N_{G_t}(u) \cap S_t = A$. (By the first item, if u exists it is unique.)
4. • A pair $(w_1, w_2) \in \mathbb{W}$ if and only if $w_1, w_2 \in W$ and $N_{G_t}(w_1) \cap S_t = N_{G_t}(w_2) \cap S_t$.
 • A pair $(w_1, +) \in \mathbb{W}$ if and only if $w_1 \in W$ and there exists a vertex $u \in V(G_t) \setminus (S_t \cup X_t)$ such that $N_{G_t}(w_1) \cap S_t = N_{G_t}(u) \cap S_t$. (By the first item, if u exists it is unique.)

For a valid tuple $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ at node t , we define $\mathbf{d}[t, \tau]$ as the minimum cardinality of a candidate solution at t . If no candidate solution exists, we define $\mathbf{d}[t, \tau] := \infty$.

We now proceed to describe a recursive algorithm used to update entries in the DP table for each node type in a nice tree-decomposition.

Leaf node. If t is a leaf node, then X_t is an empty set and the following claim is trivial.

LEMMA 3.6. *If t is a leaf node, then $\mathbf{d}[t, \tau] = 0$ for $\tau = \langle \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle$ and $\mathbf{d}[t, \tau] = \infty$ for any other τ .*

Introduce node. Let $t \in V(T)$ be an introduce node with unique child t' , such that $X_t = X_{t'} \cup \{x\}$, for some vertex $x \in V(G_t) \setminus V(G_{t'})$. Let $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ be a valid tuple at node t . We present the update rule for $\mathbf{d}[t, \tau]$ based on where x is in the set X_t . We need the following two definitions for the lemma.

DEFINITION 3.7 (introduce- Y -compatible). *Consider the case when x is in Y . Consider a tuple $\tau' = \langle Y', W', W'_0, \mathcal{Y}', \mathbb{W}' \rangle$ which is valid at t' . We say τ' is introduce- Y -compatible with τ if it satisfies the following conditions:*

1. $Y' = Y \setminus \{x\}$, $W = W' \cup (N_{G_t}(x) \setminus N_{G_t}[Y])$, $W_0 = W'_0 \cup (N_{G_t}(x) \setminus N_{G_t}[Y])$,
2. $\mathcal{Y}' = \mathcal{Y}$, and
3. $\mathbb{W} \subseteq \mathbb{W}'$ such that

- for each element of the type $(w_1, w_2) \in \mathbb{W} \setminus \mathbb{W}'$, exactly one of w_1 and w_2 is adjacent to x ; and
- for each element of the type $(w_1, +) \in \mathbb{W} \setminus \mathbb{W}'$, w_1 is adjacent to x in G_t .

Let $I_c^Y(\tau)$ be the collection of all valid tuples at t' that are introduce- Y -compatible with τ .

Consider the case when x is in W . Define $A := N_{G_t}(x) \cap Y$ as the set of neighbors of x that are included in the partial solution. If $A = \emptyset$, then x is not adjacent to any vertex in the partial solution, violating the definition of set W . In this case, set $d[t, \tau] := \infty$.

DEFINITION 3.8 (introduce- W -compatible). *Consider the case when x is in W , and suppose $A := N_{G_t}(x) \cap Y \neq \emptyset$. Consider a valid tuple $\tau' = \langle Y', W', W'_0, \mathcal{Y}', \mathbb{W}' \rangle$ at t' . We say that τ' is introduce- W -compatible with τ if*

1. $Y' = Y$, $W = W' \cup \{x\}$, $W_0 = W'_0 \cup \{x\}$,
2. $\mathcal{Y}' = \mathcal{Y}$, and
3. $\mathbb{W}$ is obtained from $\mathbb{W}'$ by the following procedure:
 - (a) If $A \in \mathcal{Y}'$, i.e., A models some $u \in V(G_{t'}) \setminus X_{t'}$ such that $N(u) \cap S_{t'} = A$ in a potential candidate solution $S_{t'}$ at t' , then add $(x, +)$ to $\mathbb{W}'$.
 - (b) For every vertex w_1 in W'_0 such that $A = N_{G_{t'}}(w_1) \cap Y$, add (x, w_1) to $\mathbb{W}'$.

Let $I_c^W(\tau)$ denote the set of all introduce- W -compatible tuples at t' .

LEMMA 3.9. *Let t be an introduce node with child t' , where $X_t = X_{t'} \cup \{x\}$, and let $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ be a valid tuple at t . Then,*

$$d[t, \tau] = \begin{cases} 1 + \min_{\tau' \in I_c^Y(\tau)} d[t', \tau'] & \text{if } x \in Y, \\ \min_{\tau' \in I_c^W(\tau)} d[t', \tau'] & \text{if } x \in W \text{ and } N_{G_t}(x) \cap Y \neq \emptyset, \\ d[t', \tau] & \text{if } x \in Z, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. As discussed earlier, we consider three mutually disjoint and exhaustive cases based on the membership of x in the sets specified by τ : whether $x \in Y$, $x \in W$, or $x \in Z$.

Case 1: x is in Y . Assume that the minimum value of $\min_{\tau' \in I_c^Y(\tau)} d[t', \tau']$ is finite and attained at τ' and is witnessed by the set $S_{t'}$. Define $S_t := S_{t'} \cup \{x\}$. We show that S_t is a valid candidate solution at t by verifying that it satisfies the conditions in Definition 3.5. We refer the readers to Definition 3.7 for the relationship between the sets specified by τ and τ' .

1. As $S_{t'}$ locates all vertices in $V(G_{t'}) \setminus X_{t'}$, set S_t , which is a superset of $S_{t'}$, is a locating-dominating set for all vertices in $V(G_t) \setminus X_t$, which is equal to $V(G_{t'}) \setminus X_{t'}$.
2. As $S_t = S_{t'} \cup \{x\}$ and $Y = Y' \cup \{x\}$, the intersection of S_t with X_t is exactly Y . Moreover, the vertices in $X_t \setminus Y$ that are dominated by S_t form the set W , as $W = W' \cup (N_{G_t}(x) \setminus N_{G_t}[Y])$. Also, vertices in W' whose neighbors in the partial solution $S_{t'}$ are all in Y' remain the same, apart from those that are adjacent to only x . Hence, $W_0 = W'_0 \cup (N_{G_t}(x) \setminus N_{G_t}[Y])$.
3. Since x is introduced in this bag, x is not adjacent to any vertex in $V(G_t) \setminus X_t$. Thus, the collection of unusable subsets $\mathcal{Y}$ at t is identical to the collection $\mathcal{Y}'$ at t' .

4. Now, consider the set W . As only change in partial dominating set is the addition of vertex x , the only new vertices that are added to W are from $N_{G_t}(x)$, and hence we have $\mathbb{W} \subseteq \mathbb{W}'$. Furthermore, for any pair (w_1, w_2) or $(w_1, +)$ which is in $\mathbb{W}'$ but not in $\mathbb{W}$, the vertex x must locate one of these vertices:

- In the case (w_1, w_2) , x is adjacent to exactly one of w_1 or w_2 .
- In the case $(w_1, +)$, x is adjacent to w_1 .

This confirms that $S_t = S_{t'} \cup \{x\}$ is a valid candidate solution for $\mathbf{d}[t, \tau]$. As x is introduced at this node, $x \notin S_{t'}$ and hence $|S_t| = |S_{t'}| + 1$. Thus we have $\mathbf{d}[t, \tau] \leq |S_t| = 1 + |S_{t'}| = 1 + \mathbf{d}[t', \tau']$.

Conversely, suppose that S_t is an optimal candidate solution at t corresponding to τ . We show that $S_{t'} := S_t \setminus \{x\}$ is a valid candidate solution at t' for τ' .

1. Since x is adjacent only to vertices in X_t within G_t , and S_t is a locating-dominating set for $V(G_t) \setminus X_t$, it follows that $S_{t'}$ is a locating-dominating set for $V(G_{t'}) \setminus X_{t'}$.
2. By construction of $S_{t'}$, we have $S_{t'} \cap X_{t'} = Y'$. Additionally, since $W = W' \cup (N_{G_t}(x) \setminus N_{G_t}[Y])$, W' captures precisely those vertices in $X_{t'}$ that are dominated by some vertex in $S_{t'}$. Using the same argument, W'_0 precisely contains the vertices in W' whose neighborhood in the partial solution is in Y' .
3. A subset $A \in \mathcal{Y}$ if and only if there exists a unique vertex $u \in V(G_t) \setminus (S_t \cup X_t)$ such that $N_{G_t}(u) \cap S_t = A$. Since x is not adjacent to any vertex in $V(G_{t'}) \setminus X_{t'}$, the collection of unusable sets remains unchanged, i.e., $\mathcal{Y}' = \mathcal{Y}$.
4. Note that x may locate certain vertex pairs in W' that were not located by $S_{t'}$. Hence, we have $\mathbb{W} \subseteq \mathbb{W}'$. More precisely,
 - For $(w_1, w_2) \in \mathbb{W}' \setminus \mathbb{W}$, this occurs only if exactly one of w_1 or w_2 is adjacent to x .
 - For $(w_1, +) \in \mathbb{W}' \setminus \mathbb{W}$, this occurs only if w_1 is adjacent to x .

Thus, $S_{t'}$ is a valid candidate solution at t' . As S_t is a valid candidate at t , and x is in Y , this implies that x is in S_t , and hence, $|S_{t'}| = |S_t| - 1$. Thus, $\mathbf{d}[t', \tau'] \leq |S_{t'}| = |S_t| - 1 = \mathbf{d}[t, \tau] - 1$. We conclude that $\mathbf{d}[t, \tau] \geq 1 + \mathbf{d}[t', \tau']$.

The above arguments establish the correctness of the recursive computation at introduce nodes when the introduced vertex x is included in the candidate solution.

Case 2: x is in W . Assume that the minimum value of $\min_{\tau' \in I^W(\tau)} \mathbf{d}[t', \tau']$ is finite and attained at τ' and witnessed by a set $S_{t'}$. We argue that $S_t := S_{t'}$ is also a valid candidate solution at t by verifying that it satisfies the conditions in Definition 3.5. We refer the reader to Definition 3.8 for the correspondence between the sets specified by τ and τ' . Moreover, as noted in the discussion preceding Definition 3.8, it suffices to consider the case when $A := N_{G_t}(x) \cap Y \neq \emptyset$.

1. Since $S_{t'}$ locates all vertices in $V(G_{t'}) \setminus X_{t'}$, the same holds for S_t with respect to $V(G_t) \setminus X_t$.
2. As no new vertex is included in the candidate solution, the intersection $S_t \cap X_t$ equals $Y = Y'$. Furthermore, since the newly introduced vertex belongs to W , we have $W' = W \setminus \{x\}$. Note that x can only be adjacent to vertices in S_t that are in X_t , and hence $W_0 = W'_0 \cup \{x\}$.
3. Again, because the candidate solution remains unchanged, the collection of unusable subsets $\mathcal{Y}$ at t is identical to $\mathcal{Y}'$ at t' .

4. As no new vertex is added to the candidate solution, no pairs from $\mathbb{W}'$ are located, and thus $\mathbb{W}' \subseteq \mathbb{W}$. Any pair in $\mathbb{W} \setminus \mathbb{W}'$ necessarily involves x . The other vertex in the pair depends on the structure of the set A . We consider two subcases:

- (a) If $(x, +) \in \mathbb{W} \setminus \mathbb{W}'$, then $A \in \mathcal{Y} = \mathcal{Y}'$; that is, there exists a vertex $u \in V(G_{t'}) \setminus X_{t'}$ such that $N(u) \cap S_{t'} = A$.
- (b) If $(x, w_1) \in \mathbb{W} \setminus \mathbb{W}'$, then $w_1 \in W'$ and $A = N_{G_{t'}}(w_1) \cap S_{t'}$.

This verifies that $S_t = S_{t'}$ is a valid candidate solution for $\mathbf{d}[t, \tau]$, and hence we obtain $\mathbf{d}[t, \tau] \leq \mathbf{d}[t', \tau']$.

Conversely, suppose that S_t is an optimal candidate solution at node t corresponding to tuple τ . We show that $S_{t'} := S_t$ is a valid candidate solution at t' for tuple τ' .

1. Since x is introduced in this bag, $S_{t'}$ locates all vertices in $V(G_{t'}) \setminus X_{t'}$.
2. The partial solution remains unchanged, so $Y = Y'$. By definition, $W = W' \cup \{x\}$, and hence W' consists of all vertices in $V(G_t)$ that are dominated by $S_{t'}$. Also, as x can only be adjacent to vertices in S_t that are in the bag, we have $W_0 = W'_0 \cup \{x\}$.
3. Again, since the partial solution remains unchanged, the set of unusable subsets of Y is preserved.
4. As $W = W' \cup \{x\}$, we again have $\mathbb{W}' \subseteq \mathbb{W}$. Every pair in $\mathbb{W} \setminus \mathbb{W}'$ must involve x , and the other component of the pair depends on how the neighborhood of x intersects Y , as detailed in Definition 3.8.

This confirms that $S_{t'} = S_t$ is a valid candidate solution at t' , and thus $\mathbf{d}[t, \tau] \geq \mathbf{d}[t', \tau']$.

Combining both directions, we establish the correctness of the recursive update at introduce nodes when the newly introduced vertex x is not included in the partial solution but is dominated by some vertex already in the partial solution.

Case 3: x is in Z : If $N_{G_t}[x] \cap Y \neq \emptyset$, then x is either in the candidate solution or adjacent to a vertex in the candidate solution, contradicting the assumption that x lies in Z . In this case, we set $\mathbf{d}[t, \tau] := \infty$. Otherwise, the new vertex x imposes no additional constraint on the partial solution. Therefore, τ is also a valid tuple at t' , and the DP value remains unchanged: $\mathbf{d}[t, \tau] = \mathbf{d}[t', \tau]$.

This completes the proof of the lemma. $\square$

Forget node. Let $t \in V(T)$ be a forget node with unique child t' , such that $X_t = X_{t'} \setminus \{x\}$, for some vertex $x \in X_{t'}$. Let $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ be a valid tuple at node t . The following claim presents the update rule for $\mathbf{d}[t, \tau]$ based on which set x belongs to in τ' . We follow the same template as before, we present definitions of sets F_c^Y and F_c^W which are *compatible* with τ before stating the lemma.

DEFINITION 3.10 (forget- Y -compatible). *Consider tuple $\tau' = \langle Y', W', W'_0, \mathcal{Y}', \mathbb{W}' \rangle$ which is valid at t' . Suppose x is in Y' . We say τ' is forget- Y -compatible with τ if it satisfies the following conditions:*

1. $Y = Y' \setminus \{x\}$, $W = W'$, $W_0 = W'_0 \setminus \{w \in W' \mid x \in N_{G_{t'}}(w)\}$,
2. $\mathcal{Y} = \mathcal{Y}' \setminus \{A \in \mathcal{Y} \mid x \in A\}$, and
3. $\mathbb{W} = \mathbb{W}'$.

Let $F_c^Y(\tau)$ denote the collection of all valid tuples at t' that are forget- Y -compatible with τ .

Consider the case when x is in W' .

DEFINITION 3.11 (forget- W -compatible). Consider tuple $\tau' = \langle Y', W', W'_0, \mathcal{Y}', \mathbb{W}' \rangle$ which is valid at t' . Suppose x is in W' and $(x, +) \notin \mathbb{W}'$. We say that τ' is forget- W -compatible with τ if we have the following:

1. $Y = Y'$, $W = W' \setminus \{x\}$, and $W_0 = W'_0 \setminus \{x\}$,
2. If $x \in W'_0$, then $\mathcal{Y}' = \mathcal{Y} \cup \{A\}$, where $A := N_{G_{t'}}(x) \cap Y'$, because $(x, +) \notin \mathbb{W}'$; otherwise $\mathcal{Y}' = \mathcal{Y}$, and
3. $\mathbb{W}$ is obtained from $\mathbb{W}'$ by replacing every tuple of the form (w_1, x) , for some w_1 in W , by $(w_1, +)$.

Let $F_c^W(\tau)$ denote the collection of all valid tuples at t' that are forget- W -compatible with τ .

LEMMA 3.12. Let t be a forget node with child t' , where $X_t = X_{t'} \setminus \{x\}$, and let $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ be a valid tuple at t . Then,

$$\mathbf{d}[t, \tau] = \begin{cases} \min_{\tau' \in F_c^Y(\tau)} \mathbf{d}[t', \tau'] & \text{if } x \in Y', \\ \min_{\tau' \in F_c^W(\tau)} \mathbf{d}[t', \tau'] & \text{if } x \in W' \text{ and } N_{G_{t'}}(x) \cap Y' \neq \emptyset; \text{ and } (x, +) \notin \mathbb{W}', \\ \infty & \text{if } x \in Z', \\ \infty & \text{otherwise.} \end{cases}$$

Proof. We consider the three mutually disjoint and exhaustive cases depending on whether x is in Y' , W' , or $X_{t'} \setminus (Y' \cup W')$ in the sets specified by τ' .

Case 1: x is in Y' . Suppose that the minimum value of $\min_{\tau' \in F_c^Y(\tau)} \mathbf{d}[t', \tau']$ is finite and attained at τ' , and let $S_{t'}$ be the corresponding witnessing set. Define $S_t := S_{t'}$. We now verify that S_t constitutes a valid candidate solution at node t by checking that it satisfies the conditions in Definition 3.5. The relationship between the sets corresponding to τ and τ' is specified by Definition 3.10.

1. Since $S_{t'}$ locates all vertices in $V(G_{t'}) \setminus X_{t'}$, and $(V(G_{t'}) \setminus X_{t'}) \cup \{x\} = V(G_t) \setminus X_t$, it follows that S_t locates all vertices in $V(G_t) \setminus X_t$.
2. As $X_t = X_{t'} \setminus \{x\}$ and $Y = Y' \setminus \{x\}$, the intersection of S_t with X_t is precisely Y . Furthermore, since the partial solution remains unchanged, the set of vertices in $X_t \setminus Y$ that are dominated by S_t is denoted by W . Every vertex in W' that is adjacent to x is now adjacent to some vertex in the partial solution which is not in the bag, and hence $W_0 = W'_0 \setminus \{w \in W' \mid x \in N_{G_{t'}}(w)\}$.
3. Since x is the forgotten vertex, the collection of unusable subsets $\mathcal{Y}$ at node t is derived from $\mathcal{Y}'$ by removing all subsets that contain x .
4. As the partial solution remains unchanged, the set of vertex pairs that are not located also remains identical.

Consequently, $S_t = S_{t'}$ is a valid candidate solution for $\mathbf{d}[t, \tau]$, and we obtain $\mathbf{d}[t, \tau] \leq \mathbf{d}[t', \tau']$.

Conversely, suppose that S_t is an optimal candidate solution at node t corresponding to τ , and that $x \in S_t$. We demonstrate that the set $S_{t'} := S_t$ constitutes a valid candidate solution at node t' for τ' .

1. Since S_t locates all vertices in $V(G_t) \setminus X_t$, and $(V(G_t) \setminus X_t) \cup \{x\} = V(G_{t'}) \setminus X_{t'}$, it follows that $S_{t'}$ locates all vertices in $V(G_{t'}) \setminus X_{t'}$.
2. As $X_t = X_{t'} \setminus \{x\}$ and $Y = Y' \setminus \{x\}$, the intersection of $S_{t'}$ with $X_{t'}$ is precisely Y' . Additionally, as the partial solution remains unchanged, the set of vertices in $X_{t'} \setminus Y'$ dominated by $S_{t'}$ is denoted by W' . Any vertex which is in W'_0 but not in W_0 must be adjacent to x , and hence $W_0 = W'_0 \setminus \{w \in W' \mid x \in N_{G_{t'}}(w)\}$.

3. Again, since the partial solutions are identical, the collection of unusable subsets at t' differs from that at t only by the exclusion of those subsets that include x .
4. Likewise, the collection of vertex pairs that must be located via the extension of the partial solution remains the same.

Hence, $S_{t'}$ is a valid candidate solution at node t' , which implies $\mathbf{d}[t, \tau] \geq \mathbf{d}[t', \tau']$.

Combining both directions of the argument, we conclude that the recursive computation at forget nodes correctly preserves the minimum when the forgotten vertex x is included in the partial solution.

Case 2: x is in W' . Suppose that the minimum value of $\min_{\tau' \in I^Y(\tau)} \mathbf{d}[t', \tau']$ is finite and attained at some τ' , and let $S_{t'}$ be the corresponding witnessing set. Define $S_t := S_{t'}$. We now verify that S_t constitutes a valid candidate solution at node t by checking that it satisfies the conditions specified in Definition 3.5. The relationship between the sets defined by τ and τ' is specified by Definition 3.11.

Note that if $(x, +) \in \mathbb{W}'$, there are two vertices in G_t , one of them being x , that have identical neighborhoods in $S_{t'} = S_t$. As no new vertex can be adjacent to either of these vertices, these two will have same neighborhood in any extension of S_t and thus, $\mathbf{d}[t, \tau] := \infty$. Hence, we may assume next that $(x, +) \notin \mathbb{W}'$.

1. As $V(G_{t'}) \setminus X_{t'}$ and $V(G_t) \setminus X_t$ differ only by the vertex x , to prove that S_t locates $V(G_t) \setminus X_t$, it suffices to prove that it locates x . This follows from the fact that $(x, +) \notin \mathbb{W}'$, i.e., no vertex $u \in V(G_{t'})$ satisfies $N_{G_t}(u) \cap S_t = N_{G_t}(x) \cap S_t$.
2. Since the partial solution remains unchanged, we have $S_t \cap X_t = Y = Y'$. Moreover, because the forgotten vertex belongs to W , we have $W = W' \setminus \{x\}$ and $W_0 = W'_0 \setminus \{x\}$.
3. If $x \in W_0$, then x is adjacent only to vertices in $A = N_{G_t}(x) \cap S_t$. Consequently, A becomes unusable for locating purposes, and we set $\mathcal{Y} = \mathcal{Y}' \cup \{A\}$. Otherwise, we retain $\mathcal{Y} = \mathcal{Y}'$.
4. Any pair present in exactly one of $\mathbb{W}$ and $\mathbb{W}'$ necessarily involves the vertex x . Since $(x, +) \notin \mathbb{W}'$, and for every pair $(w_1, x) \in \mathbb{W}'$, the vertex x is forgotten at node t , such a pair must be replaced by $(w_1, +)$ in $\mathbb{W}$, according to the convention.

Thus, $S_t = S_{t'}$ constitutes a valid candidate solution for $\mathbf{d}[t, \tau]$, and we have $\mathbf{d}[t, \tau] \leq \mathbf{d}[t', \tau']$.

Conversely, suppose that S_t is an optimal candidate solution at node t for tuple τ such that $x \notin S_t$. We now demonstrate that $S_{t'} := S_t$ is a valid candidate solution at node t' for the corresponding tuple τ' .

1. Since x is forgotten in this bag, we have $V(G_{t'}) \setminus X_{t'} \subseteq V(G_t) \setminus X_t$, and therefore $S_{t'}$ still locates all vertices in $V(G_{t'}) \setminus X_{t'}$.
2. The partial solution remains unchanged, implying $Y = Y'$. By definition, $W = W' \setminus \{x\}$, and thus W' consists of all vertices in $V(G_t)$ that are dominated by $S_{t'}$. Using similar arguments, $W_0 = W'_0 \setminus \{x\}$.
3. Define $A := S_t \cap N_{G_t}(x)$. If $A \subseteq Y$, then A is marked unusable and $A \in \mathcal{Y}$. However, at node t' , where x is included in the bag, the set A becomes usable, and hence $\mathcal{Y}' = \mathcal{Y} \setminus \{A\}$. Otherwise, the set of unusable sets remains unchanged.
4. Since $W' = W \cup \{x\}$, any discrepancy between $\mathbb{W}$ and $\mathbb{W}'$ involves the vertex x . Suppose that there exists $(w_1, +) \in \mathbb{W}$ such that $N_{G_t}(x) \cap S_t = A$. Then, at node t' , the vertices x and w_1 share identical neighborhoods in S_t . As S_t locates $V(G_t) \setminus X_t$, x is the unique vertex whose neighborhood within

S_t is A . Consequently, $\mathbb{W}'$ is obtained from $\mathbb{W}$ by replacing each such pair $(w_1, +)$ with (w_1, x) .

Hence, $S_{t'} = S_t$ is a valid candidate solution at node t' , and we have $\mathbf{d}[t, \tau] \geq \mathbf{d}[t', \tau']$.

By combining both directions, we establish the correctness of the recursive update at forget nodes in the case where the forgotten vertex x is not part of the candidate solution but is dominated by some vertex already included in the candidate solution.

Case 3: x is in Z' . If $N_{G_{t'}}[x] \cap Y' \neq \emptyset$, then either x is in the partial solution or adjacent to some vertex in it, violating the definition of set Y' or W' . In this case, set $\mathbf{d}[t, \tau] := \infty$. Otherwise, the forgotten vertex x was not adjacent to any vertex of the partial solution, and any extension of it will remain invalid, as x will remain undominated. Hence, also in this case $\mathbf{d}[t, \tau] := \infty$.

This concludes the proof of the lemma. $\square$

Join node. Let t be a join node with children t^1 and t^2 , where $X_t = X_{t^1} = X_{t^2}$, and let $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ be a valid tuple at t .

A crucial property of tree-decompositions is that the intersection of any two bags corresponding to adjacent nodes in the decomposition tree forms a separator in G (see, for example, the book [26, Lemma 7.1]). Hence, if $V(G_{t^1}) \setminus X_t$ and $V(G_{t^2}) \setminus X_t$ are both nonempty, then X_t forms a separator between the two corresponding subgraphs. We will use this implicitly in the proofs below.

We start with some useful definitions.

DEFINITION 3.13 (pairwise join-compatible). *Consider two tuples $\tau^1 = \langle Y^1, W^1, W_0^1, \mathcal{Y}^1, \mathbb{W}^1 \rangle$ and $\tau^2 = \langle Y^2, W^2, W_0^2, \mathcal{Y}^2, \mathbb{W}^2 \rangle$, valid at nodes t^1 and t^2 , respectively. We say that τ^1 and τ^2 are pairwise join-compatible, denoted by $\langle \tau^1, \tau^2 \rangle$, if the following conditions hold:*

1. $Y^1 = Y^2$,
2. $\mathcal{Y}^1 \cap \mathcal{Y}^2 = \emptyset$, and
3. for any $w \in W_0^1 \cap W_0^2$, the pair $(w, +)$ appears in at most one of the sets $\mathbb{W}^1$ and $\mathbb{W}^2$.

Note that even though $Y^1 = Y^2$, it may happen that $W^1 \setminus N(Y^1) \neq W^2 \setminus N(Y^2)$, and thus $W^1 \neq W^2$, because W^1 and W^2 , respectively, denote the vertices in X_{t^1} and X_{t^2} that are dominated by vertices in the partial solutions of G_{t^1} and G_{t^2} . Additionally, there may exist a vertex $w \in W_0^1$ such that $w \notin W_0^2$ (or vice versa), as w could be adjacent to some vertex in $V(G_{t^2}) \setminus X_{t^2}$ that is part of the partial solution.

DEFINITION 3.14 (join-compatible). *We say that τ is join-compatible with the pair $\langle \tau^1, \tau^2 \rangle$ of pairwise join-compatible tuples valid at t^1 and t^2 , respectively, if the following conditions are satisfied:*

1. $Y = Y^1 = Y^2$, $W = W^1 \cup W^2$, and $W_0 = W_0^1 \cap W_0^2$,
 2. $\mathcal{Y} = \mathcal{Y}^1 \cup \mathcal{Y}^2$, and
 3. (a) A pair (w, w') is in $\mathbb{W}$ if and only one of the following conditions is satisfied:
 - If $w \in W^1 \cap W^2$, then $(w, w') \in \mathbb{W}^1 \cap \mathbb{W}^2$.
 - If $w \in W^1 \setminus W^2$, then (w, w') is in $\mathbb{W}^1$ and not in $\mathbb{W}^2$.
(This refers to the case when w' is adjacent to some vertex in the partial solution in G_{t^1} but not adjacent to any vertex of the partial solution in G_{t^2} .)
 - If $w \in W^2 \setminus W^1$, then (w, w') is in $\mathbb{W}^2$ and not in $\mathbb{W}^1$.
- Note that for any set in $\{W^1 \cap W^2, W^1 \setminus W^2, W^2 \setminus W^1\}$, w is present in the set if and only if w' is in it as well.

- (b) A pair $(w, +)$ is in $\mathbb{W}$ if and only if of the following conditions is satisfied:
- If $w \in W^1 \cap W^2$, then consider the following subcases:
 - If $w \in W_0^1 \cap W_0^2$, then $(w, +)$ is either in $\mathbb{W}^1$ or $\mathbb{W}^2$, but not both.
 - If $w \in W_0^1$ and $w \in W^2 \setminus W_0^2$, then $(w, +)$ is in $\mathbb{W}^2$ (and may be in $\mathbb{W}^1$).
 - If $w \in W_0^2$ and $w \in W^1 \setminus W_0^1$, then $(w, +)$ is in $\mathbb{W}^1$ (and may be in $\mathbb{W}^2$).
 - If $w \in W^1 \setminus W^2$, then $(w, +)$ is in $\mathbb{W}^1$ and not in $\mathbb{W}^2$.
 - If $w \in W^2 \setminus W^1$, then $(w, +)$ is in $\mathbb{W}^2$ and not in $\mathbb{W}^1$.

Let $J(\tau)$ denote the collection of pairs $\langle \tau^1, \tau^2 \rangle$ that are join-compatible with τ .

LEMMA 3.15. Let t be a join node with children t^1, t^2 , where $X_t = X_{t^1} = X_{t^2}$, and let $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ be a valid tuple at t . Then,

$$d[t, \tau] = \min_{\langle \tau^1, \tau^2 \rangle \in J(\tau)} \{d[t^1, \tau^1] + d[t^2, \tau^2]\} - |Y|.$$

Proof. Suppose that the minimum value of $\min_{\langle \tau^1, \tau^2 \rangle \in J(\tau)} \{d[t^1, \tau^1] + d[t^2, \tau^2]\}$ is finite and attained at the pair (τ^1, τ^2) . Let S_{t^1} and S_{t^2} be the corresponding partial solutions that realize this minimum. Define $S_t := S_{t^1} \cup S_{t^2}$. We now verify that S_t is a valid candidate solution at node t by checking that it satisfies the requirements of Definition 3.5. The relationships among the sets specified by τ , τ^1 , and τ^2 are governed by Definition 3.13 and Definition 3.14.

1. By definition, S_{t^1} and S_{t^2} locate all vertices in $V(G_{t^1}) \setminus X_t$ and $V(G_{t^2}) \setminus X_t$, respectively. Since $V(G_{t^1}) \setminus X_t$ and $V(G_{t^2}) \setminus X_t$ are disjoint, and only vertices in X_t can have neighbors in both subgraphs, it follows that S_t locates any $u \in V(G_t) \setminus X_t$ provided $N_{G_t}(u) \cap S_t$ is not entirely contained in X_t . Moreover, as τ^1 and τ^2 are pairwise compatible, we have $\mathcal{Y}^1 \cap \mathcal{Y}^2 = \emptyset$. Equivalently, for any $A \subseteq Y$, there exists at most one vertex in $V(G_t) \setminus X_t$ with neighborhood A in G_t . Therefore, S_t indeed locates every vertex in $V(G_t) \setminus X_t$.
2. Since $Y = Y^1 = Y^2$, the intersection of S_t with X_t is precisely Y . Furthermore, $W = W^1 \cup W^2$ denotes exactly the vertices of X_t dominated by S_t , and $W_0 = W_0^1 \cap W_0^2$ consists of those vertices whose neighbors in S_t lie entirely within Y .
3. No vertex in $V(G_{t^1}) \setminus X_t$ is adjacent to any vertex in $V(G_{t^2}) \setminus X_t$. This implies that no vertex in $V(G_{t^1}) \setminus (S_{t^1} \cup X_t)$ is adjacent with a vertex in $S_{t^2} \setminus X_t$. A symmetric statement holds for vertices in $V(G_{t^2}) \setminus (S_{t^2} \cup X_t)$. Consequently, the set of unusable subsets at t is given by $\mathcal{Y} = \mathcal{Y}^1 \cup \mathcal{Y}^2$.
4. It remains to verify that the set of unresolved pairs in X_t with respect to S_t are given by the set $\mathbb{W}$, as prescribed in Definition 3.14.

(a) Consider $(w, w') \in \mathbb{W}$ with $w, w' \in W$. This pair is unresolved by S_t if and only if $N_{G_t}(w) \cap S_t = N_{G_t}(w') \cap S_t$. For any of the three sets $\{W^1 \cap W^2, W^1 \setminus W^2, W^2 \setminus W^1\}$, the membership of w and w' is identical.

 - If $w \in W^1 \cap W^2$, let $N_i(v) := N_{G_{t^i}}(v) \cap S_{t^i}$ for $i \in \{1, 2\}$. Then $N_{G_t}(v) \cap S_t = N_1(v) \cup N_2(v)$, and the disjointness of $S_{t^1} \setminus X_t$ and $S_{t^2} \setminus X_t$ implies $N_1(w) \cup N_2(w) = N_1(w') \cup N_2(w')$ if and only if $N_1(w) = N_1(w')$ and $N_2(w) = N_2(w')$. Thus, (w, w') is unresolved by S_t if and only if it is unresolved in both S_{t^1} and S_{t^2} , i.e., $(w, w') \in \mathbb{W}^1 \cap \mathbb{W}^2$.
 - If $w \in W^1 \setminus W^2$, then $(w, w') \notin \mathbb{W}^2$, implying it is unresolved solely in S_{t^1} , hence $(w, w') \in \mathbb{W}^1$.
 - If $w \in W^2 \setminus W^1$, the argument is symmetric to the previous case.

- (b) For a vertex $w \in W$, $(w, +) \in \mathbb{W}$ if w shares the same neighborhood in S_t as some $u \in V(G_t) \setminus (S_t \cup X_t)$. Let u_1 (respectively, u_2) be such a vertex in $V(G_{t^1}) \setminus (S_{t^1} \cup X_{t^1})$ (respectively, $V(G_{t^2}) \setminus (S_{t^2} \cup X_{t^2})$), if it exists. Define $A_1 := N_{G_{t^1}}(w) \cap S_{t^1}$ and $A_2 := N_{G_{t^2}}(w) \cap S_{t^2}$.

- If $w \in W^1 \cap W^2$, we have the following:
 - If $w \in W_0^1 \cap W_0^2$, then both u_1 and u_2 cannot exist, else it contradicts that S_t locates all vertices in $V(G_t) \setminus X_t$. Hence, $(w, +)$ lies in exactly one of $\mathbb{W}^1$ or $\mathbb{W}^2$, but not both.
 - If $w \in W_0^1$ and $w \in W^2 \setminus W_0^2$, then A_1 is in Y and is a proper subset of A_2 . Hence, a vertex in $A_2 \setminus A_1$ resolves the pair w, u_1 , if it exists. This implies that $(w, +)$ is in $\mathbb{W}^2$. Note that $(w, +)$ is in $\mathbb{W}^1$ if and only if u_1 exists.
 - If $w \in W_0^2$ and $w \in W^1 \setminus W_0^1$, the argument is symmetric.
 - If $w \in W^1 \setminus W_0^1$ and $w \in W^2 \setminus W_0^2$, then a vertex in $A_1 \setminus A_2$ resolves the pair (w, u_2) and a vertex in $A_2 \setminus A_1$ resolves the pair (w, u_1) . This implies that $(w, +)$ is *not* present in $\mathbb{W}$.
- Suppose that $w \in W^1 \setminus W^2$. In this case, pair $(w, +)$ is not present in W^2 . Hence, $u = u_1$ which implies that $(w, +)$ is in $\mathbb{W}^1$.
- If $w \in W^2 \setminus W^1$, then $(w, +)$ is in $\mathbb{W}^2$ (only and not in $\mathbb{W}^1$). This case follows using symmetric arguments as in the previous case.

Thus, S_t is a valid candidate solution corresponding to the tuple τ . Its size is given by $|S_t| = |S_{t^1}| + |S_{t^2}| - |Y|$. Minimizing over all join-compatible pairs $\langle \tau^1, \tau^2 \rangle \in J(\tau)$ yields $\mathbf{d}[t, \tau] \leq \min_{\langle \tau^1, \tau^2 \rangle \in J(\tau)} \{ \mathbf{d}[t^1, \tau^1] + \mathbf{d}[t^2, \tau^2] \} - |Y|$.

Conversely, let S_t be an optimal candidate solution for the tuple τ at node t . By definition, $|S_t| = \mathbf{d}[t, \tau]$, and S_t satisfies all conditions specified in Definition 3.5 for τ . Define $S_{t^1} := S_t \cap V(G_{t^1})$, and $S_{t^2} := S_t \cap V(G_{t^2})$. We first establish certain properties of S_{t^1} and S_{t^2} , and then define the tuples $\tau_1 := \langle Y^1, W^1, W_0^1, \mathcal{Y}^1, \mathbb{W}^1 \rangle$ and $\tau_2 := \langle Y^2, W^2, W_0^2, \mathcal{Y}^2, \mathbb{W}^2 \rangle$.

1. S_{t^1} locates (in G_{t^1}) all vertices in $V(G_{t^1}) \setminus X_{t^1}$, and S_{t^2} locates (in G_{t^2}) all vertices in $V(G_{t^2}) \setminus X_{t^2}$. This follows since S_t locates all vertices in $V(G_t)$, and no vertex in $V(G_{t^1}) \setminus X_{t^1}$ is adjacent to any vertex in $V(G_{t^2}) \setminus X_{t^2}$.
2. Define $Y^1 := S_{t^1} \cap X_{t^1}$. Let W^1 be the set of vertices in $X_{t^1} \setminus Y^1$ that are dominated by S_{t^1} . Let $W_0^1 \subseteq W^1$ be the set of vertices whose neighborhood in S_{t^1} is contained in Y^1 . Define Y^2, W^2, W_0^2 analogously for t^2 .
3. Define $\mathcal{Y}^1$ as the collection of subsets $A \subseteq Y$ such that there exists a vertex $u \in V(G_{t^1}) \setminus (S_{t^1} \cup X_{t^1})$ with $N_{G_{t^1}}(u) \cap S_t = A$. Since S_{t^1} locates every vertex in $V(G_{t^1}) \setminus X_{t^1}$, such a vertex u (if it exists) is unique. Define $\mathcal{Y}^2$ analogously.
4. • For $w, w' \in W^1$, add the pair (w, w') to $\mathbb{W}^1$ if and only if $N_{G_{t^1}}(w) \cap S_{t^1} = N_{G_{t^1}}(w') \cap S_{t^1}$.
 • For $w \in W^1$, add the pair $(w, +)$ to $\mathbb{W}^1$ if and only if there exists $u \in V(G_{t^1}) \setminus (S_{t^1} \cup X_{t^1})$ such that $N_{G_{t^1}}(w) \cap S_t = N_{G_{t^1}}(u) \cap S_{t^1}$.

Define $\mathbb{W}^2$ analogously.

By Definition 3.4, τ_1 and τ_2 are valid tuples at t^1 and t^2 , respectively. By Definition 3.5, S_{t^1} and S_{t^2} are candidate solutions at t^1 with respect to τ_1 , and at t^2 with respect to τ_2 , respectively.

Next, we prove that τ_1 and τ_2 are pairwise join-compatible by showing that they satisfy the properties in Definition 3.13.

1. By construction, $Y^1 = Y^2$.
2. Suppose that $\mathcal{Y}^1 \cap \mathcal{Y}^2 \neq \emptyset$. Then there exist vertices $u_1 \in V(G_{t^1}) \setminus X_{t^1}$ and $u_2 \in V(G_{t^2}) \setminus X_{t^2}$ such that their neighborhoods in S_t are identical and

contained in Y . This contradicts the fact that S_t locates every vertex in $V(G_t) \setminus X_t$. Therefore, $\mathcal{Y}^1 \cap \mathcal{Y}^2 = \emptyset$.

3. Suppose that there exists $w \in W_0^1 \cap W_0^2$ such that $(w, +) \in \mathbb{W}^1 \cap \mathbb{W}^2$. Then there exist $u_1 \in V(G_{t_1}) \setminus X_{t_1}$ and $u_2 \in V(G_{t_2}) \setminus X_{t_2}$ whose neighborhoods in S_t are identical to that of w and contained in Y . This again contradicts the fact that S_t locates every vertex in $V(G_t) \setminus X_t$.

Hence, τ_1 and τ_2 are pairwise join-compatible.

Next, we establish that $\langle \tau^1, \tau^2 \rangle$ is join-compatible with τ by verifying that it satisfies all the properties stated in Definition 3.14.

1. By definition, $Y = Y^1 = Y^2$.

For any vertex $w \in V(G_t)$, we have $N_{G_t}[w] \cap S_t = (N_{G_{t_1}}[w] \cap S_{t_1}) \cup (N_{G_{t_2}}[w] \cap S_{t_2})$. Consequently, w is dominated by S_t if and only if it is dominated by either S_{t_1} or S_{t_2} . By definition, W^1 and W^2 are precisely the sets of vertices in X_t dominated by S_{t_1} and S_{t_2} , respectively. Therefore, $W = W^1 \cup W^2$.

A vertex $w \in X_t$ belongs to W_0 if all its neighbors in S_t are contained in Y , i.e., $N_{G_t}(w) \cap S_t \subseteq Y$. This holds if and only if both $N_{G_{t_1}}(w) \cap S_{t_1} \subseteq Y$ and $N_{G_{t_2}}(w) \cap S_{t_2} \subseteq Y$, which are precisely the conditions for $w \in W_0^1$ and $w \in W_0^2$, respectively. Thus, $W_0 = W_0^1 \cap W_0^2$.

2. Let $A \in \mathcal{Y}$, and suppose that $u \in V(G_t) \setminus (S_t \cup X_t)$ is such that $N_{G_t}(u) \cap S_t = A$.

The set of such vertices u is the disjoint union of the following:

- those in $V(G_{t_1}) \setminus (S_{t_1} \cup X_t)$, for which $N_{G_t}(u) \cap S_t = N_{G_{t_1}}(u) \cap S_{t_1} \in \mathcal{Y}^1$;
- those in $V(G_{t_2}) \setminus (S_{t_2} \cup X_t)$, for which $N_{G_t}(u) \cap S_t = N_{G_{t_2}}(u) \cap S_{t_2} \in \mathcal{Y}^2$.

Therefore, $\mathcal{Y} = \mathcal{Y}^1 \cup \mathcal{Y}^2$:

3. (a) Since $W = W^1 \cup W^2$, consider any $w, w' \in W$. If $(w, w') \in \mathbb{W}$, then for each set in $\{W^1 \cap W^2, W^1 \setminus W^2, W^2 \setminus W^1\}$, the membership of w coincides with that of w' .

- If $w, w' \in W^1 \cap W^2$, then $(w, w') \in \mathbb{W}$ if and only if $(w, w') \in \mathbb{W}^1 \cap \mathbb{W}^2$. Indeed, $(w, w') \in \mathbb{W}$ means $N_{G_t}(w) \cap S_t = N_{G_t}(w') \cap S_t$, which is equivalent to $N_{G_{t_1}}(w) \cap S_{t_1} = N_{G_{t_1}}(w') \cap S_{t_1}$ and $N_{G_{t_2}}(w) \cap S_{t_2} = N_{G_{t_2}}(w') \cap S_{t_2}$.
- If $w, w' \in W^1 \setminus W^2$, then $(w, w') \in \mathbb{W}$ if and only if $(w, w') \in \mathbb{W}^1$.
- If $w, w' \in W^2 \setminus W^1$, the statement follows symmetrically from the previous case.

- (b) Let $w \in W$, and let u_1 (respectively, u_2) be a vertex, if it exists, in $V(G_{t_1}) \setminus (S_{t_1} \cup X_t)$ (respectively, $V(G_{t_2}) \setminus (S_{t_2} \cup X_t)$) whose neighborhood in S_{t_1} (respectively, S_{t_2}) matches that of w . Denote these neighborhoods by A_1 and A_2 , respectively. Similarly, let u (if it exists) be the vertex in $V(G_t) \setminus (S_t \cup X_t)$ whose neighborhood in S_t matches that of w . By construction, u is either u_1 or u_2 .

- Suppose that $w \in W^1 \cap W^2$; then we consider the following subcases:
 - Suppose that $w \in W_0^1 \cap W_0^2$. Vertex u exists if and only if exactly one of u_1 and u_2 exists, as otherwise it contradicts the fact that S_t locates in G_t every vertex in $V(G_t) \setminus X_t$. Hence, $(w, +) \in \mathbb{W}$ if and only if $(w, +)$ is in either in $\mathbb{W}^1$ or $\mathbb{W}^2$ (but not both).
 - Suppose that $w \in W_0^1$ and $w \in W^2 \setminus W_0^2$. This implies A_1 is in Y and is a proper subset of A_2 . Hence, a vertex in $A_2 \setminus A_1$ resolve the pair w, u_1 . This implies u exists if and only if u_2 exists. Hence, $(w, +)$ is in $\mathbb{W}$ if and only if $(w, +)$ is in $\mathbb{W}^2$. Note that depending on whether u_1 exists or not, $(w, +)$ may be present in $\mathbb{W}^1$.
 - Suppose that $w \in W_0^2$ and $w \in W^1 \setminus W_0^1$. This case follows using symmetric arguments as in the previous case.

- Suppose that $w \in W^1 \setminus W_0^1$ and $w \in W^2 \setminus W_0^2$. In this case, a vertex in $A_1 \setminus A_2$ resolves the pair w, u_2 and a vertex in $A_2 \setminus A_1$ resolves the pair w, u_1 . This implies that $w, +$ is *not* present in $\mathbb{W}$.
- Suppose that $w \in W^1 \setminus W^2$. In this case, u_2 does not exist, and hence u exists if and only if u_1 exists. Hence $(w, +)$ is in $\mathcal{W}$ if and only if $(w, +)$ is in $\mathbb{W}^1$.
- Suppose that $w \in W^2 \setminus W^1$. This case follows using symmetric arguments as in the previous case.

From the above, it follows that τ is join-compatible with the pair $\langle \tau^1, \tau^2 \rangle$.

In summary, S_{t^1} and S_{t^2} are valid candidate solutions for τ^1 and τ^2 , respectively, and $\langle \tau^1, \tau^2 \rangle$ is join-compatible with τ . Moreover, $|S_t| = |S_{t^1} \cup S_{t^2}| = |S_{t^1}| + |S_{t^2}| - |S_{t^1} \cap S_{t^2}| = |S_{t^1}| + |S_{t^2}| - |Y|$. Since $\mathbf{d}[t^1, \tau^1]$ and $\mathbf{d}[t^2, \tau^2]$ denote the sizes of a minimum candidate solutions for τ^1 and τ^2 , respectively, we have $|S_{t^1}| \geq \mathbf{d}[t^1, \tau^1]$ and $|S_{t^2}| \geq \mathbf{d}[t^2, \tau^2]$. Substituting into the above expression yields $\mathbf{d}[t, \tau] = |S_t| \geq \mathbf{d}[t^1, \tau^1] + \mathbf{d}[t^2, \tau^2] - |Y|$. This inequality holds for any $\langle \tau^1, \tau^2 \rangle \in J(\tau)$, where $J(\tau)$ denotes the set of all pairs satisfying Definition 3.14. Therefore, $\mathbf{d}[t, \tau] \geq \min_{\langle \tau'^1, \tau'^2 \rangle \in J(\tau)} \{ \mathbf{d}[t^1, \tau'^1] + \mathbf{d}[t^2, \tau'^2] \} - |Y|$. This completes the proof of the reverse inequality. By combining both directions, we conclude the correctness of the recursive update at join nodes. $\square$

We are now in a position to present the proof of Theorem 3.3 which states that LOCATING-DOMINATING SET admits an algorithm with running time $2^{2^{\mathcal{O}(\text{tw})}} \cdot n$.

Proof of Theorem 3.3. The algorithm first computes a nice tree-decomposition of width $\mathcal{O}(\text{tw})$ with $\mathcal{O}(n)$ nodes in time $2^{\mathcal{O}(\text{tw})} \cdot n$ [49, 51]. It then evaluates the DP table in a bottom-up manner, starting from the leaf nodes of the decomposition and proceeding toward the root node r . Finally, it outputs the set corresponding to $\mathbf{d}[r, \tau]$, where $\tau = \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$.

The fact that each DP-state can be computed correctly follows from Lemmas 3.6, 3.9, 3.12, and 3.15. Together with the properties of nice tree-decompositions, these lemmas imply that the set returned by the algorithm is indeed an optimal locating-dominating set of G . This establishes the correctness of the algorithm.

We now analyze the running time. The total number of possible DP-states at a node is bounded by $2^{2^{\mathcal{O}(\text{tw})}}$, and all other operations at a node can be performed in time polynomial in the number of states. Therefore, since there are $\mathcal{O}(n)$ nodes, the overall running time is $2^{2^{\mathcal{O}(\text{tw})}} \cdot n$, as claimed. This concludes the proof of Theorem 5.1. $\square$

3.2. Lower bound. In this subsection, we prove the lower bound for LOCATING-DOMINATING SET mentioned in Theorem 1.1. For convenience, we restate this part of the statement as follows.

THEOREM 3.16 (restating part of Theorem 1.1). *Unless the ETH fails, LOCATING-DOMINATING SET does not admit an algorithm running in time $2^{2^{\mathcal{O}(\text{tw})}} \cdot \text{poly}(n)$, where tw is the treewidth and n is the order of the input graph.*

To prove the above theorem, we present a reduction from a variant of 3-SATISFIABILITY (3-SAT) called (3,3)-SAT. In this variation, an input is a boolean satisfiability formula ψ in conjunctive normal form such that each clause contains *at most*⁴ 3 variables, and each variable appears at most 3 times. Consider the following reduction from an instance ϕ of 3-SAT with n variables and m clauses to an instance

⁴We remark that if each clause contains *exactly* 3 variables, and each variable appears 3 times, then the problem is polynomial-time solvable [71, Theorem 2.4]

ψ of (3,3)-SAT mentioned in [71]: For every variable x_i that appears $k > 3$ times, the reduction creates k many new variables $x_i^1, x_i^2, \dots, x_i^k$, replaces the j th occurrence of x_i by x_i^j , and adds the series of new clauses to encode $x_i^1 \Rightarrow x_i^2 \Rightarrow \dots \Rightarrow x_i^k \Rightarrow x_i^1$. For an instance ψ of 3-SAT, suppose k_i denotes the number of times a variable x_i appeared in ϕ . Then, $\sum_{i \in [n]} k_i \leq 3 \cdot m$. Hence, the reduced instance ψ of (3,3)-SAT has at most $3m$ variables and $4m$ clauses. Using the ETH [45] and the sparsification lemma [46], we have the following result.

PROPOSITION 3.17. (3,3)-SAT, with n variables and m clauses, does not admit an algorithm running in time $2^{o(m+n)}$, unless the ETH fails.

We highlight that every variable appears positively and negatively at least once. Otherwise, if a variable appears only positively (respectively, only negatively), then we can assign it **True** (respectively, **False**) and safely reduce the instance by removing the clauses containing this variable. Hence, instead of the first, second, or third appearance of the variable, we use the first positive, first negative, second positive, or second negative appearance of the variable.

Reduction. The reduction takes as input an instance ψ of (3,3)-SAT with n variables and outputs an instance (G, k) of LOCATING-DOMINATING SET such that $\text{tw}(G) = \mathcal{O}(\log(n))$. Suppose $X = \{x_1, \dots, x_n\}$ is the collection of variables and $C = \{C_1, \dots, C_m\}$ is the collection of clauses in ψ . Here, we consider $\langle x_1, \dots, x_n \rangle$ and $\langle C_1, \dots, C_m \rangle$ to be arbitrary but fixed orderings of variables and clauses in ψ . For a particular clause, the first order specifies the first, second, or third (if it exists) variable in the clause in a natural way. The second ordering specifies the first/second positive/negative appearance of variables in X in a natural way.

We will make use of *set-representation gadgets* as defined in [36]. Intuitively speaking, such gadgets form a logarithmic-size separator that allows to connect two linear-size sets of vertices and encode their interactions.

The reduction constructs a graph G as follows:

- To construct a variable gadget for x_i , it starts with two claws $\{\alpha_i^0, \alpha_i^1, \alpha_i^2, \alpha_i^3\}$ and $\{\beta_i^0, \beta_i^1, \beta_i^2, \beta_i^3\}$ centered at α_i^0 and β_i^0 , respectively. (Recall that a claw is the star $K_{1,3}$.) It then adds four vertices $x_i^1, \neg x_i^1, x_i^2, \neg x_i^2$, and the corresponding edges, as shown in Figure 2. Let A_i be the collection of these twelve vertices and $A = \bigcup_{i=1}^n A_i$. Define $X_i := \{x_i^1, \neg x_i^1, x_i^2, \neg x_i^2\}$.
- To construct a clause gadget for C_j , the reduction starts with a star graph centered at γ_j^0 and with four leaves $\{\gamma_j^1, \gamma_j^2, \gamma_j^3, \gamma_j^4\}$. It then adds three vertices c_j^1, c_j^2, c_j^3 and the corresponding edges shown in Figure 2. Let B_j be the collection of these eight vertices and define $B = \bigcup_{j=1}^m B_j$.
- Let p be the smallest positive integer such that $4n \leq \binom{2p}{p}$. Define $\mathcal{F}_p$ as the collection of subsets of $[2p]$ that contains exactly p integers (such a collection $\mathcal{F}_p$ is called a *Sperner family*). Define **set-rep**: $\bigcup_{i=1}^n X_i \rightarrow \mathcal{F}_p$ as an injective function by arbitrarily assigning a set in $\mathcal{F}_p$ to a vertex x_i^ℓ or $\neg x_i^\ell$, for every $i \in [n]$ and $\ell \in [2]$. In other words, every appearance of a literal is assigned a distinct subset in $\mathcal{F}_p$.
- The reduction adds a *connection portal* V , which is a clique on $2p$ vertices $v_1, v_2, \dots, v_{2p}$. For every vertex v_q in V , the reduction adds a pendant vertex u_q adjacent to v_q .
- For each vertex $x_i^\ell \in X$ where $i \in [n]$ and $\ell \in [2]$, the reduction adds edges (x_i^ℓ, v_q) for every $q \in \text{set-rep}(x_i^\ell)$. Similarly, it adds edges $(\neg x_i^\ell, v_q)$ for every $q \in \text{set-rep}(\neg x_i^\ell)$.

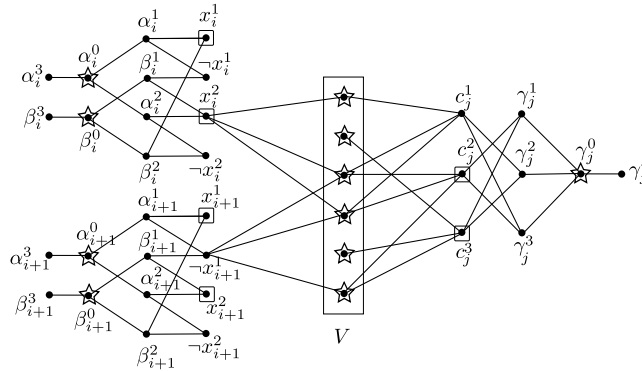


FIG. 2. Illustration of the reduction used in subsection 3.2. For the sake of clarity, we do not explicitly show the pendant vertices adjacent to vertices in V . The variable and clause gadgets are on the left side and right side of V , respectively. In this example, we consider a clause $C_j = x_i \vee \neg x_{i+1} \vee x_{i+2}$. Moreover, suppose this is the second positive appearance of x_i , and the first negative appearance of x_{i+1} , and x_i corresponds to c_j^1 and x_{i+1} corresponds to c_j^2 . Suppose V contains 6 vertices indexed from top to bottom, and the sets corresponding to these two appearances are $\{1, 3, 4\}$ and $\{3, 4, 6\}$, respectively. Vertices with a star boundary are those that we can assume to be in any locating-dominating set, without loss of generality. The square boundaries correspond to the selection of other vertices in the solution. In the above example, it corresponds to setting both x_i and x_{i+1} to **True**. On the clause side, the selection corresponds to selecting x_i to satisfy the clause C_j .

- For a clause C_j , suppose variable x_i appears positively for the ℓ th time as the r th variable in C_j . Then, the reduction adds edges across B and V such that the vertices c_j^r and x_i^ℓ have the same neighborhood in V , namely, the set $\{v_q : q \in \text{set-rep}(x_i^\ell)\}$. For example, x_i appears positively for the second time as the third variable in C_j . Then, the vertices c_j^3 and x_i^2 should have the same neighborhood in V . Similarly, it adds edges for the negative appearance of the variables.

This concludes the construction of G . The reduction sets $k = 4n + 3m + 2p$ and returns (G, k) as the reduced instance of **LOCATING-DOMINATING SET**.

We now prove the validity of the reduction in the following lemmas.

LEMMA 3.18. *If ψ be a YES-instance of (3,3)-SAT, then (G, k) is a YES-instance of **LOCATING-DOMINATING SET**.*

Proof. Suppose $\pi : X \mapsto \{\text{True}, \text{False}\}$ is a satisfying assignment of ψ . We construct a vertex subset S of G from the satisfying assignment in the following manner: Initialize S by adding all the vertices in V . For variable x_i , if $\pi(x_i) = \text{True}$, then include $\{\alpha_i^0, \beta_i^0, x_i^1, x_i^2\}$ in S ; otherwise include $\{\alpha_i^0, \beta_i^0, \neg x_i^1, \neg x_i^2\}$ in S . For any clause C_j , if its first variable is set to **True**, then include $\{\gamma_j^0, c_j^2, c_j^3\}$ in S , and if its second variable is set to **True**, then include $\{\gamma_j^0, c_j^1, c_j^3\}$ in S ; otherwise include $\{\gamma_j^0, c_j^1, c_j^2\}$ in S . If more than one variable of a clause C_j is set to **True**, we include the vertices corresponding to the smallest indexed variable set to **True**. This concludes the construction of S .

It is easy to verify that $|S| = 4n + 3m + 2p = k$. In the remainder of this proof, we argue that S is a locating-dominating set of G . To do so, we first show that S is a dominating set of G . Notice that $V \subseteq S$ dominates the pendant vertices u_q for all $q \in [2p]$ and all the vertices of the form x_i^ℓ for any $i \in [n]$ and $\ell \in [2]$, and c_j^r for any $j \in [m]$ and $r \in [3]$. Moreover, the vertices α_i^0 , β_i^0 , and γ_j^0 dominate the sets

$\{\alpha_i^1, \alpha_i^2, \alpha_i^3\}$, $\{\beta_i^1, \beta_i^2, \beta_i^3\}$, and $\{\gamma_j^1, \gamma_j^2, \gamma_j^3, \gamma_j^4\}$, respectively. This proves that S is a dominating set of G .

We now show that S is also a locating set of G . To begin with, we notice that, for $p \geq 2$, all the pendant vertices u_q for $q \in [2p]$ are located from every other vertex in G by the fact that $|N_G(u) \cap S| = 1$. Next, we divide the analysis of S being a locating set of G into the following three cases:

- First, consider the vertices within A_i 's. Each x_i^ℓ or $\neg x_i^\ell$ for any literal $x_i \in X$ and $\ell \in [2]$ have a distinct neighborhood in V ; hence, they are all pairwise located. For any $i \neq i' \in [n]$ and $\ell \neq \ell' \in [2]$, the pair $(\alpha_i^\ell, \beta_{i'}^{\ell'})$ is located by α_i^0 . Moreover, the pair $(\alpha_i^\ell, \alpha_{i'}^{\ell'})$, for all $i \in [n]$, $\ell \neq \ell' \in [2]$ are located by either $\{x_i^1, x_i^2\}$ or $\{\neg x_i^1, \neg x_i^2\}$, one of which is a subset of S .
- Second, consider the vertices within B_j 's. The set of vertices in $\{\gamma_j^0, \gamma_j^1, \gamma_j^2, \gamma_j^3, \gamma_j^4\}$ are pairwise located by three vertices in the set included in S . One of this vertex is γ_j^0 , and the other two are from $\{c_j^1, c_j^2, c_j^3\}$. For the one vertex in the above set which is not located by the vertices in S in the clause gadget, is located by its neighborhood in V . Finally, any two vertices of the form c_j^r and $c_{j'}^{r'}$ that are not located by corresponding clause vertices in S are located from one another by the fact they are associated with two different variables or two different appearances of the same variable, and hence by construction, their neighborhood in V is different.
- Third, let us consider pairs of vertices not belonging to S and where one vertex belongs to a clause gadget and the other belongs to a variable gadget. To that end, we only need to consider pairs of vertices which are of distance at most 2 from each other, that is, pairs of the form (c_j^r, x_i^ℓ) or $(c_j^r, \neg x_i^\ell)$, where $i \in [n]$, $j \in [m]$, $r \in [3]$, and $\ell \in [2]$. Without loss of generality, let us consider the pair (c_j^r, x_i^ℓ) , where $c_j^r, x_i^\ell \notin S$. This implies that the literal x_i does not appear at the ℓ th time at the r th position of the clause C_j , or else, by the assumption $c_j^r \notin S$, we would have $\pi(x_i) = \text{True}$ making $x_i^\ell \in S$, a contradiction to our assumption. Hence, by construction, the vertices c_j^r and x_i^ℓ have different neighborhoods in V and, thus, are located by S . A similar argument for $(c_j^r, \neg x_i^\ell)$ proves that the latter pair is also located by S .

This proves that S is a locating set of G . $\square$

LEMMA 3.19. *If (G, k) is a YES-instance of LOCATING-DOMINATING SET, then ψ is a YES-instance of $(3, 3)$ -SAT.*

Proof. Suppose S is a locating-dominating set of G of size $k = 4n + 3m + 2p$. Recall that every vertex v in the connection portal V is adjacent to a pendant vertex. Hence, by Observation 2.1, it is safe to assume that S contains V . This implies $|S \setminus V| = 4n + 3m$. Using the same observation, it is safe to assume that α_i^0, β_i^0 , and γ_j^0 are in S for every $i \in [n]$ and $j \in [m]$. As α_i^3 is adjacent to only α_i^0 in S , set $S \setminus \{\alpha_i^0, \beta_i^0\}$ contains at least one vertex in the closed the neighborhood of α_i^1 and one from the closed neighborhood of α_i^2 . By the construction, these two sets are disjoint. Hence, set S contains at least four vertices from variable gadget corresponding to every variable x_i in X . Using similar arguments, S has at least three vertices from clause gadget corresponding to C_j for every clause in C . The cardinality constraints mentioned above imply that S contains exactly four vertices from each variable gadget and exactly three vertices from each clause gadget. Using this observation, we prove the following two claims.

CLAIM 3.20. *We may assume that, for each variable gadget corresponding to variable x_i , S contains either $\{\alpha_i^0, \beta_i^0, x_i^1, x_i^2\}$ or $\{\alpha_i^0, \beta_i^0, \neg x_i^1, \neg x_i^2\}$.*

Proof of Claim. Note that S is a locating-dominating set of G such that $|S \cap A_i| = 4$. Let $X_i = \{x_i^1, x_i^2\}$ and $\neg X_i = \{\neg x_i^1, \neg x_i^2\}$. We prove that S contains exactly one of X_i and $\neg X_i$. First, we show that $|(S \cap A_i) \setminus \{\alpha_i^0, \alpha_i^3, \beta_i^0, \beta_i^3\}| = 2$. Define $R_i^1 = \{\alpha_i^1, x_i^1, \neg x_i^1\}$, $R_i^2 = \{\alpha_i^2, x_i^2, \neg x_i^2\}$, $G_i^1 = \{\beta_i^1, \neg x_i^1, x_i^2\}$, and $G_i^2 = \{\beta_i^2, \neg x_i^2, x_i^1\}$. Note that sets R_i^1 and R_i^2 (respectively, G_i^1 and G_i^2) are disjoint. Also, $(R_i^1 \cap G_i^2) \cup (R_i^2 \cap G_i^1) = \{x_i^1, x_i^2\}$ and $(R_i^1 \cap G_i^1) \cup (R_i^2 \cap G_i^2) = \{\neg x_i^1, \neg x_i^2\}$. To distinguish the vertices in pairs (α_i^1, α_i^2) , (α_i^1, α_i^3) and (α_i^2, α_i^3) , the set S must contain at least one vertex from each R_i^1 and R_i^2 or at least one vertex from each G_i^1 and G_i^2 . Similarly, to distinguish the vertices in pairs (β_i^1, β_i^2) , (β_i^1, β_i^3) and (β_i^2, β_i^3) , the set S must contain at least one vertex from each R_i^1 and R_i^2 or at least one vertex from each G_i^1 and G_i^2 . As both of these conditions need to hold simultaneously, S contains either X_i or $\neg X_i$. $\square$

CLAIM 3.21. *We may assume that, for each clause gadget corresponding to clause C_j , S contains either $\{\gamma_i^0, c_i^2, c_i^3\}$ or $\{\gamma_i^0, c_i^1, c_i^3\}$ or $\{\gamma_i^0, c_i^1, c_i^2\}$.*

Proof of Claim. Note that S is a dominating set that contains 3 vertices corresponding to the clause gadget corresponding to every clause in C . Moreover, c_j^1, c_j^2, c_j^3 separate the vertices in the remaining clause gadget from the rest of the graph. Also, as mentioned before, without loss of generality, C contains γ_j^0 . We define $C_j^* = \{c_j^1, c_j^2, c_j^3\}$ and prove that S contains exactly two vertices from this set. Assume that $S \cap C_j^* = \emptyset$; then S needs to include all the three vertices $\gamma_j^1, \gamma_j^2, \gamma_j^3$ to locate every pair of vertices in the clause gadget. This, however, contradicts the cardinality constraints. Assume that $|S \cap C_j^*| = 1$, and without loss of generality, suppose $S \cap C_j^* = \{c_j^2\}$. If $S \cap \{\gamma_j^1, \gamma_j^2, \gamma_j^3\} = \gamma_j^2$, then the pair (γ_j^1, γ_j^3) is not located by S , a contradiction. Now suppose $S \cap \{\gamma_j^1, \gamma_j^2, \gamma_j^3\} = \gamma_j^1$; then the pair (γ_j^2, γ_j^3) is not located by S , again a contradiction. The similar argument holds for other cases. As both cases, this leads to contradictions, we have $|S \cap C_j^*| = 2$. $\square$

Using these properties of S , we present a way to construct an assignment π for ψ . If S contains $\{\alpha_i^0, \beta_i^0, x_i^1, x_i^2\}$, then set $\pi(x_i) = \text{True}$, otherwise set $\pi(x_i) = \text{False}$. The first claim ensures that this is a valid assignment. We now prove that this assignment is also a satisfying assignment. The second claim implies that for any clause gadget corresponding to clause C_j , there is exactly one vertex *not* adjacent to any vertex in $S \cap B_j$. Suppose c_j^ℓ is such a vertex and is the ℓ th positive appearance of variable x_i . Since x_i^ℓ and c_j^ℓ have identical neighborhood in V , and S is a locating-dominating set in G , S contains x_i^ℓ , and hence $\pi(x_i) = \text{True}$. A similar reason holds when x_i appears negatively. Hence, every vertex in the clause gadget that is not dominated by vertices of S in the clause gadget corresponds to the variable set to **True** and this makes the clause satisfied. This implies π is a satisfying assignment of ψ which concludes the proof of the lemma. $\square$

We are now in a position to prove the conditional lower bounds about LOCATING-DOMINATING SET mentioned in Theorem 1.1: LOCATING-DOMINATING SET does not admit an algorithm running in time $2^{2^{o(\text{tw})}} \cdot \text{poly}(n)$.

Proof of the LOCATING-DOMINATING SET part of Theorem 1.1. Assume that there is an algorithm $\mathcal{A}$ that, given an instance (G, k) of LOCATING-DOMINATING SET, runs in time $2^{2^{o(\text{tw})}} \cdot n^{\mathcal{O}(1)}$ and correctly determines whether it is YES-instance. Consider the following algorithm that takes as input an instance ψ of (3,3)-SAT and determines whether it is a YES-instance. It first constructs an equivalent instance (G, k) of LOCATING-DOMINATING SET as mentioned in this subsection. Then, it calls

algorithm $\mathcal{A}$ as a subroutine and returns the same answer. The correctness of this algorithm follows from the correctness of algorithm $\mathcal{A}$, Lemma 3.18, and Lemma 3.19. Note that since each component of $G - V$ is of constant order, the vertex integrity (and thus treewidth) of G is $\mathcal{O}(|V|)$. By the asymptotic estimation of the central binomial coefficient, $\binom{2p}{p} \sim \frac{4^p}{\sqrt{\pi p}}$ [44]. To get the upper bound of $2p$, we scale down the asymptotic function and have that $4n \leq \frac{4^p}{2^p} = 2^p$. As we choose the smallest possible value of p such that $2^p \geq 4n$, we can choose $p = \log n + 3$. Therefore, $p = \mathcal{O}(\log(n))$. And hence, $|V| = \mathcal{O}(\log(n))$, which implies $\text{tw}(G) = \mathcal{O}(\log n)$. As the other steps, including the construction of the instance of LOCATING-DOMINATING SET, can be performed in the polynomial step, the running time of the algorithm for (3, 3)-SAT is $2^{2^{\mathcal{O}(\log(n))}} \cdot n^{\mathcal{O}(1)} = 2^{\mathcal{O}(n)} \cdot n^{\mathcal{O}(1)}$. This, however, contradicts Proposition 3.17. Hence, our assumption is wrong, and LOCATING-DOMINATING SET does not admit an algorithm running in time $2^{2^{\mathcal{O}(\text{tw})}} \cdot |V(G)|^{\mathcal{O}(1)}$, unless the ETH fails. $\square$

4. Locating-Dominating Set parameterized by the solution size. In this section, we study the parameterized complexity of LOCATING-DOMINATING SET when parameterized by the solution size k . In the first subsection, we formally prove that the problem admits a kernel with $\mathcal{O}(2^k)$ vertices, and hence a simple FPT algorithm running in time $2^{\mathcal{O}(k^2)}$. In the second subsection, we prove that both results mentioned above are optimal under the ETH.

4.1. Upper bound.

PROPOSITION 4.1. LOCATING-DOMINATING SET admits a kernel with $\mathcal{O}(2^k)$ vertices and an algorithm running in time $2^{\mathcal{O}(k^2)} + \mathcal{O}(k \log n)$.

Proof. Slater proved that for any graph G on n vertices with a locating-dominating set of size k , we have $n \leq 2^k + k - 1$ [68]. Hence, if $n > 2^k + k - 1$, we can return a trivial NO instance (this check takes time $\mathcal{O}(k \log n)$). Otherwise, we have a kernel with $\mathcal{O}(2^k)$ vertices. In this case, we can enumerate all subsets of vertices of size k and, for each of them, check in quadratic time if it is a valid solution. Overall, this takes time $\binom{n}{k} n^2$; since $n \leq 2^k + k - 1$, this is $\binom{2^{\mathcal{O}(k)}}{k} \cdot 2^{\mathcal{O}(k)}$, which is $2^{\mathcal{O}(k^2)}$. $\square$

4.2. Lower bound. In this section, we prove Theorem 1.2, which states that both the results mentioned in the previous subsection are optimal under the ETH. We restate it for the reader's convenience.

THEOREM 1.2. Unless the ETH fails, LOCATING-DOMINATING SET does not admit

- an algorithm running in time $2^{\mathcal{O}(k^2)} \cdot \text{poly}(n)$, or
- a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{\mathcal{O}(k)}$ vertices.

Reduction. To prove the theorem, we present a reduction that takes as input an instance ψ , with n variables, of 3-SAT and returns an instance (G, k) of LOCATING-DOMINATING SET such that $|V(G)| = 2^{\mathcal{O}(\sqrt{n})}$ and $k = \mathcal{O}(\sqrt{n})$. By adding dummy variables in each set, we can assume that $\sqrt{n}$ is an even integer. Suppose the variables are named $x_{i,j}$ for $i, j \in [\sqrt{n}]$.

We will make use of *bit-representation gadgets*, as formalized in [36], which are useful to ensure that certain sets of vertices are located from each other and also from the rest of the graph. For a set X to be located, a corresponding bit-representation gadget is of logarithmic size (in terms of the size of X) and assign a distinct set of neighbors of the gadget to each vertex of X . By attaching pendant vertices to each

vertex in the bit-representation gadget, we ensure (using Observation 2.1) that they can be assumed to be part of any solution and, hence, locate X .

The reduction constructs graph G as follows:

- It partitions the variables of ψ into $\sqrt{n}$ many *buckets* $X_1, X_2, \dots, X_{\sqrt{n}}$ such that each bucket contains exactly $\sqrt{n}$ many variables. Let $X_i = \{x_{i,j} \mid j \in [\sqrt{n}]\}$ for all $i \in [\sqrt{n}]$.
 - For every X_i , it constructs set A_i of $2^{\sqrt{n}}$ new vertices, $A_i = \{a_{i,\ell} \mid \ell \in [2^{\sqrt{n}}]\}$. Each vertex in A_i corresponds to a unique assignment of variables in X_i . Let A be the collection of all the vertices added in this step.
 - For every X_i , the reduction adds a path on three vertices b_i° , b'_i , and b_i^* with edges (b_i°, b'_i) and (b'_i, b_i^*) . Suppose B is the collection of all the vertices added in this step.
 - For every X_i , it makes b_i° adjacent to every vertex in A_i .
- For every clause C_j , the reduction adds a pair of vertices c_j°, c_j^* . For a vertex $a_{i,\ell} \in A_i$ for some $i \in [\sqrt{n}]$, and $\ell \in [2^{\sqrt{n}}]$, if the assignment corresponding to vertex $a_{i,\ell}$ satisfies clause C_j , then it adds edge $(a_{i,\ell}, c_j^\circ)$.
- The reduction adds a bit-representation gadget to locate set A . Once again, informally speaking, it adds some supplementary vertices such that it is safe to assume that these vertices are present in a locating-dominating set, and they locate every vertex in A . More precisely, we have the following:
 - First, set $q := \lceil \log(|A|) \rceil + 1$. This value for q allows to uniquely represent each integer in $[|A|]$ by its bit-representation in binary (starting from 1 and not 0).
 - For every $i \in [q]$, the reduction adds two vertices $y_{i,1}$ and $y_{i,2}$ and edge $(y_{i,1}, y_{i,2})$.
 - For every integer $q' \in [|A|]$, let $\text{bit}(q')$ denote the binary representation of q' using q bits. Connect $a_{i,\ell} \in A$ with $y_{i,1}$ if the i th bit in $\text{bit}((i-1) \cdot 2^{\sqrt{n}} + \ell)$ is 1.
 - It adds two vertices, $y_{0,1}$ and $y_{0,2}$, and edge $(y_{0,1}, y_{0,2})$. It also makes every vertex in A adjacent to $y_{0,1}$.

Let $\text{bit-rep}(A)$ be the collection of vertices $y_{i,1}$ added in this step, and let Y be the collection of vertices $y_{i,1}$ and $y_{i,2}$ added in this step (over all $i \in \{0\} \cup [q]$).
- Finally, the reduction adds a bit-representation gadget to locate set C . However, it adds the vertices in such a way that for any pair c_j°, c_j^* , the supplementary vertices adjacent to them are identical.
 - The reduction sets $p := \lceil \log(|C|/2) \rceil + 1$, and for every $i \in [p]$, it adds two vertices $z_{i,1}$ and $z_{i,2}$ and edge $(z_{i,1}, z_{i,2})$.
 - For every integer $j \in [|C|/2]$, let $\text{bit}(j)$ denote the binary representation of j using p bits. Connect $c_j^\circ, c_j^* \in C$ with $z_{i,1}$ if the i th bit in $\text{bit}(j)$ is 1.
 - It adds two vertices, $z_{0,1}$ and $z_{0,2}$, and edge $(z_{0,1}, z_{0,2})$. It also makes every vertex in C adjacent to $y_{0,1}$.

Let $\text{bit-rep}(C)$ be the collection of the vertices $z_{i,1}$ added in this step, and let Z be the collection of vertices $z_{i,1}$ and $z_{i,2}$ added in this step (over all $i \in \{0\} \cup [p]$).

This completes the construction; see Figure 3 for an illustration.

The reduction sets

$$k = |B|/3 + (\lceil \log(|A|) \rceil + 1 + 1) + (\lceil \log(|C|/2) \rceil + 1 + 1) + \sqrt{n} = \mathcal{O}(\sqrt{n})$$

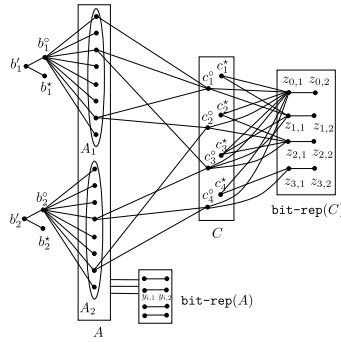


FIG. 3. An illustrative example of the graph constructed by the reduction used in subsection 4.2. Suppose an instance ψ of 3-SAT has $n = 9$ variables and 4 clauses. We do not show the third variable bucket and explicit edges across A and $\text{bit-rep}(A)$ for brevity. Vertices with a star boundary are those that we can assume to be in any locating-dominating set, without loss of generality. The square boundaries correspond to the selection of other vertices in the solution.

as $|B| = 3\sqrt{n}$, $|A| = \sqrt{n} \cdot 2^{\sqrt{n}}$, and $|C| = \mathcal{O}(n^3)$, and returns (G, k) as a reduced instance.

The vertices of G are naturally partitioned into the five sets A , B , C , Y , and Z defined above. Furthermore, we partition B into B' , B° , and B^* as follows: $B' = \{b_i^* \mid i \in [\sqrt{n}]\}$, $B^\circ = \{b_i^\circ \mid i \in [\sqrt{n}]\}$, and $B^* = \{b_i^* \mid i \in [\sqrt{n}]\}$. Define Y_1 , Y_2 , Z_1 , and Z_2 in a similar way. Note that B^* , Y_2 , and Z_2 together contain all pendant vertices.

LEMMA 4.2. *If ψ is a YES-instance of 3-SAT, then (G, k) is a YES-instance of LOCATING-DOMINATING SET.*

Proof. Suppose $\pi : X \mapsto \{\text{True}, \text{False}\}$ is a satisfying assignment for ψ . Using this assignment, we construct a locating-dominating set S of G of size at most k . Initialize set S by adding all the vertices in $B' \cup Y_1 \cup Z_1$. At this point, the cardinality of S is $k - \sqrt{n}$. We add the remaining $\sqrt{n}$ vertices as follows: Partition X into $\sqrt{n}$ parts $X_1, \dots, X_i, \dots, X_{\sqrt{n}}$ as specified in the reduction, and define π_i for every $i \in [\sqrt{n}]$ by restricting the assignment π to the variables in X_i . By the construction of G , there is a vertex $a_{i,\ell}$ in A corresponding to the assignment π_i . Add that vertex to S . It is easy to verify that the size of S is at most k . We next argue that S is a locating-dominating set.

The vertices in B are located from all other vertices by the vertices in B' . Moreover, pairs of the form $\{b_i^\circ, b_i^*\}$ are located by the vertices in $A \cap Y$.

Consider set A . By the property of a bit-representation gadget, every vertex in A is adjacent to a unique set of vertices in $\text{bit-rep}(A)$. Consider a vertex $a_{i,\ell}$ in A such that the bit-representation of $((i-1) \cdot 2^{\sqrt{n}} + \ell)$ contains a single 1 at the j th location. Hence, both $y_{j,2}$ and $a_{i,\ell}$ are adjacent to the same vertex, viz. $y_{j,1}$ in $\text{bit-rep}(A) \setminus \{y_{0,1}\}$. However, this pair of vertices is located by $y_{0,1}$ which is adjacent to $a_{i,\ell}$ and not to $y_{j,2}$. Also, as the bit-representation of vertices starts from 1, there is no vertex in A which is adjacent to only $y_{0,1}$ in $\text{bit-rep}(A)$. Hence, $\text{bit-rep}(A)$ locates all pairs of vertices in $A \cup Y$.

Using similar arguments for C , $\text{bit-rep}(C)$, and Z and the properties of bit-representation gadgets, we can conclude that $\text{bit-rep}(A) \cup \text{bit-rep}(C)$ locates all pairs of vertices of G , apart from the pairs of the form (b_j°, b_j^*) and (c_j°, c_j^*) .

TABLE 1

Partition of $V(G)$ and the corresponding set that dominates and locates the vertices in each part.

Set	B'	B°	$B^\star$	A	Y_1	Y_2	C°	$C^\star$	Z_1	Z_2
Dominated by	B'	B'	B'	Y_1	Y_1	Y_1	Z_1	Z_1	Z_1	Z_1
Located by	—	$B' + A$	B'	Y_1	—	Y_1	$Z_1 + A$	Z_1	—	Z_1

By the construction, the sets mentioned in the second row of Table 1 dominate the vertices mentioned in the sets in the respective first rows. Hence, S is a dominating set. We need to prove that the location of only those vertices of a set which are dominated by other vertices of the same set. First, consider the vertices in $B^\circ \cup B^\star$. Recall that every vertex of the form b_i° and $b_i^\star$ is adjacent to b'_i for every $i \in [\sqrt{n}]$. Hence, the only pair of vertices that needs to be located are of the form b_i° and $b_i^\star$. However, as S contains at least one vertex from A_i , a vertex in S is adjacent to b_i° and not adjacent to $b_i^\star$. Now, consider the vertices in A and Y_2 . Note that every vertex in Y_2 is adjacent to precisely one vertex in Y_1 . However, every vertex in A is adjacent to at least two vertices in Y_1 (one of which is $y_{0,1}$). Hence, every vertex in $A \cup Y_2$ is located. Using similar arguments, every vertex in $C \cup Z_2$ is also located.

The only thing that remains to argue is that every pair of vertices c_j° and $c_j^\star$ is located. As π is a satisfying assignment, at least one of its restrictions, say π_i , is a satisfying assignment for clause C_j . By the construction of the graph, the vertex corresponding to π_i is adjacent to c_j° but not adjacent to $c_j^\star$. Also, such a vertex is present by the construction of S . Hence, there is a vertex in $S \cap A_i$ that locates c_j° from $c_j^\star$. This concludes the proof that S is a locating-dominating set of G of size k . Hence, if ψ is a YES-instance of 3-SAT, then (G, k) is a YES-instance of LOCATING-DOMINATING SET. $\square$

LEMMA 4.3. *If (G, k) is a YES-instance of LOCATING-DOMINATING SET, then ψ is a YES-instance of 3-SAT*

Proof. Suppose S is a locating-dominating set of G of size at most k . We construct a satisfying assignment π for ψ . Recall that $B_i^\star$, Y_2 , and Z_2 contain exactly all pendant vertices of G . By Observation 2.1, it is safe to assume that every vertex in B' , Y_1 , and Z_1 is present in S .

Consider the vertices in B° and $B^\star$. As mentioned before, every vertex of the form b_i° and $b_i^\star$ is adjacent to b'_i , which is in S . By the construction of G , only the vertices in A_i are adjacent to b_i° but not adjacent to $b_i^\star$. Hence, S contains at least one vertex in $A_i \cup \{b_i^\circ, b_i^\star\}$. As the number of vertices in $S \setminus (B' \cup Y_1 \cup Y_2)$ is at most $\sqrt{n}$, S contains exactly one vertex from $A_i \cup \{b_i^\circ, b_i^\star\}$ for every $i \in [\sqrt{n}]$. Suppose S contains a vertex from $\{b_i^\circ, b_i^\star\}$. As the only purpose of this vertex is to locate a vertex in this set, it is safe to replace this vertex with any vertex in A_i . Hence, we can assume that S contains exactly one vertex in A_i for every $i \in [\sqrt{n}]$.

For every $i \in [\sqrt{n}]$, let $\pi_i : X_i \mapsto \{\text{True}, \text{False}\}$ be the assignment of the variables in X_i corresponding to the vertex of $S \cap A_i$. We construct an assignment $\pi : X \mapsto \{\text{True}, \text{False}\}$ such that π restricted to X_i is identical to π_i . As X_i is a partition of variables in X , and every vertex in A_i corresponds to a valid assignment of variables in X_i , it is easy to see that π is a valid assignment. It remains to argue that π is a satisfying assignment. Consider a pair of vertices c_j° and $c_j^\star$ corresponding to clause C_j . By the construction of G , both these vertices have identical neighbors in Z_1 , which

is contained in S . The only vertices that are adjacent to c_j° and are not adjacent to c_j^* are in A_i for some $i \in [\sqrt{n}]$ and correspond to some assignment that satisfies clause C_j . As S is a locating-dominating set of G , there is at least one vertex in $S \cap A_i$ that locates c_j° from c_j^* . Alternately, there is at least one vertex in $S \cap A_i$ that corresponds to an assignment that satisfies clause C_j . This implies that if (G, k) is a YES-instance of LOCATING-DOMINATING SET, then ϕ is a YES-instance of 3-SAT. $\square$

We are now in a position to prove Theorem 1.2, which states that unless the ETH fails, LOCATING-DOMINATING SET does not admit an algorithm running in time $2^{o(k^2)} \cdot n^{\mathcal{O}(1)}$ or a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{o(k)}$ vertices.

Proof of Theorem 1.2. Assume that there exists an algorithm, say $\mathcal{A}$, that takes as input an instance (G, k) of LOCATING-DOMINATING SET and correctly concludes whether it is a YES-instance in time $2^{o(k^2)} \cdot |V(G)|^{\mathcal{O}(1)}$. Consider algorithm $\mathcal{B}$ that takes as input an instance ψ of 3-SAT, uses the reduction above to get an equivalent instance (G, k) of LOCATING-DOMINATING SET, and then uses $\mathcal{A}$ as a subroutine. The correctness of algorithm $\mathcal{B}$ follows from Lemma 4.2, Lemma 4.3, and the correctness of algorithm $\mathcal{A}$. From the description of the reduction and the fact that $k = \sqrt{n}$, the running time of algorithm $\mathcal{B}$ is $2^{\mathcal{O}(\sqrt{n})} + 2^{o(k^2)} \cdot (2^{\mathcal{O}(\sqrt{n})})^{\mathcal{O}(1)} = 2^{o(n)}$. This, however, contradicts the ETH. Hence, LOCATING-DOMINATING SET does not admit an algorithm with running time $2^{o(k^2)} \cdot |V(G)|^{\mathcal{O}(1)}$ unless the ETH fails.

For the second part of Theorem 1.2, assume that such a kernelization algorithm exists. Consider the following algorithm for 3-SAT. Given a 3-SAT formula on n variables, it uses the above reduction to get an equivalent instance of (G, k) such that $|V(G)| = 2^{\mathcal{O}(\sqrt{n})}$ and $k = \mathcal{O}(\sqrt{n})$. Then, it uses the assumed kernelization algorithm to construct an equivalent instance (H, k') such that H has $2^{o(k)}$ vertices and $k' \leq k$. Finally, it uses a brute-force algorithm, running in time $|V(H)|^{\mathcal{O}(k')}$, to determine whether the reduced instance, equivalently the input instance of 3-SAT, is a YES-instance. The correctness of the algorithm follows from the correctness of the respective algorithms and our assumption. The total running time of the algorithm is $2^{\mathcal{O}(\sqrt{n})} + (|V(G)| + k)^{\mathcal{O}(1)} + |V(H)|^{\mathcal{O}(k')} = 2^{\mathcal{O}(\sqrt{n})} + (2^{\mathcal{O}(\sqrt{n})})^{\mathcal{O}(1)} + (2^{o(\sqrt{n})})^{\mathcal{O}(\sqrt{n})} = 2^{o(n)}$. This, however, contradicts the ETH. Hence, LOCATING-DOMINATING SET does not admit a polynomial-time kernelization algorithm that reduces the solution size and returns a kernel with $2^{o(k)}$ vertices unless the ETH fails. $\square$

5. Modifications for Test Cover. In this section, we present the small (but crucial) modifications required to obtain similar results as mentioned in previous section for TEST COVER. As in this problem one only need to “locate” elements, the reductions in this case are often simpler than the corresponding reductions for LOCATING-DOMINATING SET to prove a conditional lower bound. We restate the problem definition for the readers’ convenience.

TEST COVER

Input: A set of items U , a collection of subsets of U called *tests* and denoted by $\mathcal{F}$, and an integer k .

Question: Does there exist a collection of at most k tests such that for each pair of items, there is a test that contains exactly one of the two items?

We also recall the incidence (bipartite) graph G on n vertices with bipartition $\langle R, B \rangle$ of $V(G)$ such that sets R and B contain a vertex for every set in $\mathcal{F}$ and for every item in U , respectively, and $r \in R$ and $b \in B$ are adjacent if and only if the set corresponding to r contains the element corresponding to b . We note that description of the incidence (bipartite) graph G is sufficient to describe the instance of TEST COVER. We find it notationally cleaner to work with encoding of the instance.

5.1. Parameterization by treewidth.

5.1.1. Upper bound. In this section, we outline the DP approach for TEST COVER and use it to prove the following theorem.

THEOREM 5.1. *TEST COVER admits an algorithm running in time $2^{2^{\mathcal{O}(\text{tw})}} \cdot n^{\mathcal{O}(1)}$, where tw is the treewidth and n is the order of the incidence graph of the input.*

Let G be a bipartite graph on n vertices with a bipartition $V(G) = \langle R, B \rangle$. The objective is to decide whether there exists a set of at most k vertices from R such that, for every pair $x, y \in B$, there exists $r \in R$ with $rx \in E(G)$ but $ry \notin E(G)$. Without loss of generality, we assume that we are given a nice tree-decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G of width at most $2\text{tw}(G) + 1$.

The overall structure of the algorithm and the motivation for defining the DP-states follows the framework in subsection 3.1. For completeness, we recall the relevant notions, adapted to the present setting. As before, for each $t \in V(T)$, let G_t denote the subgraph of G corresponding to the subtree of T rooted at t . We define $R_t := R \cap V(G_t)$ and $B_t := B \cap V(G_t)$.

For each node t , the DP-state is defined in terms of a *valid tuple*, which encodes the interaction between the partial solution and the vertices in the current bag.

DEFINITION 5.2 (valid tuple). *Let $t \in V(T)$ be a node in the tree-decomposition. A tuple $\langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ is valid at t if*

- $Y \subseteq R \cap X_t$ and $W \subseteq B \cap X_t$, with $N(Y) \cap X_t \subseteq W$ and $W_0 \subseteq W$;
- $\mathcal{Y}$ is a family of subsets of Y ;
- $\mathbb{W}$ is a collection of pairs (w_1, w_2) with $w_1, w_2 \in W$, or $(w_1, +)$ with $w_1 \in W$.

DEFINITION 5.3 (candidate solution). *Let $\tau = \langle Y, W, W_0, \mathcal{Y}, \mathbb{W} \rangle$ be a valid tuple at node t . A set $S_t \subseteq R_t$ is a candidate solution for τ if*

1. S_t locates all vertices in $B_t \setminus X_t$; that is, for any distinct $u, v \in B_t \setminus (S_t \cup X_t)$, both $N_{G_t}(u) \cap S_t$ and $N_{G_t}(v) \cap S_t$ are nonempty and distinct.
2. $Y = S_t \cap X_t$, and $W = N(S_t) \cap X_t$. Moreover, every vertex $w \in W \setminus W_0$ has a neighbor in $S_t \setminus Y$.
3. $A \in \mathcal{Y}$ if and only if there exists a unique $u \in B_t \setminus (S_t \cup X_t)$ such that $N_{G_t}(u) \cap S_t = A$.
4. For $\mathbb{W}$:
 - $(w_1, w_2) \in \mathbb{W}$ if and only if $w_1, w_2 \in W$ and $N_{G_t}(w_1) \cap S_t = N_{G_t}(w_2) \cap S_t$;
 - $(w_1, +) \in \mathbb{W}$ if and only if $w_1 \in W$ and there exists a unique $u \in V(G_t) \setminus (S_t \cup X_t)$ with $N_{G_t}(w_1) \cap S_t = N_{G_t}(u) \cap S_t$.

For a valid tuple τ at node t , we define $\mathbf{d}[t, \tau]$ as the minimum size of a candidate solution for τ ; if none exists, set $\mathbf{d}[t, \tau] := \infty$.

The subsequent case analysis and recursive algorithms (and their proof of correctness) follow exactly as in subsection 3.1, with the additional restriction that $Y \subseteq R$ and $W \subseteq B$. Any DP-state violating these constraints can be discarded immediately. For instance, if a vertex $x \in B$ is introduced at t , the case $x \in Y$ is ignored. Since this condition can be checked locally, the remainder of the argument proceeds as before, establishing Theorem 5.1.

5.1.2. Lower bound. In this subsection, we prove the lower bound mentioned in Theorem 1.1, which we restate for convenience.

THEOREM 5.4 (restating part of Theorem 1.1). *Unless the ETH fails, TEST COVER does not admit an algorithm running in time $2^{2^{o(\text{tw})}}$ · poly(n), where tw is the treewidth and n is the order of the incidence graph of the input.*

The reduction is a modification of the reduction presented in subsection 3.2.

Reduction. For notational convenience, instead of specifying an instance of TEST COVER, we specify the incidence graph as mentioned in the definition. The reduction takes as input an instance ψ of (3,3)-SAT with n variables and outputs an instance $(G, \langle R, B \rangle, k)$ of TEST COVER such that $\text{tw}(G) = \mathcal{O}(\log(n))$. Suppose $X = \{x_1, \dots, x_n\}$ is the collection of variables and $C = \{C_1, \dots, C_m\}$ is the collection of clauses in ψ . Here, we consider $\langle x_1, \dots, x_n \rangle$ and $\langle C_1, \dots, C_m \rangle$ to be arbitrary but fixed orderings of variables and clauses in ψ . For a particular clause, the first order specifies the first, second, or third (if it exists) variable in the clause in a natural way. The second ordering specifies the first/second positive/negative appearance of variables in X in a natural way.

The reduction constructs a graph G after initializing $R = B = \emptyset$. We again make use of the set-representation gadget from [36], as in subsection 3.2:

- To construct a variable gadget for x_i , it adds a blue vertex β_i to B , two red vertices α_i^1, α_i^2 to R , and four blue vertices $x_i^1, \neg x_i^1, x_i^2, \neg x_i^2$ to B . It then adds the edges as shown in Figure 4. Let A_i be the collection of these seven vertices. Define $X_i := \{x_i^1, \neg x_i^1, x_i^2, \neg x_i^2\}$.
- To construct a clause gadget for C_j , the reduction adds three blue vertices $\{\gamma_j^1, \gamma_j^2, \gamma_j^3\}$ to B , three red vertices $\{\delta_j^1, \delta_j^2, \delta_j^3\}$ to R , and three vertices c_j^1, c_j^2, c_j^3 to B . It adds the edges as shown in Figure 4. Let B_j be the collection of these nine vertices.
- Let p be the smallest positive integer such that $4n \leq \binom{2p}{p}$. Define $\mathcal{F}_p$ as the collection of subsets of $[2p]$ that contains exactly p integers (such a collection $\mathcal{F}_p$ is called a Sperner family). Define **set-rep**: $\bigcup_{i=1}^n X_i \mapsto \mathcal{F}_p$ as an injective function by arbitrarily assigning a set in $\mathcal{F}_p$ to a vertex x_i^ℓ or $\neg x_i^\ell$ for every $i \in [n]$ and $\ell \in [2]$. In other words, every appearance of a literal is assigned a distinct subset in $\mathcal{F}_p$.

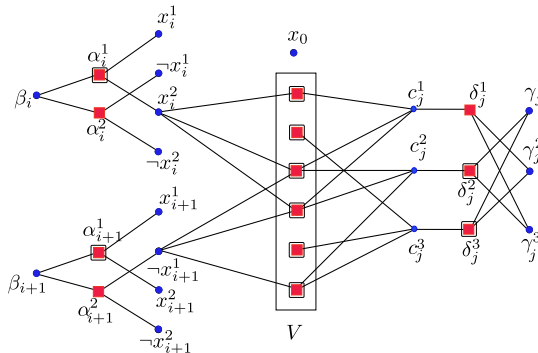


FIG. 4. An illustrative example of the graph constructed by the reduction. Red (squared) nodes denote the tests, whereas blue (filled circle) nodes the elements. For clarity, we do not explicitly show the pendant blue vertices adjacent to vertices in V . The variable and clause gadgets are on the left and right sides of V , respectively. This is an adaptation of the same example mentioned in subsection 3.2. Red vertices with a thick boundary are part of a solution.

- The reduction adds a connection portal V , which is an independent set on $2p$ red vertices $v_1, v_2, \dots, v_{2p}$, to R . For every vertex v_q in V , the reduction adds a pendant blue vertex u_q adjacent to the red vertex v_q .
- For every vertex $x_i^\ell \in X$, where $i \in [n]$ and $\ell \in [2]$, the reduction adds edges (x_i^ℓ, v_q) for every $q \in \text{set-rep}(x_i^\ell)$. Similarly, it adds edges $(\neg x_i^\ell, v_q)$ for every $q \in \text{set-rep}(\neg x_i^\ell)$.
- For a clause C_j , suppose variable x_i appears positively for the ℓ th time as the r th variable in C_j . For example, x_i appears positively for the second time as the third variable in C_j . Then, the reduction adds edges across B and V such that the vertices c_j^r and x_i^ℓ have the same neighborhood in V , namely, the set $\{v_q : q \in \text{set-rep}(x_i^\ell)\}$. Similarly, it adds edges for the negative appearance of the variables.
- The reduction adds an isolated blue vertex x_0 .

This concludes the construction of G . The reduction sets $k = n + 2m + 2p$ and returns $(G, \langle R, B \rangle, k)$ as the reduced instance of TEST COVER.

We now prove the correctness of the reduction in the following lemma.

LEMMA 5.5. *ψ is a YES-instance of (3,3)-SAT if and only if $(G, \langle R, B \rangle, k)$ is a YES-instance of TEST COVER.*

Proof. Let us first assume that ψ be a YES-instance of (3,3)-SAT. Moreover, suppose $\pi : X \mapsto \{\text{True}, \text{False}\}$ is a satisfying assignment of ψ . We construct a vertex subset S of G from the satisfying assignment on ψ in the following manner: Initialize S by adding all the vertices in V . For variable x_i , if $\pi(x_i) = \text{True}$, then include α_i^1 in S ; otherwise include α_i^2 in S . For any clause C_j , if its first variable is set to **True**, then include $\{\delta_j^2, \delta_j^3\}$ in S , and if its second variable is set to **True**, then include $\{\delta_j^1, \delta_j^3\}$ in S , otherwise include $\{\delta_j^1, \delta_j^2\}$ in S . If more than one variable of a clause C_j is set to **True**, we include the vertices corresponding to the smallest indexed variable set to **True**. This concludes the construction of S .

It is easy to verify that $|S| = n + 2m + 2p = k$. To prove that $(G, \langle R, B \rangle, k)$ is a YES-instance of TEST COVER, we show that, apart from the vertex x_0 which has an empty neighborhood in S , every other blue vertex of G has a (nonempty) unique neighborhood in S . To begin with, we see that the vertex β_i for each $i \in [n]$ has a unique neighbor in S , namely, either α_i^1 or α_i^2 . Similarly, each pendant blue vertex u_q , for $q \in [2p]$, has a unique neighbor in S , namely, the vertex v_q . Since $V \subset S$, by the construction of G , the vertices of $\bigcup_{i=1}^n X_i$ have pairwise distinct neighborhoods of size p in S . Similarly, again by construction, the vertices of $\bigcup_{i=1}^n \{c_j^1, c_j^2, c_j^3\}$ have pairwise distinct neighborhoods of size p in S . On the other hand, since two vertices of $\{\delta_j^1, \delta_j^2, \delta_j^3\}$ belong to S for each $j \in [m]$, it implies that the vertices of $\bigcup_{j=1}^m \{\gamma_j^1, \gamma_j^2, \gamma_j^3\}$ have pairwise distinct neighborhoods in S . Now, looking at a pair of the form x_i^ℓ (or equivalently $\neg x_i^\ell$) and c_j^r , the pair has distinct neighborhoods in $V \subset S$ if c_j^r is not the ℓ th occurrence of the literal x_i . Conversely, if c_j^r is the ℓ th occurrence of x_i , then $\delta_j^r \notin S$ implies that $\pi(x_i) = \text{True}$, which forces $\alpha_i^1 \in S$. In other words, the pair x_i^ℓ and c_j^r has distinct neighborhoods in S . Finally, pairs of the form β_i, x_i^ℓ and c_j^r, γ_j^s have pairwise distinct neighborhoods in S since the vertices x_i^ℓ and c_j^r have neighbors in S and β_i and γ_j^s do not. This proves the forward direction of the claim.

We now assume that $(G, \langle R, B \rangle, k)$ is a YES-instance of TEST COVER and that S is a test cover of G of size $k = n + 2m + 2p$. Since x_0 is undominated by S , every other blue vertex must be dominated by S . Hence, we can assume that $V \subset S$ since each v_q is adjacent to a pendant vertex u_q , where $q \in [2p]$, which must be covered. Similarly, for each $i \in [n]$, either α_i^1 or α_i^2 belongs to S in order for S to cover β_i .

Moreover, for every $j \in [m]$, at least two vertices of $\{\delta_j^1, \delta_j^2, \delta_j^3\}$ must be in S in order for the blue vertices in $\{\gamma_j^1, \gamma_j^2, \gamma_j^3\}$ to have pairwise distinct neighborhoods in S . This implies that $|\{\alpha_i^1, \alpha_i^2\} \cap S| = 1$ for all $i \in [n]$. Thus, setting $\pi(x_i) = \text{True}$ if $\alpha_i \in S$, and $\pi(x_i) = \text{False}$ otherwise, ensures a valid truth assignment for ψ . Moreover, we have $|\{\delta_j^1, \delta_j^2, \delta_j^3\} \cap S| = 2$ for all $j \in [m]$. Thus, if $\delta_j^r \notin S$ for some $j \in [m]$ and $r \in [3]$, it implies that $\alpha_i^1 \in S$ (respectively, $\alpha_i^2 \in S$), where α_i^r is the r th occurrence of x_i (respectively, $\neg x_i$). This further implies that $\pi(x_i) = \text{True}$ (respectively, $\pi(x_i) = \text{False}$) and hence the clause C_j is satisfied. This proves that the assignment on ψ is a satisfying one and, thus, proves the reverse direction of the claim. $\square$

We are now in a position to prove the part of Theorem 1.1 that states TEST COVER does not admit an algorithm running in time $2^{2^{o(\text{tw})}} \cdot \text{poly}(n)$, unless the ETH fails.

Proof of the TEST COVER part of Theorem 1.1. Assume that there is an algorithm $\mathcal{A}$ that, given a reduced instance $(G, \langle R, B \rangle, k)$ of TEST COVER, runs in time $2^{2^{o(\text{tw})}} \cdot n^{\mathcal{O}(1)}$ and correctly determines whether it is YES-instance. Consider the following algorithm that takes as input an instance ψ of (3,3)-SAT and determines whether it is a YES-instance. It first constructs an equivalent instance $(G, \langle R, B \rangle, k)$ of TEST COVER as mentioned in this subsection. Then, it calls algorithm $\mathcal{A}$ as a subroutine and returns the same answer. The correctness of this algorithm follows from the correctness of algorithm $\mathcal{A}$ and of the reduction (Lemma 5.5). Note that since each component of $G - V$ is of constant order, the vertex integrity (and thus treewidth) of G is $\mathcal{O}(|V|)$. By the asymptotic estimation of the central binomial coefficient, $\binom{2p}{p} \sim \frac{4^p}{\sqrt{\pi \cdot p}}$ [44]. To get the upper bound of $2p$, we scale down the asymptotic function and have that $4n \leq \frac{4^p}{2^p} = 2^p$. As we choose the smallest possible value of p such that $2^p \geq 4n$, we can choose $p = \log n + 3$. Therefore, $p = \mathcal{O}(\log(n))$. And hence, $|V| = \mathcal{O}(\log(n))$ which implies $\text{tw}(G) = \mathcal{O}(\log n)$. As the other steps, including the construction of the instance of TEST COVER can be performed in the polynomial step, the running time of the algorithm for (3,3)-SAT is $2^{2^{o(\log(n))}} \cdot n^{\mathcal{O}(1)} = 2^{o(n)} \cdot n^{\mathcal{O}(1)}$. This, however, contradicts Proposition 3.17. Hence, our assumption is wrong, and TEST COVER does not admit an algorithm running in time $2^{2^{o(\text{tw})}} \cdot |V(G)|^{\mathcal{O}(1)}$, unless the ETH fails. $\square$

5.2. Parameterization by the solution size. In this subsection, we present a reduction that is very close to the reduction used in the proof of Theorem 1.2 to prove Theorem 1.3, which is restated here for convenience.

THEOREM 1.3. *Unless the ETH fails, TEST COVER does not admit*

- *an algorithm running in time $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, or*
- *a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{2^{o(k)}}$ vertices.*

Reduction. For notational convenience, instead of specifying an instance of TEST COVER, we specify the incidence graph as mentioned in the beginning of the section. The reduction takes as input an instance ψ , with n variables and m clauses, of 3-SAT and returns a reduced instance $(G, \langle R, B \rangle, k)$ of TEST COVER and $k = \mathcal{O}(\log(n) + \log(m)) = \mathcal{O}(\log(n))$, using the sparsification lemma [46].

We again make use of bit-representation gadgets of [36], as in subsection 4.2.

The reduction constructs graph G as follows:

- The reduction adds some dummy variables to ensure that $n = 2^{2q}$ for some positive integer q which is a power of 2. This ensures that $r = \log_2(n) = 2q$ and $s = \frac{n}{r}$ both are integers. It partitions the variables of ψ into r many

buckets $X_1, X_2, \dots, X_r$ such that each bucket contains s many variables. Let $X_i = \{x_{i,j} \mid j \in [s]\}$ for all $i \in [r]$.

- For every X_i , the reduction constructs a set A_i of 2^s many red vertices, that is, $A_i = \{a_{i,\ell} \mid \ell \in [2^s]\}$. Each vertex in A_i corresponds to a unique assignment of the variables in X_i . Moreover, let $A = \cup_{i=1}^r A_i$.
- Corresponding to each X_i , let the reduction add a blue vertex b_i and the edges $(b_i, a_{i,\ell})$ for all $i \in [r]$ and $\ell \in [2^s]$. Let $B' = \{b_i \mid i \in [r]\}$.
- For every clause C_j , the reduction adds a pair of blue vertices c_j°, c_j^* . For a vertex $a_{i,\ell} \in A_i$ with $i \in [r]$, and $\ell \in [2^s]$, if the assignment corresponding to vertex $a_{i,\ell}$ satisfies the clause C_j , then the reduction adds the edge $(a_{i,\ell}, c_j^\circ)$. Let $C = \{c_j^\circ, c_j^* \mid j \in [m]\}$.
- The reduction adds a bit-representation gadget to locate set C . However, it adds the vertices in such a way that for any pair c_j°, c_j^* , the supplementary vertices adjacent to them are identical.
 - The reduction sets $p := \lceil \log(m) \rceil + 1$ and for every $i \in [p]$, it adds two vertices, a red vertex $z_{i,1}$ and a blue vertex $z_{i,2}$, and the edge $(z_{i,1}, z_{i,2})$.
 - For every integer $j \in [m]$, let $\text{bit}(j)$ denote the binary representation of j using p bits. Connect $c_j^\circ, c_j^* \in C$ with $z_{i,1}$ if the i th bit in $\text{bit}(j)$ is 1.
 - Add two vertices $z_{0,1}$ and $z_{0,2}$, and edge $(z_{0,1}, z_{0,2})$. Also make every vertex in C adjacent to $z_{0,1}$. Let Z be the collection of all the vertices added in this step, and $\text{bit-rep}(C)$ be the set of red vertices in Z .
- The reduction adds an isolated blue vertex b_0 (whose purpose is to remain uncovered, thereby forcing all other blue vertices to be covered by some solution test).

This completes the construction. We refer to Figure 5 for an illustration of the construction. The reduction sets $k = r + p = \mathcal{O}(\log(n)) + \mathcal{O}(\log(m)) = \mathcal{O}(\log(n))$, and returns $(G, \langle R, B \rangle, k)$ as an instance of TEST COVER.

We now prove the correctness of the reduction.

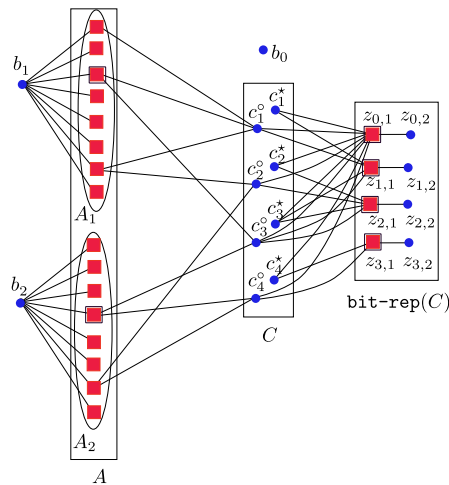


FIG. 5. An illustrative example of the graph constructed by the reduction in subsection 5.2. Red vertices with a thick boundary are part of the solution. Red (squared) vertices denote the tests whereas blue (filled circle) vertices the elements. Vertices with a thick boundary are in a potential solution.

LEMMA 5.6. ψ is a YES-instance of 3-SAT if and only if $(G, \langle R, B \rangle, k)$ is a YES-instance of TEST COVER.

Proof. Assume first that ψ is a YES-instance of 3-SAT. We construct a subset $R' \subset R$ such that $|R'| \leq k$ in the following manner. Since ψ is a YES-instance of 3-SAT, there exists a satisfying assignment $\alpha : X \rightarrow \{\text{True}, \text{False}\}$ of ψ . Then, the assignment α restricted to each bucket X_i gives an assignment $\alpha_i : X_i \rightarrow \{\text{True}, \text{False}\}$ of the variables in X_i . By our construction of G , the assignment α_i corresponds to a particular $a_{i,\ell} \in A_i$. We let each $a_{i,\ell}$ corresponding to each α_i be in R' . We complete R' by including in it all vertices of $\text{bit-rep}(C)$. Then, clearly, $R' \subset R$ and $|R'| = k$.

We first show that R' dominates the set $B \setminus \{b_0\}$. To start with, the set $\text{bit-rep}(C)$ dominates the blue vertices of $Z \cup C$. Moreover, the vertices of B are dominated by the vertices $a_{i,\ell}$ picked in R' . Thus, R' dominates $B \setminus \{b_0\}$. We now show that R' locates every pair of vertices in B . Since b_0 is the only uncovered vertex, it is located. Each blue pendant vertex $z_{i,2}$ has a unique neighbor $z_{i,1}$ in R' , and no other blue vertex is only dominated by $z_{i,1}$ (since all vertices in C are dominated by at least two vertices in R' thanks to $z_{0,1}$). Hence, the vertices of Z are located by R' . The pairs $(c_j^\circ, c_{j'}^\circ)$, where $j \neq j'$, are located by R' on account of each such vertex having a unique neighborhood of size at least 2 in $\text{bit-rep}(C)$. The same argument applies to the pairs of the form $(c_j^\circ, c_{j'}^*)$ and $(c_j^*, c_{j'}^*)$, where $j \neq j'$. Each vertex b_i of B' is the only one that is adjacent to only one vertex of $R' \cap A$, and all of them are adjacent to a different one, so B' is located by R' . Each pair $(b_i, b_{i'})$ is located by the vertices $a_{i,\ell}$ picked in R' that dominate them. Finally, the pairs (c_j°, c_j^*) are located by the vertices in $A \cap R'$, since α is a satisfying assignment, and thus each vertex c_j° is dominated by at least one vertex of $A \cap R'$. This proves that R' , indeed, is a set of tests of the graph G .

Let us now assume that (G, k) is a YES-instance of TEST COVER. Therefore, let $R' \subseteq R$ be a test cover of G such that $|R'| \leq k$. Since b_0 is an isolated vertex of G , the set R' dominates $B \setminus \{b_0\}$ and locates every pair of B . Since, for each $i \in [p]$, the vertex $z_{i,2}$ is pendant and dominated by R' , R' must contain $z_{i,1}$. This implies that R' must contain at most r vertices from A . The only vertices of R' that can dominate vertex b_i of B' are those in A_i , where $i \in [r]$. Since there are r such vertices, the set R' must contain at exactly one vertex $a_{i,\ell}$, say, from A_i , for each $i \in [r]$. Since each $a_{i,\ell} \in R'$ corresponds to a partial assignment $\alpha_i : X_i \rightarrow \{\text{True}, \text{False}\}$, therefore, defining $\alpha : X \rightarrow \{\text{True}, \text{False}\}$ by $\alpha(x_r) = \alpha_i(x_r)$ if $x_r \in X_i$ gives an assignment α on ψ . Moreover, α is a satisfying assignment on ψ since, for each $j \in [m]$, the pair (c_j°, c_j^*) is located by some vertex $a_{i,\ell} \in A \cap R'$. Hence, by the construction of G , the corresponding assignment α_i satisfies the clause C_j . This implies that the assignment α satisfies each clause of ψ and hence, is a satisfying assignment of ψ . $\square$

We can now prove Theorem 1.3, that unless the ETH fails, TEST COVER does not admit an algorithm running in time $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, or a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{2^{o(k)}}$ vertices.

Proof of Theorem 1.3. Assume that there is an algorithm $\mathcal{A}$ that, given an instance $(G, \langle R, B \rangle, k)$ of TEST COVER, runs in time $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$ and correctly determines whether it is a YES-instance. Consider the following algorithm that takes as input an instance ψ of 3-SAT and determines whether it is a YES-instance. It first constructs an equivalent instance $(G, \langle R, B \rangle, k)$ of TEST COVER as mentioned in this subsection. Note that such an instance can be constructed in time $2^{\mathcal{O}(n/\log(n))}$ as the

most time consuming step is to enumerate all the possible assignment of $\mathcal{O}(n/\log(n))$ many variables in each bucket.

Then, it calls algorithm $\mathcal{A}$ as a subroutine and returns the same answer. The correctness of this algorithm follows from the correctness of algorithm $\mathcal{A}$ and the correctness of the reduction (Lemma 5.6). Moreover, from the reduction, we have $k = \mathcal{O}(\log(n))$. As the other steps, including the construction of the instance of TEST COVER, can be performed in time $2^{\mathcal{O}(n/\log(n))} \cdot n^{\mathcal{O}(1)} = 2^{o(n)}$, the running time of the algorithm for 3-SAT is $2^{2^{o(\log(n))}} \cdot (|R| + |B|)^{\mathcal{O}(1)} = 2^{o(n)} \cdot n^{\mathcal{O}(1)}$. This, however, contradicts ETH (using the sparsification lemma). Hence, our assumption is wrong, and TEST COVER does not admit an algorithm running in time $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, unless the ETH fails.

For the second part of Theorem 1.3, assume that a kernelization algorithm $\mathcal{B}$ exists that takes as input an instance $(G, \langle R, B \rangle, k)$ of TEST COVER and returns an equivalent instance with $2^{2^{o(k)}}$ vertices. Then, the brute-force enumerating all the possible solutions works in time $\binom{2^{2^{o(k)}}}{k} \cdot (|R| + |B|)^{\mathcal{O}(1)}$, which is $2^{k \cdot 2^{o(k)}} \cdot (|R| + |B|)^{\mathcal{O}(1)}$, which is $2^{2^{o(k)}} \cdot (|R| + |B|)^{\mathcal{O}(1)}$, contradicting the first result, and hence the ETH. Hence, TEST COVER does not admit a polynomial-time kernelization algorithm that returns a kernel with $2^{2^{o(k)}}$ vertices unless the ETH fails. $\square$

6. Conclusion. We presented several results that advance our understanding of the algorithmic complexity of LOCATING-DOMINATING SET and TEST COVER, which we showed to have very interesting and rare parameterized complexities. Moreover, we believe the techniques used in this article can be applied to other identification problems to obtain relatively rare conditional lower bounds. The process of establishing such lower bounds boils down to designing *bit-representation gadgets* and *set-representation gadgets* for the problem in question.

Apart from the broad question of designing such lower bounds for other identification problems, we mention an interesting problem left open by our work. Can our tight double-exponential lower bound for LOCATING-DOMINATING SET parameterized by vertex integrity (and thus, treedepth, pathwidth, and treewidth) be applied to the feedback vertex set number? In our reductions, after removing a central connecting gadget of logarithmic size, we obtain a collection of small connected components, and thus the graph has small vertex integrity. However, these components are not acyclic, which is why the feedback vertex set number is unbounded. Note that a single-exponential time algorithm with respect to feedback *edge* set number was presented by the authors in [13].

This type of question can of course also be studied for other parameters: see [10, 13] for works on other structural parameterizations of LOCATING-DOMINATING SET and TEST COVER.

As we have two algorithms for LOCATING-DOMINATING SET running in time $2^{\mathcal{O}(k^2)} + \mathcal{O}(k \log n)$ and $2^{2^{\mathcal{O}(\text{tw})}} n$, respectively, and they are optimal under ETH, it would be interesting to determine whether an algorithm running in time, say, $2^{\mathcal{O}(k + \text{tw})} n^{\mathcal{O}(1)}$ exists.

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