



Metric dimension and geodetic set parameterized by vertex cover[☆]

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ABSTRACT

For a graph G , a subset $S \subseteq V(G)$ is called a *resolving set* of G if, for any two vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that $d(w, u) \neq d(w, v)$. The **Metric Dimension** problem takes as input a graph G on n vertices and a positive integer k , and asks whether there exists a resolving set of size at most k . In another *metric-based graph problem*, **Geodetic Set**, the input is a graph G and an integer k , and the objective is to determine whether there exists a subset $S \subseteq V(G)$ of size at most k such that, for any vertex $u \in V(G)$, there are two vertices $s_1, s_2 \in S$ such that u lies on a shortest path from s_1 to s_2 . These two classical problems are intractable with respect to the solution size, as well as most structural parameters, including the feedback vertex set number and pathwidth. The very few existing tractable results state that they are both FPT with respect to the vertex cover number vc . Specifically, we observe that both problems admit an FPT algorithm running in $2^{O(\text{vc}^2)} \cdot n^{O(1)}$ time, and a kernelization algorithm that outputs a kernel with $2^{O(\text{vc})}$ vertices. For these problems, we also prove that, unless the Exponential Time Hypothesis fails, the established running time is asymptotically the best possible.

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1. Introduction

In this article, we study two *metric-based* graph problems, one of which is defined through distances, while the other relies on shortest paths. Metric-based graph problems are ubiquitous in computer science; for example, the classical (SINGLE-SOURCE) SHORTEST PATH, (GRAPHIC) TRAVELING SALESPERSON or STEINER TREE problems fall into this category. Those are fundamental problems, often stemming from applications in network design, for which a considerable amount of algorithmic research has been done. Metric-based graph packing and covering problems, like DISTANCE DOMINATION [29] or SCATTERED SET [30], have recently gained a lot of attention. Their non-local nature leads to non-trivial algorithmic properties that differ from most graph problems of a more local nature.

We focus here on the METRIC DIMENSION and GEODETIC SET problems, which arise from network monitoring and network design, respectively. As noted in the introduction of [6] and the conclusion of [31], and recently demonstrated in [3,19], these two metric-based graph covering problems share many algorithmic properties. They have far-reaching applications, as exemplified by, e.g., the recent work [3] where it was shown that enumerating minimal solution sets for METRIC DIMENSION and GEODETIC SET in (general) graphs and split graphs, respectively, is equivalent to the enumeration of minimal transversals in hypergraphs, whose solvability in total-polynomial time is arguably the most important open problem in algorithmic enumeration. Formally, these two problems are defined as follows.

METRIC DIMENSION

Input: A graph G on n vertices and a positive integer k .

Question: Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ with $d(w, u) \neq d(w, v)$?

GEODETIC SET

Input: A graph G on n vertices and a positive integer k .

Question: Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any vertex $u \in V(G)$, there exist two distinct vertices $s_1, s_2 \in S$ such that u lies on a shortest path from s_1 to s_2 ?

METRIC DIMENSION dates back to the 1970s [26,39], whereas GEODETIC SET was introduced in 1993 [25]. The non-local nature of these problems has since posed interesting algorithmic challenges. METRIC DIMENSION was first shown to be NP-complete in general graphs in Garey and Johnson's book [22], and this was later extended to many restricted graph classes (see 'Related work' below). GEODETIC SET was proven to be NP-complete in the seminal paper [25], and later shown to be NP-hard on restricted graph classes as well.

As these two problems are NP-hard even in very restricted cases, it is natural to ask for ways to confront this hardness. In this direction, the parameterized complexity paradigm allows for a more refined analysis of a problem's complexity. In this setting, we associate each instance I of a problem with a parameter ℓ , and are interested in algorithms running in $f(\ell) \cdot |I|^{\mathcal{O}(1)}$ time for some computable function f . Parameterized problems that admit such an algorithm are called fixed-parameter tractable (FPT for short) with respect to the considered parameter. Under standard complexity assumptions, parameterized problems that are hard for the complexity class $W[1]$ or $W[2]$ do not admit such algorithms. A parameter may originate from the formulation of the problem itself (called a *natural parameter*) or it can be a property of the input (called a *structural parameter*).

This approach, however, had limited success in the case of these two problems. In the seminal paper [28], METRIC DIMENSION was proven to be $W[2]$ -hard parameterized by the solution size k , even in subcubic bipartite graphs. Similarly, GEODETIC SET is $W[2]$ -hard parameterized by the solution size [15,31], even on chordal bipartite graphs. These initial hardness results drove the ensuing meticulous study of the problems under structural parameterizations. We present an overview of such results in 'Related work' below. In this article, we focus on the *vertex cover number*, denoted by vc , of the input graph and prove the following positive results.

Theorem 1. METRIC DIMENSION and GEODETIC SET admit

- FPT algorithms running in $2^{\mathcal{O}(\text{vc}^2)} \cdot n^{\mathcal{O}(1)}$ time, and
- kernelization algorithms that output kernels with $2^{\mathcal{O}(\text{vc})}$ vertices.

The second set of results follows from simple reduction rules, and was also observed in [28] for METRIC DIMENSION. The first set of results builds on the second set by using a simple, but critical observation. For METRIC DIMENSION, this also improves upon the $2^{2^{\mathcal{O}(\text{vc})}} \cdot n^{\mathcal{O}(1)}$ algorithm mentioned in [28]. Our main technical contribution, however, is in proving that these results are optimal assuming the Exponential Time Hypothesis (ETH).

Theorem 2. Unless the ETH fails, METRIC DIMENSION and GEODETIC SET do not admit

- FPT algorithms running in $2^{o(\text{vc}^2)} \cdot n^{\mathcal{O}(1)}$ time, nor
- kernelization algorithms that do not increase the solution size and output kernels with $2^{o(\text{vc})}$ vertices,

even on graphs of bounded diameter.

Both of these statements constitute a rare set of results in the existing literature. We know of only one other problem in the literature that admits a lower bound of the form $2^{o(vc^2)} \cdot n^{O(1)}$ and a matching upper bound [1] - whereas such results parameterized by pathwidth are mentioned in [37] and [38]. Very recently, the authors in [7] also proved a similar result with respect to solution size. Similarly, the list of known problems with exponential lower bounds on the number of vertices in kernelized instances is very short.⁵ To the best of our knowledge, the only known results of this kind (that is, ETH-based lower bounds on the number of vertices in a kernel) are for EDGE CLIQUE COVER [13], BICLIQUE COVER [9], STEINER TREE [36], STRONG METRIC DIMENSION [19], B-NCTD⁺ [8], LOCATING DOMINATING SET [7], and TELEPHONE BROADCASTING [41].⁶ For METRIC DIMENSION, the above also improves a result of [24], which states that METRIC DIMENSION parameterized by $k + vc$ does not admit a polynomial kernel unless the polynomial hierarchy collapses to its third level. Indeed, the result of [24] does not rule out a kernel of super-polynomial or sub-exponential size.

In a recent work [19], the present set of authors proved that, unless the ETH fails, METRIC DIMENSION and GEODETIC SET on graphs of bounded diameter do not admit $2^{2^{o(tw)}} \cdot n^{O(1)}$ -time algorithms, thereby establishing one of the first such results for NP-complete problems. Note that $n > vc > fvs > tw$ and $n > vc > td > pw > tw$ in the parameter hierarchy, where n is the number of vertices, fvs is the feedback vertex set number, td is the treedepth, and tw is the treewidth of the graph. The authors further proved that their lower bound also holds for fvs and td in the case of METRIC DIMENSION, and for td in the case of GEODETIC SET [19]. Note that a simple brute-force algorithm enumerating all possible candidates runs in $2^{O(n)}$ time for both of these problems. Thus, the next natural question is whether such a lower bound for METRIC DIMENSION and GEODETIC SET can be extended to larger parameters, in particular vc . Our first results answer this question in the negative. Together with the lower bounds with respect to vc , this establishes the boundary between parameters yielding single-exponential and double-exponential running times for METRIC DIMENSION and GEODETIC SET.

Before moving forward, we highlight the parallels and differences between [19] and our work. Their aim was to establish double-exponential lower bounds for NP-complete problems, and to do so they focused on the restriction of the problems to graphs of bounded treewidth and diameter. Our objective is to closely examine one of the very few tractable results for METRIC DIMENSION and GEODETIC SET on general graphs by focusing on the vertex cover parameter. While we use some gadgets from [19], overall our reductions significantly differ from the corresponding reductions in that article. Note that we need to “control” the vertex cover number of the reduced graph, whereas the corresponding reductions in [19] only need to “control” the treewidth.

Related work. We mention here results concerning structural parameterizations of METRIC DIMENSION and GEODETIC SET, and refer the reader to the full version of [19] for a more comprehensive overview of applications and related work regarding these two problems.

As previously mentioned, METRIC DIMENSION is W[2]-hard parameterized by the solution size k , even in subcubic bipartite graphs [28]. Several other parameterizations have been studied for this problem, on which we elaborate next (see also [21, Figure 1]). Through careful algorithmic design, kernelization, and/or meta-theorems, it was proven that there is an XP algorithm parameterized by the feedback edge set number [18], and FPT algorithms parameterized by the max leaf number [17], the modular-width and the treelength plus the maximum degree [2], the treedepth and the clique-width plus the diameter [23], and the distance to cluster (co-cluster, respectively) [21]. Recently, an FPT algorithm parameterized by the treewidth in chordal graphs was given in [5]. On the negative side, METRIC DIMENSION is W[1]-hard parameterized by the pathwidth even on graphs of constant degree [4], para-NP-hard parameterized by the pathwidth [34], and W[1]-hard parameterized by the combined parameter feedback vertex set number plus pathwidth [21].

The parameterized complexity of GEODETIC SET was first addressed in [31], in which they observed that the reduction from [15] implies that the problem is W[2]-hard parameterized by the solution size (even for chordal bipartite graphs). This motivated the authors of [31] to investigate structural parameterizations of GEODETIC SET. They proved the problem to be W[1]-hard for the combined parameters solution size, feedback vertex set number, and pathwidth, and FPT for the parameters treedepth, modular-width (more generally, clique-width plus diameter), and feedback edge set number. Recently, the hardness result for the combined parameters was strengthened, i.e., GEODETIC SET remains NP-hard on graphs of constant pathwidth and feedback vertex set number [40]. The problem was also shown to be FPT on chordal graphs when parameterized by the treewidth [6].

2. Preliminaries

For an integer a , we let $[a] = \{1, \dots, a\}$.

Graph theory. We generally use standard graph-theoretic notation and refer the reader to [14] for any undefined notation. For an undirected graph G , the sets $V(G)$ and $E(G)$ denote its set of vertices and edges, respectively. Two vertices $u, v \in V(G)$ are *adjacent* or *neighbors* if $(u, v) \in E(G)$. The *open neighborhood* of a vertex $u \in V(G)$, denoted by $N(u) := N_G(u)$, is the set of vertices that are neighbors of u . The *closed neighborhood* of a vertex $u \in V(G)$ is denoted by $N[u] := N_G[u] :=$

⁵ For the definition of a kernelized instance and kernelization algorithm, refer to Section 2 or [12].

⁶ POINT LINE COVER also does not admit a kernel with $O(k^{2-\epsilon})$ vertices, for any $\epsilon > 0$, unless $NP \subseteq coNP/poly$ [32].

$N_G(u) \cup \{u\}$. For any $X \subseteq V(G)$ and $u \in V(G)$, $N_X(u) = N_G(u) \cap X$. Any two vertices $u, v \in V(G)$ are *true twins* if $N[u] = N[v]$, and are *false twins* if $N(u) = N(v)$. Observe that if u and v are true twins, then $(u, v) \in E(G)$, but if they are false twins, then $(u, v) \notin E(G)$. For a subset S of $V(G)$, we say that the vertices in S are true (false, respectively) twins if, for any $u, v \in S$, u and v are true (false, respectively) twins. The *distance* between two vertices $u, v \in V(G)$ in G , denoted by $d(u, v) := d_G(u, v)$, is the length of a (u, v) -shortest path in G . For a subset S of $V(G)$, we define $N[S] = \bigcup_{v \in S} N[v]$ and $N(S) = N[S] \setminus S$. For a graph G , a set $X \subseteq V(G)$ is said to be a *vertex cover* if $V(G) \setminus X$ is an independent set. We denote by $\nu_c(G)$ the size of a minimum vertex cover in G . When G is clear from the context, we simply write ν_c .

Metric dimension and geodetic set. A subset of vertices $S \subseteq V(G)$ *resolves* a pair of vertices $u, v \in V(G)$ if there exists a vertex $w \in S$ such that $d(w, u) \neq d(w, v)$. A subset of vertices $S \subseteq V(G)$ is a *resolving set* of G if it resolves all pairs of vertices $u, v \in V(G)$. A vertex $u \in V(G)$ is *distinguished* by a subset of vertices $S \subseteq V(G)$ if, for any $v \in V(G) \setminus \{u\}$, there exists a vertex $w \in S$ such that $d(w, u) \neq d(w, v)$.

Observation 1. Let G be a graph. For any (true or false) twins $u, v \in V(G)$ and any $w \in V(G) \setminus \{u, v\}$, $d(u, w) = d(v, w)$, and so, for any resolving set S of G , $S \cap \{u, v\} \neq \emptyset$.

Proof. As $w \in V(G) \setminus \{u, v\}$, and u and v are (true or false) twins, the shortest (u, w) - and (v, w) -paths contain a vertex of $N := N(u) \setminus \{v\} = N(v) \setminus \{u\}$, and $d(u, w) = d(v, w)$. Hence, any resolving set S of G contains at least one of u and v . $\square$

A subset $S \subseteq V(G)$ is a *geodetic set* if for every $u \in V(G)$, the following holds: there exist $s_1, s_2 \in S$ such that u lies on a shortest path from s_1 to s_2 . The following simple observation is used throughout the paper. Recall that a vertex is *simplicial* if its neighborhood forms a clique. Observe that any simplicial vertex v does not belong to any shortest path between any pair x, y of vertices (both distinct from v). Hence, the following observation follows:

Observation 2 ([10]). If a graph G contains a simplicial vertex v , then v belongs to any geodetic set of G . Specifically, every degree-1 vertex belongs to any geodetic set of G .

Parameterized complexity. An instance of a parameterized problem Π comprises an input I , which is an input of the classical instance of the problem, and an integer ℓ , which is called the parameter. A problem Π is said to be *fixed-parameter tractable* or in FPT if given an instance (I, ℓ) of Π , we can decide whether or not (I, ℓ) is a Yes-instance of Π in $f(\ell) \cdot |I|^{\mathcal{O}(1)}$ time, for some computable function f whose value depends only on ℓ .

A *kernelization* algorithm for Π is a polynomial-time algorithm that takes as input an instance (I, ℓ) of Π and returns an *equivalent* instance (I', ℓ') of Π , where $|I'|, \ell' \leq f(\ell)$, for some function f that depends only on the initial parameter ℓ . If such an algorithm exists for Π , we say that Π admits a kernel of size $f(\ell)$. If f is a polynomial or exponential function of ℓ , we say that Π admits a polynomial or exponential kernel, respectively. If Π is a graph problem, then I contains a graph, say G , and I' contains a graph, say G' . In this case, we say that Π admits a kernel with $f(\ell)$ vertices if the number of vertices of G' is at most $f(\ell)$.

It is typical to describe a kernelization algorithm as a series of reduction rules. A *reduction rule* is a polynomial-time algorithm that takes as an input an instance of a problem and outputs another (usually reduced) instance. A reduction rule is said to be *applicable* on an instance if the output instance is different from the input instance. A reduction rule is *safe* if the input instance is a Yes-instance if and only if the output instance is a Yes-instance.

The Exponential Time Hypothesis roughly states that n -variable 3-SAT cannot be solved in $2^{o(n)}$ time. For more on parameterized complexity and related terminologies, we refer the reader to [12].

3-PARTITIONED-3-SAT. Our lower bound proofs consist of reductions from the 3-PARTITIONED-3-SAT problem. This version of 3-SAT was introduced in [33] and is defined as follows.

3-PARTITIONED-3-SAT

Input: A formula ψ in 3-CNF form, together with a partition of the set of its variables into three disjoint sets $X^\alpha, X^\beta, X^\gamma$, with $|X^\alpha| = |X^\beta| = |X^\gamma| = N$, and such that no clause contains more than one variable from each of X^α, X^β , and X^γ .

Question: Determine whether ψ is satisfiable.

The authors of [33] also proved the following.

Proposition 1 ([33, Theorem 3]). Unless the ETH fails, 3-PARTITIONED-3-SAT does not admit an algorithm running in $2^{o(N)}$ time.

3. METRIC DIMENSION: lower bounds regarding vertex cover

In this section, we first prove the following theorem.

Theorem 3. *There is an algorithm that, given an instance ψ of 3-PARTITIONED-3-SAT on $3N$ variables, runs in $2^{\mathcal{O}(\sqrt{N})}$ time, and constructs an equivalent instance (G, k) of METRIC DIMENSION such that $\text{vc}(G) + k = \mathcal{O}(\sqrt{N})$, $|V(G)| = 2^{\mathcal{O}(\sqrt{N})}$, and the diameter of G is upper-bounded by a constant.*

The above theorem, along with the arguments that are standard to prove the ETH-based lower bounds, immediately implies the following results.

Corollary 1. *Unless the ETH fails, METRIC DIMENSION does not admit an algorithm running in $2^{o(\text{vc}^2)} \cdot n^{\mathcal{O}(1)}$ time, even on graphs of bounded diameter.*

Corollary 2. *Unless the ETH fails, METRIC DIMENSION does not admit a kernelization algorithm that does not increase the solution size k and outputs a kernel with $2^{o(k+\text{vc})}$ vertices, even on graphs of bounded diameter.*

Proof. For the sake of contradiction, assume that such a kernelization algorithm exists. Consider the following algorithm for 3-PARTITIONED-3-SAT. Given a 3-PARTITIONED-3-SAT formula on $3N$ variables, it uses Theorem 3 to obtain an equivalent instance (G, k) of METRIC DIMENSION such that $\text{vc}(G) + k = \mathcal{O}(\sqrt{N})$, $|V(G)| = 2^{\mathcal{O}(\sqrt{N})}$, and G has bounded diameter. Then, it uses the assumed kernelization algorithm to construct an equivalent instance (H, k') such that H has $2^{o(\text{vc}(G)+k)}$ vertices and $k' \leq k$. Finally, it uses a brute-force algorithm, running in $|V(H)|^{\mathcal{O}(k')}$ time, to determine whether the reduced instance, or equivalently the input instance of 3-PARTITIONED-3-SAT, is a Yes-instance. The correctness of the algorithm follows from the correctness of the respective algorithms and our assumption. The total running time of the algorithm is

$$2^{\mathcal{O}(\sqrt{N})} + (|V(G)| + k)^{\mathcal{O}(1)} + |V(H)|^{\mathcal{O}(k')} = 2^{\mathcal{O}(\sqrt{N})} + (2^{\mathcal{O}(\sqrt{N})})^{\mathcal{O}(1)} + (2^{o(\sqrt{N})})^{\mathcal{O}(\sqrt{N})} = 2^{o(N)}.$$

But, by Proposition 1, this contradicts the ETH. $\square$

Before presenting the reduction, we first introduce some preliminary tools.

3.1. Preliminary tools

3.1.1. Set identifying gadget

We redefine a gadget introduced in [19]. Suppose we are given a graph G' and a subset $X \subseteq V(G')$ of its vertices. Further, suppose that we want to add a vertex set X^+ to G' in order to obtain a new graph G such that (1) each vertex in $X \cup X^+$ will be distinguished by vertices in X^+ that must be in any resolving set S of G , and (2) no vertex in X^+ can resolve any pair of vertices in $V(G) \setminus (X \cup X^+)$ that are in the same distance class with respect to X . The graph induced by the vertices of X^+ , along with the edges connecting X^+ to G' , is referred to as the Set Identifying Gadget for the set X [19].

Given a graph G' and a non-empty subset $X \subseteq V(G')$ of its vertices, to construct such a graph G , we add vertices and edges to G' as follows:

- The vertex set X^+ that we are aiming to add is the union of a set $\text{bit-rep}(X)$ and a special vertex denoted by $\text{nullifier}(X)$.
- Let $X := \{x_i \mid i \in [|X|]\}$ and set $q := \lceil \log(|X| + 2) \rceil + 1$. We select this value for q to (1) uniquely represent each integer in $[|X|]$ by its bit-representation in binary (note that we start from 1 and not 0), (2) ensure that the only vertex whose bit-representation contains all 1's is $\text{nullifier}(X)$, and (3) reserve one spot for an additional vertex y_* .
- For every $i \in [q]$, add three vertices y_i^a, y_i, y_i^b , and add the path (y_i^a, y_i, y_i^b) .
- Add three vertices y_*, y_*, y_* , and add the path (y_*, y_*, y_*) . Add all the edges to make $\{y_i \mid i \in [q]\} \cup \{y_*\}$ a clique. Make y_* adjacent to each vertex $v \in X$. We denote $\text{bit-rep}(X) := \{y_i, y_i^a, y_i^b \mid i \in [q]\} \cup \{y_*, y_*, y_*\}$ and its subset $\text{bits}(X) := \{y_i^a, y_i^b \mid i \in [q]\} \cup \{y_*, y_*\}$ for convenience in a later case analysis.
- For every integer $j \in [|X|]$, let $\text{bin}(j)$ denote the binary representation of j using q bits. Connect x_j with y_i if the i^{th} bit (going from left to right) in $\text{bin}(j)$ is 1.
- Add a vertex, denoted by $\text{nullifier}(X)$, and make it adjacent to every vertex in $\{y_i \mid i \in [q]\} \cup \{y_*\}$. One can think of $\text{nullifier}(X)$ as the only vertex whose bit-representation contains all 1's.
- For every vertex $u \in V(G) \setminus (X \cup X^+)$ such that u is adjacent to some vertex in X , add an edge between u and $\text{nullifier}(X)$. We add this vertex to ensure that vertices in $\text{bit-rep}(X)$ do not resolve certain pairs in $V(G)$ that we later refer to as “critical pairs”.

This completes the construction of G . The properties of G are not proven yet, but just given as an intuition behind its construction. See Fig. 1 for an illustration.

3.1.2. Gadget to add critical pairs

Any resolving set needs to resolve *all* pairs of vertices in the input graph. As we will see, some pairs, which we call critical pairs, are harder to resolve than others. In fact, the non-trivial part will be to resolve all of the critical pairs.

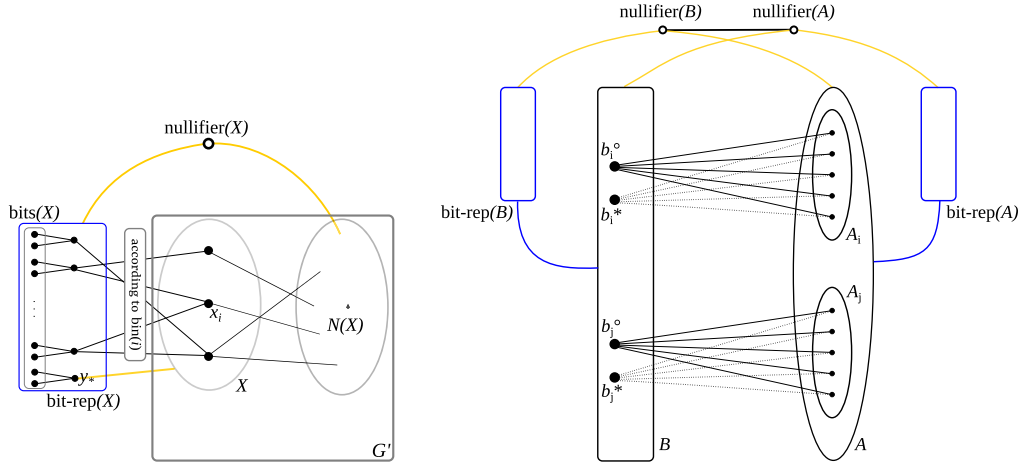


Fig. 1. Set Identifying Gadget (left). The blue box represents $\text{bit-rep}(X)$. The yellow lines represent that all possible edges exist between $\text{bit-rep}(X) \setminus \text{bits}(X)$ and $\text{nullifier}(X)$, $\text{nullifier}(X)$ and $N(X)$, and y_* and X . Note that G' is not necessarily restricted to the graph induced by the vertices in $X \cup N(X)$. **Vertex Selector Gadget (right).** For $X \in \{B, A\}$, the blue box represents $\text{bit-rep}(X)$, the blue link represents the edges with respect to the binary representation, and the yellow line represents that $\text{nullifier}(X)$ is adjacent to each vertex in $\text{bit-rep}(X) \setminus \text{bits}(X)$. Dotted lines highlight absent edges. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Suppose that we need to have $m \in \mathbb{N}$ critical pairs in a graph G , say $\langle c_i^\circ, c_i^* \rangle$ for every $i \in [m]$. Define $C^\circ := \{c_i^\circ \mid i \in [m]\}$, $C^* := \{c_i^* \mid i \in [m]\}$, and $C := \{c_i^\circ, c_i^* \mid i \in [m]\}$. To make the pair $\langle c_i^\circ, c_i^* \rangle$ critical for every $i \in [m]$, we first attach a Set Identifying Gadget for C° to the vertices of C° (as was done for the set X above). Then, we add the necessary edges to ensure that, for all $i \in [m]$, c_i° and c_i^* have the same neighbors in $\text{bit-rep}(C^\circ)$. Furthermore, we make $\text{nullifier}(C^\circ)$ adjacent to every vertex in $N(C^*) \setminus (C \cup \text{bit-rep}(C^\circ) \cup \text{nullifier}(C^\circ))$. Finally, since this gadget is with respect to C , for simplicity, we rename $\text{bit-rep}(C^\circ)$ and $\text{nullifier}(C^\circ)$ to $\text{bit-rep}(C)$ and $\text{nullifier}(C)$, respectively. Hence, $\text{bit-rep}(C)$ can resolve any pair of the form $\langle c_i^\circ, c_\ell^\circ \rangle$, $\langle c_i^\circ, c_\ell^* \rangle$, or $\langle c_i^*, c_\ell^* \rangle$ as long as $i \neq \ell$. As before, $\text{bit-rep}(C)$ can also resolve all pairs with one vertex in $C \cup \text{bit-rep}(C) \cup \{\text{nullifier}(C)\}$, but no critical pair of vertices. Again, when these facts will be used, they will be proven formally.

3.1.3. Vertex selector gadgets

Suppose that we are given a collection of sets $A_1, A_2, \dots, A_q$ of vertices in a graph G , and we want to ensure that any resolving set of G includes at least one vertex from A_i for every $i \in [q]$. In the following, we construct a gadget that achieves a slightly weaker objective.

- Let $A := \bigcup_{i \in [q]} A_i$. Add a set identifying gadget for A as mentioned in Subsection 3.1.1.
- For every $i \in [q]$, add two vertices b_i° and b_i^* . Let $B := \{b_i^\circ, b_i^* \mid i \in [q]\}$. Use the gadget mentioned in Subsection 3.1.2 to make all the pairs of the form $\langle b_i^\circ, b_i^* \rangle$ critical pairs.
- For every $a \in A_i$, add an edge (a, b_i°) . We highlight that we do not make a adjacent to b_i^* by a dotted line in Fig. 1. Finally, add the edges $(b_i^*, \text{nullifier}(A))$ and $(\text{nullifier}(A), \text{nullifier}(B))$. Note that the edges $(a, \text{nullifier}(B))$ and $(b_i^\circ, \text{nullifier}(A))$ are also present by the definitions of the gadgets.

This completes the construction.

Note that the only vertices that can resolve a critical pair $\langle b_i^\circ, b_i^* \rangle$, apart from b_i° and b_i^* , are the vertices in A_i . Hence, every resolving set contains at least one vertex in $\{b_i^\circ, b_i^*\} \cup A_i$. Again, when used, these facts will be proven formally.

3.2. Reduction

Consider an instance ψ of 3-PARTITIONED-3-SAT with $X^\alpha, X^\beta, X^\gamma$ the partition of the variable set. By adding dummy variables in each of these sets, we can assume that $\sqrt{N}$ is an integer greater than 2. From ψ , we construct the graph G as follows. We describe the construction of the part of the graph G that corresponds to X^α , with the parts corresponding to X^β and X^γ being analogous. We rename the variables in X^α to $x_{i,j}^\alpha$ for $i, j \in [\sqrt{N}]$.

- We partition the variables of X^α into buckets $X_1^\alpha, X_2^\alpha, \dots, X_{\sqrt{N}}^\alpha$ such that each bucket contains $\sqrt{N}$ variables. Let $X_i^\alpha := \{x_{i,j}^\alpha \mid j \in [\sqrt{N}]\}$ for all $i \in [\sqrt{N}]$.

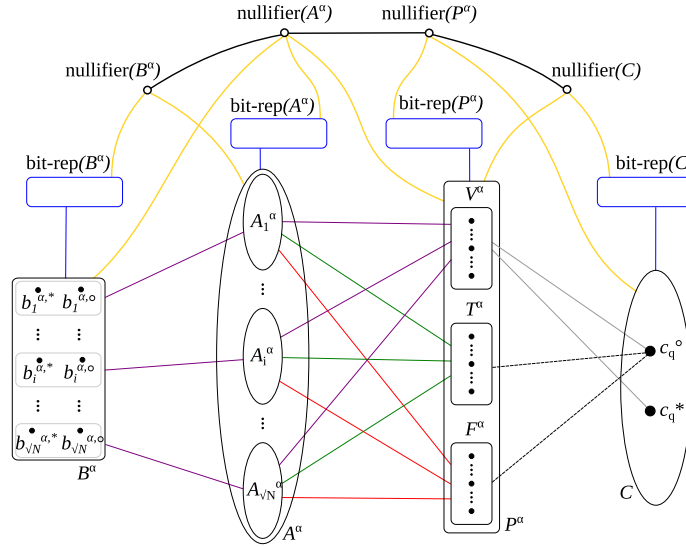


Fig. 2. Overview of the reduction. Sets in ellipses are independent sets and sets in rectangles are cliques. For $X \in \{B^\alpha, A^\alpha, P^\alpha, C\}$, the blue rectangle attached to it via the blue edge represents $\text{bit-rep}(X)$. We omit $\text{bits}(X)$ for legibility. The yellow line represents that $\text{nullifier}(X)$ is adjacent to every vertex in the set. Note the exception of $\text{nullifier}(P^\alpha)$ which is not adjacent to any vertex in A^α . Purple lines between two sets denote adjacencies with respect to indexing, i.e., $b_{i,*,*}^{\alpha,*}$ is adjacent only with all the vertices in A_i^α , and all the vertices in A_i^α are adjacent to v_i^α in validation portal V^α . Gray lines also indicate adjacencies with respect to indexing, but in a complementary way. If C_q contains a variable in X_i^α , then c_q^o and c_q^* are adjacent to all vertices $v_j^\alpha \in V^\alpha$ such that $j \neq i$. Green and red lines between the A^α and T^α and F^α roughly transfer, for each $a_{i,\ell}^\alpha \in A^\alpha$, the underlying assignment structure as formally described in Section 3.2. Similarly, we add edges for each $c_i^\alpha \in C$ depending on the assignment satisfying the variable from the part X^δ of a clause c_i , and in which block B_j^δ it lies, putting either an edge (c_i^α, t_j^α) or (c_i^α, f_j^α) accordingly ($\delta \in \{\alpha, \beta, \gamma\}$).

- For every X_i^α , we construct a set A_i^α of $2^{\sqrt{N}}$ new vertices, where $A_i^\alpha := \{a_{i,\ell}^\alpha \mid \ell \in [2^{\sqrt{N}}]\}$. Each vertex in A_i^α corresponds to a certain possible assignment of variables in X_i^α . Let A^α be the collection of all the vertices added in the above step, that is, $A^\alpha := \{a_{i,\ell}^\alpha \in A_i^\alpha \mid i \in [\sqrt{N}] \text{ and } \ell \in [2^{\sqrt{N}}]\}$. We add a set identifying gadget as mentioned in Subsection 3.1.1 in order to resolve every pair of vertices in A^α .
- For every X_i^α , we construct a pair $\langle b_{i,*,*}^{\alpha,o}, b_{i,*,*}^{\alpha,*} \rangle$ of vertices. Then, we add a gadget to make the pairs $\{\langle b_{i,*,*}^{\alpha,o}, b_{i,*,*}^{\alpha,*} \rangle \mid i \in [\sqrt{N}]\}$ critical as mentioned in Subsection 3.1.2. Let $B^\alpha := \{b_{i,*,*}^{\alpha,o}, b_{i,*,*}^{\alpha,*} \mid i \in [\sqrt{N}]\}$ be the collection of vertices in the critical pairs. We add edges in B^α to make it a clique.
- We would like that any resolving set contains at least one vertex in A_i^α for every $i \in [\sqrt{N}]$, but instead we add the construction from Subsection 3.1.3 that achieves the slightly weaker objective as mentioned there. However, for every A_i^α , instead of adding two new vertices, we use $\langle b_{i,*,*}^{\alpha,o}, b_{i,*,*}^{\alpha,*} \rangle$ as the necessary critical pair. Thus, for every $i \in [\sqrt{N}]$, $b_{i,*,*}^{\alpha,o}$ is adjacent to every vertex in A_i^α . Furthermore, $\text{nullifier}(B^\alpha)$ is adjacent to every vertex in A^α , $\text{nullifier}(A^\alpha)$ is adjacent to every vertex in B^α , and there is also the edge $(\text{nullifier}(B^\alpha), \text{nullifier}(A^\alpha))$.
- We add *portals* that transmit information from vertices corresponding to assignments, i.e., vertices in A^α , to critical pairs corresponding to clauses, i.e., vertices in C which we define soon. A portal is a clique on $\sqrt{N}$ vertices in the graph G . We add three portals, the *truth portal* (denoted by T^α), *false portal* (denoted by F^α), and *validation portal* (denoted by V^α). Let $T^\alpha := \{t_1^\alpha, t_2^\alpha, \dots, t_{\sqrt{N}}^\alpha\}$, $F^\alpha := \{f_1^\alpha, f_2^\alpha, \dots, f_{\sqrt{N}}^\alpha\}$, and $V^\alpha := \{v_1^\alpha, v_2^\alpha, \dots, v_{\sqrt{N}}^\alpha\}$. Moreover, let $P^\alpha := V^\alpha \cup T^\alpha \cup F^\alpha$.
- We add edges between A^α and the portals as follows. For $i \in [\sqrt{N}]$ and $\ell \in [2^{\sqrt{N}}]$, consider a vertex $a_{i,\ell}^\alpha$ in A_i^α . Recall that this vertex corresponds to an assignment $\pi : X_i^\alpha \mapsto \{\text{True}, \text{False}\}$, where X_i^α is the collection of variables $\{x_{i,j}^\alpha \mid j \in [\sqrt{N}]\}$. If $\pi(x_{i,j}^\alpha) = \text{True}$, then we add the edge $(a_{i,\ell}^\alpha, t_j^\alpha)$. Otherwise, $\pi(x_{i,j}^\alpha) = \text{False}$, and we add the edge $(a_{i,\ell}^\alpha, f_j^\alpha)$. We add the edge $(a_{i,\ell}^\alpha, v_i^\alpha)$ for every $\ell \in [2^{\sqrt{N}}]$.
- We add a set identifying gadget for P^α as mentioned in Subsection 3.1.1. However, while there is an edge $(\text{nullifier}(P^\alpha), \text{nullifier}(A^\alpha))$ and an edge between $\text{nullifier}(A^\alpha)$ and every vertex of P^α , we *do not* add edges between $\text{nullifier}(P^\alpha)$ and A^α , as can be seen in Fig. 2. Lastly, we add edges in P^α to make it a clique.

Then, we repeat the above steps to construct $B^\beta, A^\beta, P^\beta, B^\gamma, A^\gamma, P^\gamma$.

Now, we are ready to proceed through the final steps to complete the construction.

- For every clause C_q in ψ , we introduce a pair of vertices $\langle c_q^o, c_q^* \rangle$. Let C be the collection of vertices in such pairs. Then, we add a gadget as was described in Subsection 3.1.2 to make each pair $\langle c_q^o, c_q^* \rangle$ a critical one.

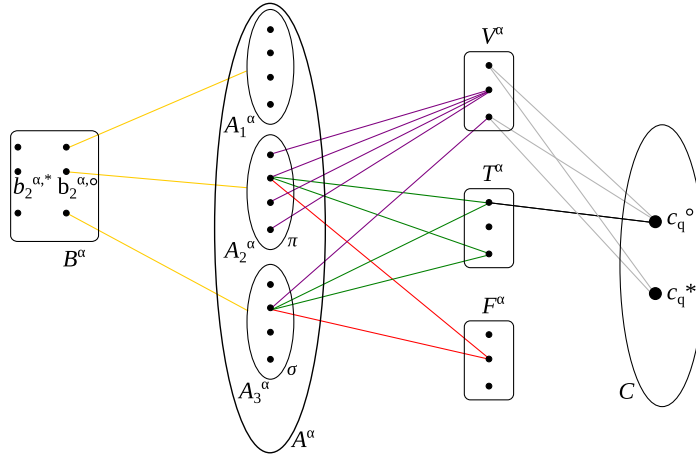


Fig. 3. A toy example illustrating the core ideas in the reduction. Note that bit-rep and nullifier are omitted for the sets.

- For each $\delta \in \{\alpha, \beta, \gamma\}$, we add the edge $(\text{nullifier}(P^\delta), \text{nullifier}(C))$, an edge between $\text{nullifier}(P^\delta)$ and each vertex of C , and an edge between $\text{nullifier}(C)$ and each vertex of P^δ . We add edges between C and the portals as follows for each $\delta \in \{\alpha, \beta, \gamma\}$. Consider a clause C_q in ψ and the corresponding critical pair $\langle c_q^o, c_q^* \rangle$ in C . As ψ is an instance of 3-PARTITIONED-3-SAT, there is at most one variable in X^δ that appears in C_q . If C_q does not contain a variable in X^δ , then we make c_q^o and c_q^* adjacent to every vertex in V^δ , and they are not adjacent to any vertex in $T^\delta \cup F^\delta$. Otherwise, suppose that C_q contains the variable $x_{i,j}^\delta$ for some $i, j \in [\sqrt{N}]$. The first subscript decides the edges between $\langle c_q^o, c_q^* \rangle$ and the validation portal, whereas the second one decides the edges between $\langle c_q^o, c_q^* \rangle$ and either the truth portal or false portal in the following sense. We add all edges of the form $(v_{i'}^\delta, c_q^o)$ and $(v_{i'}^\delta, c_q^*)$ for every $i' \in [\sqrt{N}]$ such that $i' \neq i$. If $x_{i,j}^\delta$ appears as a positive literal in C_q , then we add the edge (t_j^δ, c_q^o) . Otherwise, $x_{i,j}^\delta$ appears as a negative literal in C_q , and we add the edge (f_j^δ, c_q^o) .

This concludes the construction of G . See Fig. 2 for an illustration of part of G . The reduction returns (G, k) as an instance of METRIC DIMENSION where

$$k = 3 \cdot \left(\sqrt{N} + (\lceil \log(|B^\alpha|/2 + 2) \rceil + 1) + (\lceil \log(|A^\alpha| + 2) \rceil + 1) + (\lceil \log(|P^\alpha| + 2) \rceil + 1) \right) + \lceil \log(|C|/2 + 2) \rceil + 1.$$

We give an informal description of the proof of correctness here. See Fig. 3. Suppose $\sqrt{N} = 3$ and the vertices in the sets are indexed from top to bottom. For the sake of clarity, we do not show all the edges and only show 4 out of 8 vertices in each A_i^α for $i \in [3]$. We also omit bit-rep and nullifier for these sets. The vertex selection gadget and the budget k ensure that exactly one vertex in $\{b_i^{\alpha,o}, b_i^{\alpha,*}\} \cup A_i^\alpha$ is selected for every $i \in [3]$. If a resolving set contains a vertex from A_i^α , then it corresponds to selecting an assignment of variables in X_i^α . For example, the vertex $a_{2,2}^\alpha$ corresponds to the assignment $\pi : X_2^\alpha \mapsto \{\text{True}, \text{False}\}$. Suppose $X_2^\alpha = \{x_{2,1}^\alpha, x_{2,2}^\alpha, x_{2,3}^\alpha\}$, $\pi(x_{2,1}^\alpha) = \pi(x_{2,3}^\alpha) = \text{True}$, and $\pi(x_{2,2}^\alpha) = \text{False}$. Hence, $a_{2,2}^\alpha$ is adjacent to the first and third vertex in the truth portal T^α , whereas it is adjacent to the second vertex in the false portal F^α . Suppose the clause C_q contains the variable $x_{2,1}^\alpha$ as a positive literal. Note that c_q^o and c_q^* are at distance 2 and 3, respectively, from $a_{2,2}^\alpha$. Hence, the vertex $a_{2,2}^\alpha$, corresponding to the assignment π that satisfies clause C_q , resolves the critical pair $\langle c_q^o, c_q^* \rangle$. Now, suppose there is another assignment $\sigma : X_3^\alpha \mapsto \{\text{True}, \text{False}\}$ such that $\sigma(x_{3,1}^\alpha) = \sigma(x_{3,3}^\alpha) = \text{True}$ and $\sigma(x_{3,2}^\alpha) = \text{False}$. As ψ is an instance of 3-PARTITIONED-3-SAT and C_q contains a variable in $X_2^\alpha (\subseteq X^\alpha)$, C_q does not contain a variable in $X_3^\alpha (\subseteq X^\alpha)$. Hence, σ does not satisfy C_q . Let $a_{3,2}^\alpha$ be the vertex in A_3^α corresponding to σ . The connections via the validation portal V^α ensure that both c_q^o and c_q^* are at distance 2 from $a_{3,2}^\alpha$, and hence, $a_{3,2}^\alpha$ cannot resolve the critical pair $\langle c_q^o, c_q^* \rangle$. Hence, finding a resolving set in G corresponds to finding a satisfying assignment for ψ . These intuitions are formalized in the following subsection.

3.3. Correctness of the reduction

Suppose, given an instance ψ of 3-PARTITIONED-3-SAT with m clauses, that the reduction in the previous subsection returns (G, k) as an instance of METRIC DIMENSION. We first prove the following lemma which will be helpful in proving the correctness of the reduction.

Lemma 1. For any resolving set S of G and for all $X \in \{C\} \cup \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$, the following hold:

- (a) S contains at least one vertex from each pair of false twins in $\text{bits}(X)$.

- (b) Vertices in $\text{bits}(X) \cap S$ resolve any non-critical pair of vertices $\langle u, v \rangle$ when $u \in X \cup X^+$ and $v \in V(G)$.
(c) Vertices in $X^+ \cap S$ cannot resolve any critical pair of vertices $\langle b_i^{\delta', \circ}, b_i^{\delta', *}, c_q^{\circ}, c_q^* \rangle$ for all $i \in [\sqrt{N}]$, $\delta' \in \{\alpha, \beta, \gamma\}$, and $q \in [m]$.

Proof. (a) By Observation 1, the statement follows for all $X \in \{C\} \cup \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$.

- (b) For all $X \in \{C\} \cup \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$, note that $\text{nullifier}(X)$ is distinguished by $\text{bits}(X) \cap S$ since it is the only vertex in G that is at distance 2 from every vertex in $\text{bits}(X)$. We now do a case analysis for the remaining non-critical pairs of vertices $\langle u, v \rangle$ assuming that $\text{nullifier}(X) \notin \{u, v\}$ (also, suppose that both u and v are not in S , as otherwise, they are obviously distinguished):

Case 1: $u, v \in X \cup X^+$.

Case 1(i): $u \in X$ and $v \in \text{bit-rep}(X)$. Without loss of generality, $y_\star^a \in S \cap \text{bits}(X)$ (by Lemma 1(a)). Then, $d(y_\star^a, u) = 2$ and, for all $v \in \text{bits}(X) \setminus \{y_\star^a, y_\star^b\}$, $d(y_\star^a, v) = 3$. Without loss of generality, let y_i be adjacent to u and let $y_i^a \in S \cap \text{bits}(X)$ (by Lemma 1(a)). Then, for $v = y_\star^b$, $3 = d(y_i^a, v) \neq d(y_i^a, u) = 2$. If $v \in \text{bit-rep}(X) \setminus \text{bits}(X)$, then, without loss of generality, $v = y_j$ and $y_j^a \in S \cap \text{bits}(X)$ (by Lemma 1(a)), and $1 = d(y_j^a, v) < d(y_j^a, u)$.

Case 1(ii): $u, v \in X$ or $u, v \in \text{bit-rep}(X) \setminus \text{bits}(X)$. In the first case, let j be the bit in the binary representation of the subscript of u that is not equal to the j^{th} bit in the binary representation of the subscript of v (such a j exists since $\langle u, v \rangle$ is not a critical pair). In the second case, without loss of generality, let $u = y_i$ and $v = y_j$. Without loss of generality, $y_j^a \in S \cap \text{bits}(X)$ (by Lemma 1(a)). Then, in both cases, $d(y_j^a, u) \neq d(y_j^a, v)$.

Case 1(iii): $u, v \in \text{bits}(X)$. Without loss of generality, $u = y_i^b$ and $y_i^a \in S$ (by Lemma 1(a)). Then, $2 = d(y_i^a, u) \neq d(y_i^a, v) = 3$.

Case 1(iv): $u \in \text{bits}(X)$ and $v \in \text{bit-rep}(X) \setminus \text{bits}(X)$. Without loss of generality, $v = y_i$ and $y_i^a \in S$ (by Lemma 1(a)). Then, $1 = d(y_i^a, v) < d(y_i^a, u)$.

Case 2: $u \in X \cup X^+$ and $v \in V(G) \setminus (X \cup X^+)$. For each $u \in X \cup X^+$, there exists $w \in \text{bits}(X) \cap S$ such that $d(u, w) \leq 2$, while, for each $v \in V(G) \setminus (X \cup X^+)$ and $w \in \text{bits}(X) \cap S$, we have $d(v, w) \geq 3$.

- (c) For all $X \in \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$, $u \in X^+$, $v \in \{c_q^{\circ}, c_q^*\}$, and $q \in [m]$, we have that $d(u, v) = d(u, \text{nullifier}(P^\delta)) + 1$. Further, for $X = C$ and all $u \in X^+$ and $q \in [m]$, either $d(u, c_q^{\circ}) = d(u, c_q^*) = 1$, $d(u, c_q^{\circ}) = d(u, c_q^*) = 2$, or $d(u, c_q^{\circ}) = d(u, c_q^*) = 3$ by the construction in Subsection 3.1.2 and since $\text{bit-rep}(X) \setminus \text{bits}(X)$ is a clique. Hence, for all $X \in \{C\} \cup \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$, vertices in $X^+ \cap S$ cannot resolve a pair of vertices $\langle c_q^{\circ}, c_q^* \rangle$ for any $q \in [m]$.

For all $\delta' \in \{\alpha, \beta, \gamma\}$, if $v \in B^{\delta'}$, then, for all $X \in \{C\} \cup \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$ such that $\delta \neq \delta'$, and $u \in X^+$, we have that $d(u, v) = d(u, \text{nullifier}(A^{\delta'})) + 1$. Similarly, for all $\delta' \in \{\alpha, \beta, \gamma\}$, if $v \in B^{\delta'}$, then, for all $X \in \{A^{\delta'}, P^{\delta'}\}$ and $u \in X^+$, we have that $d(u, v) = d(u, \text{nullifier}(A^{\delta'})) + 1$. Lastly, for each $\langle b_i^{\delta', \circ}, b_i^{\delta', *} \rangle$, $\delta' \in \{\alpha, \beta, \gamma\}$, and $i \in [\sqrt{N}]$, if $X = B^{\delta'}$, then, for all $u \in X^+$, either $d(u, b_i^{\delta', \circ}) = d(u, b_i^{\delta', *}) = 1$, $d(u, b_i^{\delta', \circ}) = d(u, b_i^{\delta', *}) = 2$, or $d(u, b_i^{\delta', \circ}) = d(u, b_i^{\delta', *}) = 3$ by the construction in Subsection 3.1.2 and since $\text{bit-rep}(X) \setminus \text{bits}(X)$ is a clique. $\square$

Lemma 2. If ψ is a satisfiable 3-PARTITIONED-3-SAT formula, then G admits a resolving set of size k .

Proof. Suppose $\pi : X^\alpha \cup X^\beta \cup X^\gamma \mapsto \{\text{True}, \text{False}\}$ is a satisfying assignment for ψ . We construct a resolving set S of size k for G using this assignment.

Initially, set $S = \emptyset$. For every $\delta \in \{\alpha, \beta, \gamma\}$ and $i \in [\sqrt{N}]$, consider the assignment π restricted to the variables in X_i^δ . By the construction, there is a vertex in A_i^δ that corresponds to this assignment. Include that vertex in S . For each $X \in \{C\} \cup \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$, we add one vertex from each pair of the false twins in $\text{bits}(X)$ to S . Note that $|S| = k$ and that every vertex in S is distinguished by itself.

In the remaining part of the proof, we show that S is a resolving set of G . First, we prove that all critical pairs are resolved by S in the following claim.

Claim 1. All critical pairs are resolved by S .

Proof. For each $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, the critical pair $\langle b_i^{\delta, \circ}, b_i^{\delta, *} \rangle$ is resolved by the vertex $S \cap A_i^\delta$ by the construction. For each $q \in [m]$, the clause C_q is satisfied by the assignment π . Thus, there is a variable x in C_q that satisfies C_q according to π . Suppose that $x \in X_i^\delta$. Let $a_{i,\ell}^\delta$ be the vertex in A_i^δ corresponding to the restriction of π to the variables in X_i^δ . Then, by the construction, $d(a_{i,\ell}^\delta, c_q^{\circ}) = 2 < 3 = d(a_{i,\ell}^\delta, c_q^*)$. Thus, every critical pair $\langle c_q^{\circ}, c_q^* \rangle$ is resolved by S . $\square$

Then, every vertex pair in $V(G)$ is resolved by S by Claim 1 in conjunction with Lemma 1(b). $\square$

Lemma 3. If G admits a resolving set of size k , then ψ is a satisfiable 3-PARTITIONED-3-SAT formula.

Proof. Assume that G admits a resolving set S of size k . First, we prove some properties of S . By Lemma 1(a), for each $\delta \in \{\alpha, \beta, \gamma\}$, we have that

$$\begin{aligned} |S \cap \text{bits}(A^\delta)| &\geq \lceil \log(|A^\delta| + 2) \rceil + 1, & |S \cap \text{bits}(P^\delta)| &\geq \lceil \log(|P^\delta| + 2) \rceil + 1, \\ |S \cap \text{bits}(B^\delta)| &\geq \lceil \log(|B^\delta|/2 + 2) \rceil + 1, & |S \cap \text{bits}(C)| &\geq \lceil \log(|C|/2 + 2) \rceil + 1. \end{aligned}$$

Hence, any resolving set S of G already has size at least

$$3 \cdot (\lceil \log(|B^\alpha|/2 + 2) \rceil + 1) + (\lceil \log(|A^\alpha| + 2) \rceil + 1) + (\lceil \log(|P^\alpha| + 2) \rceil + 1) + \lceil \log(|C|/2 + 2) \rceil + 1.$$

Now, for each $\delta \in \{\alpha, \beta, \gamma\}$ and $i \in [\sqrt{N}]$, consider the critical pair $\langle b_i^{\delta, \circ}, b_i^{\delta, \star} \rangle$. By the construction, only $v \in A_i^\delta \cup \{b_i^{\delta, \circ}, b_i^{\delta, \star}\}$ resolves a pair $\langle b_i^{\delta, \circ}, b_i^{\delta, \star} \rangle$. Indeed, for all $X \in \{C\} \cup \{B^{\delta'}, A^{\delta'}, P^{\delta'} \mid \delta' \in \{\alpha, \beta, \gamma\}\}$, no vertex in X^+ can resolve such a pair by Lemma 1(c). Also, for all $X \in \{C\} \cup \{A^{\delta''}, P^{\delta''} \mid \delta'' \in \{\alpha, \beta, \gamma\}\}$ such that $\delta \neq \delta''$, and $u \in X$, we have that $d(u, b_i^{\delta, \circ}) = d(u, b_i^{\delta, \star}) = d(u, \text{nullifier}(A^\delta)) + 1$. Furthermore, for any $a \in A_j^\delta$ with $j \in [\sqrt{N}]$ such that $i \neq j$, we have that $d(a, b_i^{\delta, \circ}) = d(a, b_i^{\delta, \star}) = 2$ by the construction since B^δ is a clique. Lastly, every vertex in $B^\delta \setminus \{b_i^{\delta, \circ}, b_i^{\delta, \star}\}$ is at distance 1 from both $b_i^{\delta, \circ}$ and $b_i^{\delta, \star}$. Hence, since any resolving set S of G of size at most k can only admit at most another $3\sqrt{N}$ vertices, we have that $|S \cap \text{bits}(A^\delta)| = \lceil \log(|A^\delta| + 2) \rceil + 1$, $|S \cap \text{bits}(P^\delta)| = \lceil \log(|P^\delta| + 2) \rceil + 1$, $|S \cap \text{bits}(B^\delta)| = \lceil \log(|B^\delta|/2 + 2) \rceil + 1$, $|S \cap \text{bits}(C)| = \lceil \log(|C|/2 + 2) \rceil + 1$, and any resolving set S of G of size at most k contains one vertex from $A_i^\delta \cup \{b_i^{\delta, \circ}, b_i^{\delta, \star}\}$ for all $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$. Hence, any resolving set S of G of size at most k is of size exactly k .

Next, we construct an assignment $\pi : X^\alpha \cup X^\beta \cup X^\gamma \mapsto \{\text{True}, \text{False}\}$ in the following way. For each $\delta \in \{\alpha, \beta, \gamma\}$ and $i \in [\sqrt{N}]$, if $a_{i, \ell}^\delta \in S$ for some $\ell \in [2\sqrt{N}]$, then let $\pi_i^\delta : X_i^\delta \mapsto \{\text{True}, \text{False}\}$ correspond to the underlying assignment of $a_{i, \ell}^\delta$ for the variables in X_i^δ . For each $\delta \in \{\alpha, \beta, \gamma\}$ and $i \in [\sqrt{N}]$, if $S \cap A_i^\delta = \emptyset$, then one of $b_i^{\delta, \circ}, b_i^{\delta, \star}$ is in S , and we let $\pi_i^\delta : X_i^\delta \mapsto \{\text{True}, \text{False}\}$ be an arbitrary assignment of the variables in X_i^δ . Then, $\pi : \bigcup_{\delta \in \{\alpha, \beta, \gamma\}, i \in [\sqrt{N}]} X_i^\delta \mapsto \{\text{True}, \text{False}\}$ is the constructed assignment.

We now prove that the constructed assignment π satisfies every clause in ψ . Since S is a resolving set, it follows that, for all $q \in [m]$, there exists $v \in S$ such that $d(v, c_q^\circ) \neq d(v, c_q^\star)$. Notice that, for any v in $\text{bits}(A^\delta), \text{bits}(B^\delta), \text{bits}(P^\delta)$ for any $\delta \in \{\alpha, \beta, \gamma\}$ or in $\text{bits}(C)$, we have $d(v, c_q^\circ) = d(v, c_q^\star)$ by Lemma 1(c). Further, for any $v \in B^\delta$ and any $\delta \in \{\alpha, \beta, \gamma\}$, we have that $d(v, c_q^\circ) = d(v, c_q^\star) = d(v, \text{nullifier}(P^\delta)) + 1$. Thus, $v \in S \cap \bigcup_{\delta \in \{\alpha, \beta, \gamma\}} A^\delta$. Without loss of generality, suppose that c_q° and $c_q^\star$ are resolved by $a_{i, \ell}^\alpha$. So, $d(a_{i, \ell}^\alpha, c_q^\circ) \neq d(a_{i, \ell}^\alpha, c_q^\star)$. By the construction, the only case where $d(a_{i, \ell}^\alpha, c_q^\circ) \neq d(a_{i, \ell}^\alpha, c_q^\star)$ is when C_q contains a variable $x \in X_i^\alpha$ and $\pi(x)$ satisfies C_q . Thus, we get that the clause C_q is satisfied by the assignment π .

Since S resolves all pairs $\langle c_q^\circ, c_q^\star \rangle$ in $V(G)$, the assignment π constructed above satisfies every clause C_q in ψ . $\square$

Proof of Theorem 3. In Subsection 3.2, we presented a reduction that takes an instance ψ of 3-PARTITIONED-3-SAT and returns an equivalent instance (G, k) of METRIC DIMENSION (by Lemmas 2 and 3) in $2^{\mathcal{O}(\sqrt{N})}$ time, where

$$\begin{aligned} k &= 3 \cdot \left(\sqrt{N} + (\lceil \log(|B^\alpha|/2 + 2) \rceil + 1) + (\lceil \log(|A^\alpha| + 2) \rceil + 1) + (\lceil \log(|P^\alpha| + 2) \rceil + 1) \right) + \\ &\quad (\lceil \log(|C|/2 + 2) \rceil + 1) = \mathcal{O}(\sqrt{N}). \end{aligned}$$

Note that $|V(G)| = 2^{\mathcal{O}(\sqrt{N})}$ and the diameter of G is upper-bounded by a constant since $\text{nullifier}(C)$ is at distance at most 5 from any other vertex in G . Further, note that taking all the vertices in B^δ and P^δ for all $\delta \in \{\alpha, \beta, \gamma\}$, and $X^+ \setminus \text{bits}(X)$ for all $X \in \{C\} \cup \{B^\delta, A^\delta, P^\delta \mid \delta \in \{\alpha, \beta, \gamma\}\}$, results in a vertex cover of G . Hence,

$$\begin{aligned} \text{vc}(G) &\leq 3 \cdot ((\lceil \log(|B^\alpha|/2 + 2) \rceil + 2) + (\lceil \log(|A^\alpha| + 2) \rceil + 2) + (\lceil \log(|P^\alpha| + 2) \rceil + 2)) + \\ &\quad 3 \cdot (|B^\alpha| + |P^\alpha|) + (\lceil \log(|C|/2 + 2) \rceil + 2) = \mathcal{O}(\sqrt{N}). \end{aligned}$$

Thus, $\text{vc}(G) + k = \mathcal{O}(\sqrt{N})$. $\square$

4. GEODETIC SET: lower bounds regarding vertex cover

In this section, we follow the same template as in Section 3 and first prove the following theorem.

Theorem 4. There is an algorithm that, given an instance ψ of 3-PARTITIONED-3-SAT on $3N$ variables, runs in $2^{\mathcal{O}(\sqrt{N})}$ time, and constructs an equivalent instance (G, k) of GEODETIC SET such that $\text{vc}(G) + k = \mathcal{O}(\sqrt{N})$, $|V(G)| = 2^{\mathcal{O}(\sqrt{N})}$, and the diameter of G is upper-bounded by a constant.

The proofs of the following two corollaries are analogous to the ones for METRIC DIMENSION.

Corollary 3. *Unless the ETH fails, GEODETIC SET does not admit an algorithm running in $2^{o(vc^2)} \cdot n^{O(1)}$ time, even on graphs of bounded diameter.*

Corollary 4. *Unless the ETH fails, GEODETIC SET does not admit a kernelization algorithm that does not increase the solution size k and outputs a kernel with $2^{o(k+vc)}$ vertices, even on graphs of bounded diameter.*

4.1. Reduction

Consider an instance ψ of 3-PARTITIONED-3-SAT with $X^\alpha, X^\beta, X^\gamma$ the partition of the variable set, where $|X^\alpha| = |X^\beta| = |X^\gamma| = N$. By adding dummy variables in each of these sets, we can assume that $\sqrt{N}$ is an integer greater than 2. Further, let $\mathcal{C} = \{C_1, \dots, C_m\}$ be the set of all the clauses of ψ . From ψ , we construct the graph G as follows. We describe the construction for the part of the graph G corresponding to X^α , with the parts corresponding to X^β and X^γ being analogous. We rename the variables in X^α to $x_{i,j}^\alpha$ for $i, j \in [\sqrt{N}]$.

- We partition the variables of X^α into buckets $X_1^\alpha, X_2^\alpha, \dots, X_{\sqrt{N}}^\alpha$ such that each bucket contains $\sqrt{N}$ many variables. Let $X_i^\alpha := \{x_{i,j}^\alpha \mid j \in [\sqrt{N}]\}$ for all $i \in [\sqrt{N}]$.
- For every bucket X_i^α , we add an independent set A_i^α of $2\sqrt{N}$ new vertices, and two isolated edges $(a_{i,1}^\alpha, b_{i,1}^\alpha)$ and $(a_{i,2}^\alpha, b_{i,2}^\alpha)$. Each vertex in A_i^α corresponds to a possible assignment of variables in X_i^α . For all $i \in [\sqrt{N}]$ and $u \in A_i^\alpha$, we make both $a_{i,1}^\alpha$ and $a_{i,2}^\alpha$ adjacent to u . Let $B^\alpha := \{a_{i,j}^\alpha, b_{i,j}^\alpha \mid i \in [\sqrt{N}], j \in \{1, 2\}\}$.
- We add three independent sets T^α, F^α , and V^α on $\sqrt{N}$ vertices each: $T^\alpha := \{t_i^\alpha \mid i \in [\sqrt{N}]\}$, $F^\alpha := \{f_i^\alpha \mid i \in [\sqrt{N}]\}$, and $V^\alpha := \{v_i^\alpha \mid i \in [\sqrt{N}]\}$.
- For each $i \in [\sqrt{N}]$, we connect v_i^α with all the vertices in A_i^α .
- For each $i \in [\sqrt{N}]$, we add edges between A_i^α and T^α and between A_i^α and F^α as follows. Consider a vertex $w \in A_i^\alpha$. Recall that this vertex corresponds to an assignment $\pi : X_i^\alpha \mapsto \{\text{True}, \text{False}\}$. If $\pi(x_{i,j}^\alpha) = \text{True}$, then we add the edge (w, t_j^α) , and otherwise, we add the edge (w, f_j^α) .
- For each $i \in [\sqrt{N}]$, we add a special vertex g_i^α (also referred to as a g -vertex later on) that is adjacent to each vertex in $T^\alpha \cup F^\alpha$. Further, g_i^α is also adjacent to both $a_{i,1}^\alpha$ and $a_{i,2}^\alpha$.

This finishes the first part of the construction. The second step is to connect the three previously constructed parts for X^α, X^β , and X^γ .

- We introduce a vertex set $U := \{u_i \mid i \in [\sqrt{N}]\}$ that forms a clique. Then, for each u_i , we add an edge to a new vertex u'_i . Thus, we have a matching $\{(u_i, u'_i) \mid i \in [\sqrt{N}]\}$. Let $U' := \{u'_i \mid i \in [\sqrt{N}]\}$.
- For each $\delta \in \{\alpha, \beta, \gamma\}$, we add edges so that the vertices of $U \cup V^\delta$ almost form a complete bipartite graph, i.e., $E(G)$ contains edges between all pairs (v, w) where $v \in U$ and $w \in V^\delta$, except for the matching $\{(v_i^\delta, u_i) \mid i \in [\sqrt{N}]\}$.
- For each $\delta \in \{\alpha, \beta, \gamma\}$ and $i \in [\sqrt{N}]$, we make g_i^δ adjacent to each vertex in U .
- For each $C_q \in \mathcal{C}$, we add a new vertex c_q . Let $\mathcal{C} := \{c_q \mid q \in [m]\}$. Since we are considering an instance of 3-PARTITIONED-3-SAT, for each $\delta \in \{\alpha, \beta, \gamma\}$, there is at most one variable in C_q that lies in X^δ . If there is one, then without loss of generality, let it be $x_{i,j}^\delta$ and do the following. Make c_q adjacent to u_i and if setting $x_{i,j}^\delta$ to True (False, respectively) satisfies C_q , then $(c_q, t_j^\delta) \in E(G)$ ($(c_q, f_j^\delta) \in E(G)$, respectively).

This concludes the construction of G . See Fig. 4 for an illustration of part of G . The reduction returns (G, k) as an instance of GEODETIC SET where $k = 10\sqrt{N}$.

4.2. Correctness of the reduction

Suppose, given an instance ψ of 3-PARTITIONED-3-SAT, that the reduction above returns (G, k) as an instance of GEODETIC SET. We first observe the following which will be useful in proving the correctness of the reduction.

Observation 3. Let G be a connected graph. For any three distinct vertices $u, v, w \in V(G)$, if $d(u, v) = x$ for some positive integer x , then w cannot be on a shortest path between u and v if $d(u, w) \geq x$ and/or $d(v, w) \geq x$.

Proof. Toward a contradiction, suppose that w is on a shortest path between u and v and, without loss of generality, $d(u, w) \geq x$. Then, $d(u, v) = d(u, w) + d(w, v) > x$, contradicting that $d(u, v) = x$. $\square$

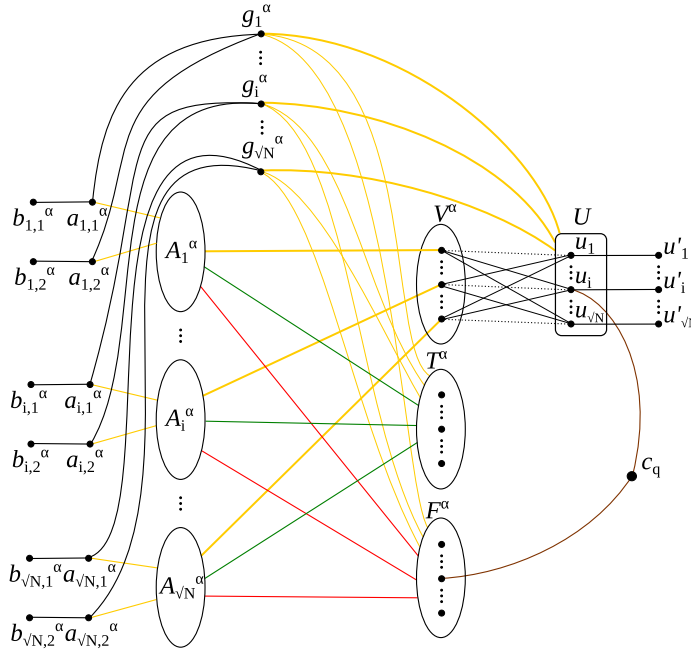


Fig. 4. Overview of the reduction. Sets with elliptical boundaries are independent sets, and sets with rectangular boundaries are cliques. For each $\delta \in \{\alpha, \beta, \gamma\}$, the sets V^δ and U almost form a complete bipartite graph, except for the matching (marked by dotted edges) that is excluded. Yellow lines from a vertex to a set denote that this vertex is connected to all the vertices in that set. The green and red lines between the A_i^α and $T^\alpha \cup F^\alpha$ transfer, in some sense, for each $w \in A_i^\alpha$, the underlying assignment structure as formally described in Section 4.1. For all $i, j \in [\sqrt{N}]$, $q \in [m]$, and $\delta \in \{\alpha, \beta, \gamma\}$, if assigning True (False, respectively) to $x_{i,j}^\delta$ satisfies C_q , then we add the edges (c_q, t_j^δ) ((c_q, f_j^δ) , respectively) and (c_q, u_i) .

We now prove several lemmas which will be helpful in proving the correctness of the reduction.

Lemma 4. *The shortest paths between any two vertices in $B^\alpha \cup B^\beta \cup B^\gamma \cup U \cup U'$ do not cover any vertices in $C \cup V^\alpha \cup V^\beta \cup V^\gamma$.*

Proof. Since, for all $i \in [\sqrt{N}]$, $j \in \{1, 2\}$, and $\delta \in \{\alpha, \beta, \gamma\}$, the shortest path from $b_{i,j}^\delta$ (u_i' , respectively) to any other vertex in G first passes through $a_{i,j}^\delta$ (u_i , respectively), it suffices to prove the statement of the lemma for the shortest paths between any two vertices in $U \cup \{a_{i,j}^\delta \mid i \in [\sqrt{N}], j \in \{1, 2\}, \delta \in \{\alpha, \beta, \gamma\}\}$.

For all $i, i' \in [\sqrt{N}]$ such that $i \neq i'$, we have $d(u_i, u_{i'}) = 1$. Hence, the shortest paths between any two vertices in U do not cover any vertices in $C \cup V^\alpha \cup V^\beta \cup V^\gamma$.

For all $i, i' \in [\sqrt{N}]$, $j \in \{1, 2\}$, and $\delta, \delta' \in \{\alpha, \beta, \gamma\}$ such that $\delta \neq \delta'$, we have $d(a_{i,j}^\delta, a_{i,j}^{\delta'}) = 2$, $d(a_{i,j}^\delta, u_{i'}) = 2$, $d(a_{i,j}^\delta, w) \geq 2$ for all $w \in V^\delta$, $d(a_{i,j}^\delta, w') = 3$ for all $w' \in V^{\delta'}$, and $d(a_{i,j}^\delta, c) = 3$ for all $c \in C$. Furthermore, for all $i, i' \in [\sqrt{N}]$, $j, j' \in \{1, 2\}$, and $\delta, \delta' \in \{\alpha, \beta, \gamma\}$ such that $i \neq i'$ and/or $\delta \neq \delta'$, we have $d(a_{i,j}^\delta, a_{i',j'}^{\delta'}) = 4$ and any shortest path realizing this distance does not contain any vertices in $C \cup V^\alpha \cup V^\beta \cup V^\gamma$. Thus, the distance between any two vertices in $U \cup \{a_{i,j}^\delta \mid i \in [\sqrt{N}], j \in \{1, 2\}, \delta \in \{\alpha, \beta, \gamma\}\}$ is either at most 2 or exactly 4 (in which case the shortest paths do not contain any vertices in $C \cup V^\alpha \cup V^\beta \cup V^\gamma$), while the distance from any vertex in $\{a_{i,j}^\delta \mid i \in [\sqrt{N}], j \in \{1, 2\}, \delta \in \{\alpha, \beta, \gamma\}\}$ to any vertex in $C \cup V^\alpha \cup V^\beta \cup V^\gamma$ is at least 2. Hence, by Observation 3, we have that the shortest paths between any two vertices in $U \cup \{a_{i,j}^\delta \mid i \in [\sqrt{N}], j \in \{1, 2\}, \delta \in \{\alpha, \beta, \gamma\}\}$ do not cover any vertices in $C \cup V^\alpha \cup V^\beta \cup V^\gamma$. $\square$

Lemma 5. *For all $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, v_i^δ can only be covered by a shortest path from a vertex in $A_i^\delta \cup \{v_i^\delta\}$ to another vertex in G .*

Proof. As stated in the proof of Lemma 4, we do not need to consider any shortest path with an endpoint that is a degree-1 vertex. First, we show that v_i^δ cannot be covered by a shortest path with one endpoint in U and the other not in $A_i^\delta \cup \{v_i^\delta\}$. To cover this case, by Lemma 4, we just need to consider all shortest paths between a vertex $u_{i'} \in U$ and any other vertex $z \in V(G) \setminus (U \cup U' \cup B^\alpha \cup B^\beta \cup B^\gamma \cup A_i^\delta \cup \{v_i^\delta\})$. Note that $d(z, v_i^\delta) \geq 2$, and so, if $d(u_{i'}, z) \leq 2$, then v_i^δ is not covered by the pair $\langle u_{i'}, z \rangle$ by Observation 3. Further, $d(u_{i'}, w) \leq 3$ for all $w \in V(G)$, and, for all $z \in V(G) \setminus (U \cup U' \cup B^\alpha \cup B^\beta \cup B^\gamma \cup A_i^\delta \cup \{v_i^\delta\})$,

$d(u_i, z) = 3$ if and only if $z \in (A_i^\alpha \cup A_i^\beta \cup A_i^\gamma) \setminus A_i^\delta$. It is easy to verify that, for all $z \in (A_i^\alpha \cup A_i^\beta \cup A_i^\gamma) \setminus A_i^\delta$, we have $d(z, v_i^\delta) \geq 3$. Hence, by Observation 3, v_i^δ cannot be covered by a shortest path with one endpoint in U and the other not in $A_i^\delta \cup \{v_i^\delta\}$.

For all $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, we have that $N(v_i^\delta) = A_i^\delta \cup (U \setminus \{u_i\})$. Hence, since v_i^δ cannot be covered by a shortest path with one endpoint in U and the other not in $A_i^\delta \cup \{v_i^\delta\}$, any shortest path whose endpoints are in $V(G) \setminus (A_i^\delta \cup \{v_i^\delta\})$ that could cover v_i^δ cannot have any of its endpoints in $N(v_i^\delta)$, and thus, must have length at least 4. In particular, this proves the following property that we put as a claim to make reference to later.

Claim 2. For any shortest path whose endpoints are not in $N[v_i^\delta]$, if its first and second endpoints are at distance at least ℓ_1 and ℓ_2 , respectively, from any vertex in $N(v_i^\delta)$, then it cannot cover v_i^δ if it has length less than $\ell_1 + \ell_2 + 2$.

We finish with a case analysis of the possible pairs $\langle u, v \rangle$ to prove that no such shortest path covering v_i^δ exists using Claim 2. We note that, by the arguments above, we do not need to consider the case where u and/or v is a degree-1 vertex, nor the case where both u and v are in $U \cup B^\alpha \cup B^\beta \cup B^\gamma$, nor the case where one of u and v is in U . We now proceed with the case analysis assuming that $u, v \notin N[v_i^\delta]$.

- **Case 1: $u \in A_i^{\delta'}$ for any $i' \in [\sqrt{N}]$ and $\delta' \in \{\alpha, \beta, \gamma\}$ such that $A_i^{\delta'} \neq A_i^\delta$.** First, u is at distance at least 2 from any vertex in $N(v_i^\delta)$, and hence, if $d(u, v) \leq 4$, then we are done by Claim 2. Note that $d(u, v) \leq 4$ since u is at distance 2 from every vertex in $U \setminus \{u_i\}$, and every vertex in G that is not a degree-1 vertex is also at distance at most 2 from all but at most one of the vertices in U .
- **Case 2: $u \in C$ or u is a g-vertex.** By Claim 2 and the previous case, it suffices to note that $d(u, v) \leq 3$ as long as $v \notin A_i^{\delta'}$ for any $i' \in [\sqrt{N}]$ and $\delta' \in \{\alpha, \beta, \gamma\}$ since v cannot be a degree-1 vertex.
- **Case 3: $u \in T^{\delta'} \cup F^{\delta'}$ for any $\delta' \in \{\alpha, \beta, \gamma\}$ such that $\delta' \neq \delta$.** Similarly to the first case, u is at distance at least 2 from any vertex in $N(v_i^\delta)$ and at distance 2 from every vertex in U , and thus, $d(u, v) \leq 4$ and we are done by Claim 2.
- **Case 4: $u \in T^\delta \cup F^\delta$.** First, if $v \notin A_i^{\delta'}$ for any $i' \in [\sqrt{N}]$ and $\delta' \in \{\alpha, \beta, \gamma\}$, and v is a vertex with a superscript δ , then $d(u, v) \leq 3$. Otherwise, based on the cases taken care of previously, the vertex v is either $v_{i',j}^{\delta'}$ or $a_{i',j}^{\delta'}$ for some $i' \in [\sqrt{N}]$, $j \in \{1, 2\}$, and $\delta' \in \{\alpha, \beta, \gamma\}$. In the former case, $d(u, v) \leq 3$. In the latter case, $d(u, v) \leq 4$, but any path between u and v that passes through v_i^δ has length at least 5. Thus, we are done by the previous cases and Claim 2.
- **Case 5: $u \in V^{\delta'}$ for any $\delta' \in \{\alpha, \beta, \gamma\}$.** For any $\delta'' \in \{\alpha, \beta, \gamma\}$, if $v \in B^{\delta''} \cup V^{\delta''}$ and v is not a degree-1 vertex, then $d(u, v) \leq 3$, and we are done by the previous cases and Claim 2. $\square$

Lemma 6. If G admits a geodetic set of size k , then ψ is a satisfiable 3-PARTITIONED-3-SAT formula.

Proof. Assume that G admits a geodetic set S of size k . Then, let us consider the set $S' = \{u_i' \mid i \in [\sqrt{N}]\} \cup \{b_{i,1}^\delta, b_{i,2}^\delta \mid \delta \in \{\alpha, \beta, \gamma\}, i \in [\sqrt{N}]\}$ of all the degree-1 vertices in G . By Observation 2, $S' \subseteq S$. By Lemma 5, for each $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, there is at least one vertex from $A_i^\delta \cup \{v_i^\delta\}$ in S . Since $|S'| = 7\sqrt{N}$ and $k = 10\sqrt{N}$, for each $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, S contains exactly 1 vertex from $A_i^\delta \cup \{v_i^\delta\}$. We do a case analysis of the possible pairs $\langle u, v \rangle$ of distinct vertices in S , to determine which shortest paths between pairs of vertices in S can cover vertices in C .

- **Case 1: $u, v \in S'$.** By Lemma 4, the shortest paths between vertices in S' do not cover vertices in C .
- **Case 2: $u \in V^\delta$ for some $\delta \in \{\alpha, \beta, \gamma\}$.** Every vertex in S is at distance at least 2 from every vertex in C , and hence, any shortest path between two vertices in S that covers C must be of length at least 4. Furthermore, every vertex in $\{b_{i,1}^{\delta'}, b_{i,2}^{\delta'} \mid \delta' \in \{\alpha, \beta, \gamma\}, i \in [\sqrt{N}]\}$ is at distance at least 4 from every vertex in C , and hence, any shortest path between a vertex in S and a vertex in $\{b_{i,1}^{\delta'}, b_{i,2}^{\delta'} \mid \delta' \in \{\alpha, \beta, \gamma\}, i \in [\sqrt{N}]\}$ that covers C must be of length at least 6. However, every vertex in V^δ is at distance at most 3 from every vertex in $S \setminus \{b_{i,1}^{\delta'}, b_{i,2}^{\delta'} \mid \delta' \in \{\alpha, \beta, \gamma\}, i \in [\sqrt{N}]\}$ and distance at most 4 from every vertex in $\{b_{i,1}^{\delta'}, b_{i,2}^{\delta'} \mid \delta' \in \{\alpha, \beta, \gamma\}, i \in [\sqrt{N}]\}$. Thus, no shortest path between u and v covers a vertex in C in this case.
- **Case 3: $u \in A_i^\delta$ for some $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, and $v \in \{b_{i',1}^{\delta'}, b_{i',2}^{\delta'} \mid \delta' \in \{\alpha, \beta, \gamma\}, i' \in [\sqrt{N}]\}$.** As stated in Case 2, any shortest path between a vertex in S and a vertex in $\{b_{i',1}^{\delta'}, b_{i',2}^{\delta'} \mid \delta' \in \{\alpha, \beta, \gamma\}, i' \in [\sqrt{N}]\}$ that covers C must be of length at least 6. However, $d(u, v) \leq 5$, and thus, no shortest path between u and v covers a vertex in C in this case.
- **Case 4: $u \in A_i^\delta$ for some $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, and $v \in (U' \setminus u_i')$.** As stated in Case 2, any shortest path between two vertices in S that covers C must be of length at least 4. However, $d(u, v) = 3$, and thus, no shortest path between u and v covers a vertex in C in this case.
- **Case 5: $u \in A_i^\delta$ for some $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, and $v = u_i'$.** Without loss of generality, $u \in A_i^\alpha$. Note that $d(u, v) = 4$ and recall from Case 2 that any shortest path between two vertices in S that covers C must be of length at least 4. For all $q \in [m]$, for there to be a path of length 4 between u and v that covers c_q , there must be $w \in T^\alpha \cup F^\alpha$ such that both (u, w) and (w, c_q) are in $E(G)$.

- **Case 6:** $u \in A_i^\delta$ for some $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, and $v \in A_{i'}^{\delta'}$ for some $i' \in [\sqrt{N}]$ and $\delta' \in \{\alpha, \beta, \gamma\}$. Without loss of generality, $u \in A_i^\alpha$. Note that $d(u, v) \leq 4$ and recall from Case 2 that any shortest path between two vertices in S that covers C must be of length at least 4. For all $q \in [m]$, for there to be a path of length 4 between u and v that covers c_q , there must be $w \in T^\alpha \cup F^\alpha$ such that both (u, w) and (w, c_q) are in $E(G)$ (there are additional necessary conditions, but this one is sufficient for our purposes).

By this exhaustive case analysis, the shortest paths between a pair $\langle u, v \rangle$ of distinct vertices in S can cover a vertex in C only in Cases 5 and 6 above, which both share a common necessary condition for this to be the case. Since S is a geodetic set, this necessary condition is met for each $q \in [m]$. That is, for each $q \in [m]$, there exists $w \in T^\delta \cup F^\delta$ and $u \in A_i^\delta \cap S$ for some $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$ such that (u, w) and (w, c_q) are in $E(G)$. By the construction, both (u, w) and (w, c_q) are in $E(G)$ if and only if the assignment corresponding to u that sets the variable in X_i^δ corresponding to w to True or False satisfies the clause C_q of the instance ψ . Then, setting $\pi : X^\alpha \cup X^\beta \cup X^\gamma \rightarrow \{\text{True}, \text{False}\}$ to be the assignment retrieved by combining the partial assignments corresponding to each such vertex $u \in S \cap \bigcup_{i \in [\sqrt{N}], \delta \in \{\alpha, \beta, \gamma\}} A_i^\delta$, and that is completed by selecting an arbitrary assignment for the variables in the buckets X_i^δ where $A_i^\delta \cap S = \emptyset$, we have that π satisfies all the clauses in $\mathcal{C}$. $\square$

Lemma 7. *If ψ is a satisfiable 3-PARTITIONED-3-SAT formula, then G admits a geodetic set of size k .*

Proof. Suppose $\pi : X^\alpha \cup X^\beta \cup X^\gamma \rightarrow \{\text{True}, \text{False}\}$ is a satisfying assignment for ψ . We construct a geodetic set S of size k for G using this assignment. Initially, let

$$S = \{u'_i \mid i \in [\sqrt{N}]\} \cup \{b_{i,1}^\delta, b_{i,2}^\delta \mid \delta \in \{\alpha, \beta, \gamma\}, i \in [\sqrt{N}]\}.$$

At this point, $|S| = 7\sqrt{N}$. Now, for each $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, we add one vertex from A_i^δ to S in the following way. For the bucket of variables X_i^δ , consider how the variables of X_i^δ are assigned by π , and denote this assignment restricted to X_i^δ by π_i^δ . Since A_i^δ contains a vertex for each of the possible $2^{\sqrt{N}}$ assignments and each of those corresponds to a certain assignment of X_i^δ , we will find $w \in A_i^\delta$ that matches the assignment π_i^δ . Then, we include this w in S as well. At the end, $|S| = 10\sqrt{N}$.

Now, we show that S is indeed a geodetic set of G . First, recall that the vertices of S are covered by any shortest path between them and any other vertex in S . Further, recall that the neighbors of the degree-1 vertices in S are also covered by the shortest paths between their degree-1 neighbor in S and any other vertex in S . In the following case analysis, we omit the cases just described above. In each case, we consider sets of vertices that we want to cover by shortest paths between pairs of vertices in S .

- **Case 1:** $\{A_i^\delta \cup \{g_i^\delta\} \mid i \in [\sqrt{N}], \delta \in \{\alpha, \beta, \gamma\}\}$. For each $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, there is a shortest path of length 4 between $b_{i,1}^\delta$ and $b_{i,2}^\delta$ that contains g_i^δ (any vertex in A_i^δ , respectively).
- **Case 2:** $T^\delta \cup F^\delta$ for all $\delta \in \{\alpha, \beta, \gamma\}$. For each $\delta \in \{\alpha, \beta, \gamma\}$ and $i, i' \in [\sqrt{N}]$ such that $i \neq i'$, there is a shortest path of length 6 between $b_{i,1}^\delta$ and $b_{i',1}^\delta$ that is as follows. First, it goes to $a_{i,1}^\delta$ and then through g_i^δ to $w \in T^\delta \cup F^\delta$, and then through $g_{i'}^\delta$ to $a_{i',1}^\delta$, before finishing at $b_{i',1}^\delta$. Since, for each $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, g_i^δ is adjacent to all the vertices in $T^\delta \cup F^\delta$, the described path of length 6 covers $T^\delta \cup F^\delta$.
- **Case 3:** $C \cup V^\delta$ for all $\delta \in \{\alpha, \beta, \gamma\}$. For all $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, consider the vertex $w \in A_i^\delta \cap S$. Recall that this w corresponds to the assignment π_i^δ , i.e., π that is restricted to the subset of variables X_i^δ . First, there is a shortest path of length 4 between w and u'_i that contains $v_i^\delta, u_{i'}$ (for some $i' \in [\sqrt{N}]$ such that $i \neq i'$), and u_i . Second, for each $q \in [m]$, there exists $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$ such that there is a shortest path of length 4 from $w \in A_i^\delta$ to u'_i that covers c_q . Consider the variable that satisfied the clause C_q of the initial instance ψ under the assignment π , and, without loss of generality, let it be $x_{i,j}^\delta$. Then, consider the bucket X_i^δ and select $w \in A_i^\delta$ such that w corresponds to π_i^δ . Since w corresponds to π_i^δ , if $\pi(x_{i,j}^\delta) = \text{True}$ ($\pi(x_{i,j}^\delta) = \text{False}$, respectively), then $(w, t_j^\delta) \in E(G)$ ($(w, f_j^\delta) \in E(G)$, respectively) and, since $x_{i,j}^\delta = \text{True}$ ($x_{i,j}^\delta = \text{False}$, respectively) satisfies C_q , $(t_j^\delta, c_q) \in E(G)$ ($(f_j^\delta, c_q) \in E(G)$, respectively) as well. Thus, we have a shortest path of length 4 that goes from w to t_j^δ (f_j^δ in the latter case) to c_q to u_i to u'_i . This way, all the vertices in C are also covered.

This covers all the vertices in $V(G)$, and thus, S is a geodetic set of G . $\square$

Proof of Theorem 4. In Section 4.1, we presented a reduction that takes an instance ψ of 3-PARTITIONED-3-SAT and returns an equivalent instance (G, k) of GEODETIC SET (by Lemmas 6 and 7) in $2^{\mathcal{O}(\sqrt{N})}$ time, where $k = 10\sqrt{N}$. Note that $|V(G)| = 2^{\mathcal{O}(\sqrt{N})}$ and the diameter of G is upper-bounded by a constant since each vertex in U is at distance at most 3 from any

other vertex in G . Further, note that taking all the vertices in B^δ , V^δ , T^δ , F^δ , U , and g_i^δ for all $i \in [\sqrt{N}]$ and $\delta \in \{\alpha, \beta, \gamma\}$, results in a vertex cover of G . Hence,

$$\text{vc}(G) \leq 3 \cdot (|B^\alpha| + |V^\alpha| + |T^\alpha| + |F^\alpha| + \sqrt{N}) + |U| = \mathcal{O}(\sqrt{N}).$$

Thus, $\text{vc}(G) + k = \mathcal{O}(\sqrt{N})$. $\square$

5. Algorithms for vertex cover parameterization

5.1. Algorithm for METRIC DIMENSION

To prove Theorem 1 for METRIC DIMENSION, we first show that the following reduction rule is safe. Let $X \subseteq V(G)$ be a vertex cover of G and let $I := V(G) \setminus X$.

Reduction Rule 1. If there exist three vertices $u, v, x \in I$ such that u, v, x are false twins, then delete x and decrease k by one.

Proof that Reduction Rule 1 is safe. We show that G has a resolving set of size k if and only if the reduced graph $G' := G - \{x\}$ has a resolving set of size $k - 1$. We first consider the forward direction. Let S be a resolving set of G of size k . By Observation 1, without loss of generality, $u, x \in S$. By Observation 1, $d(w, v) = d(w, u) = d(w, x)$ for any vertex $w \in V(G) \setminus \{u, v, x\}$. Hence, any pair of vertices in $V(G) \setminus \{u, v, x\}$ that is resolved by x in G is also resolved by u in G' , and thus, $S \setminus \{x\}$ is a resolving set of G' of size $k - 1$. The correctness of the reverse direction trivially follows from the fact that we can add x into a resolving set of $G - \{x\}$ to obtain a resolving set of G . $\square$

Lemma 8. METRIC DIMENSION, parameterized by the vertex cover number vc , admits a polynomial-time kernelization algorithm that returns an instance with $2^{\mathcal{O}(\text{vc})}$ vertices.

Proof. Given a graph G , let $X \subseteq V(G)$ be a minimum vertex cover of G . If such a vertex cover is not given, then we can find a 2-factor approximate vertex cover in polynomial time. By the definition of a vertex cover, the vertices of I are pairwise non-adjacent.

The kernelization algorithm exhaustively applies Reduction Rule 1. Now, consider an instance on which Reduction Rule 1 is not applicable. If $k \leq 0$, then Reduction Rule 1 was applied at least once (since the initial value of k in the problem input is a positive integer), and thus, we conclude that it is a No-instance since every reduced instance contains two vertices that are false twins and one of them must be in any resolving set by Observation 1. Otherwise, for any $Y \subseteq X$, there are at most two vertices $u, v \in I$ such that $N(u) = N(v) = Y$. This implies that the number of vertices in the reduced instance is at most $|X| + 2 \cdot 2^{|X|} = 2^{\text{vc}+1} + \text{vc}$. $\square$

Next, we present an XP-algorithm parameterized by the vertex cover number. This algorithm, along with the kernelization algorithm above, implies⁷ that METRIC DIMENSION admits an algorithm running in $2^{\mathcal{O}(\text{vc}^2)} \cdot n^{\mathcal{O}(1)}$ time.

Lemma 9. METRIC DIMENSION admits an algorithm running in $n^{\mathcal{O}(\text{vc})}$ time.

Proof. The algorithm starts by computing a minimum vertex cover X of G in $2^{\mathcal{O}(\text{vc})} \cdot n^{\mathcal{O}(1)}$ time using an FPT algorithm for the VERTEX COVER problem, for example the one in [11] or [27]. Then, in polynomial time, it computes a largest subset F of I such that, for every vertex u in F , $I \setminus F$ contains a false twin of u . By Observation 1, if there are false twins in I , say u, v , then any resolving set contains at least one of them. Hence, there exists an optimal resolving set that contains F . The algorithm enumerates every subset of vertices of size at most $|X|$ in $V(G) \setminus F$. If there exists a subset $A \subseteq V(G) \setminus F$ such that $A \cup F$ is a resolving set of G of size at most k , then it returns $A \cup F$. Otherwise, it returns No.

In order to prove that the algorithm is correct, we prove that $X \cup F$ is a resolving set of G . It is easy to see that, for a pair of distinct vertices u, v , if $u \in X \cup F$ and $v \in V(G)$, then the pair is resolved by u . It remains to argue that every pair of distinct vertices in $(I \setminus F) \times (I \setminus F)$ is resolved by $X \cup F$. Note that, for any two vertices $u, v \in I \setminus F$, $N(u) \neq N(v)$ as otherwise u can be moved to F , contradicting the maximality of F . Hence, without loss of generality, there is a vertex in X that is adjacent to u , but not adjacent to v , resolving the pair $\langle u, v \rangle$. This implies the correctness of the algorithm. The running time of the algorithm easily follows from its description. $\square$

⁷ Note that the application of Reduction Rule 1 does not increase the vertex cover number.

5.2. Algorithm for GEODETIC SET

To prove Theorem 1 for GEODETIC SET, we start with the following fact about false twins.

Lemma 10. *If a graph G contains a set T of false twins that are not simplicial, then any minimum-size geodetic set contains at most four vertices of T .*

Proof. Let $T = \{t_1, \dots, t_h\}$ be a set of false twins in a graph G , that are not simplicial. Thus, T forms an independent set, and there are two non-adjacent vertices x, y in the neighborhood of the vertices in T . Toward a contradiction, assume that $h \geq 5$ and G has a minimum-size geodetic set S that contains at least five vertices of T ; without loss of generality, assume $\{t_1, \dots, t_5\} \subseteq S$. We claim that $S' = (S \setminus \{t_1, t_2, t_3\}) \cup \{x, y\}$ is still a geodetic set, contradicting the choice of S as a minimum-size geodetic set of G .

To see this, notice that any vertex from $V(G) \setminus T$ that is covered by some pair of vertices in $T \cap S$ is also covered by t_4 and t_5 . Similarly, any vertex from $V(G) \setminus T$ covered by some pair $\langle t_i, z \rangle$ in $(S \cap T) \times (S \setminus T)$, is still covered by t_4 and z . Moreover, x and y cover all vertices of T , since they are at distance 2 from each other and all vertices in T are their common neighbors. Thus, S' is a geodetic set, as claimed. $\square$

Lemma 11. *GEODETIC SET, parameterized by the vertex cover number vc , admits a polynomial-time kernelization algorithm that returns an instance with $2^{\mathcal{O}(\text{vc})}$ vertices.*

Proof. Given a graph G , let $X \subseteq V(G)$ be a minimum-size vertex cover of G . If this vertex cover is not given, then we can find a 2-factor approximate vertex cover in polynomial time. The kernelization algorithm exhaustively applies the following reduction rules in a sequential manner.

Reduction Rule 2. If there exist three simplicial vertices in G that are false twins or true twins, then delete one of them from G and decrease k by one.

Reduction Rule 3. If there exist six vertices in G that are false twins but are not simplicial, then delete one of them from G .

To see that Reduction Rule 2 is correct, assume that G contains three simplicial vertices u, v, w that are twins (false or true). We show that G has a geodetic set of size k if and only if the reduced graph G' , obtained from G by deleting u , has a geodetic set of size $k - 1$. For the forward direction, let S be a geodetic set of G of size k . By Observation 2, S contains each of u, v, w . Let $S' = S \setminus \{u\}$. This set of size $k - 1$ is a geodetic set of G' . Indeed, any vertex of G' that was covered in G by u and some other vertex z of S , is also covered by v and z (if $z \neq w$) or v and w (if $z = w$) in G' . Conversely, if G' has a geodetic set S'' of size $k - 1$, then it is clear that $S'' \cup \{u\}$ is a geodetic set of size k in G .

For Reduction Rule 3, assume that G contains six false twins $T = \{t_1, \dots, t_6\}$ that are not simplicial, and let G' be the reduced graph obtained from G by deleting t_1 . We show that G has a geodetic set of size k if and only if G' has a geodetic set of size k . For the forward direction, let S be a minimum-size geodetic set of size (at most) k of G . By Lemma 10, S contains at most four vertices from T ; without loss of generality, t_1 and t_2 do not belong to S . Since the distances among all pairs of vertices in G' are the same as in G , S is still a geodetic set of G' . Conversely, let S' be a minimum-size geodetic set of G' of size (at most) k . Again, by Lemma 10, we may assume that one vertex among $t_2, \dots, t_6$ is not in S' , say, without loss of generality, that it is t_2 . Note that S' covers (in G) all vertices of G' . Thus, t_2 is covered by two vertices x, y of S' . But then, t_1 is also covered by x and y , since we can replace t_2 by t_1 in any shortest path between x and y . Hence, S' is also a geodetic set of G .

Now, consider an instance on which the reduction rules cannot be applied. If $k \leq 1$, then we conclude that it is a No-instance since either the initial budget was 1 or Reduction Rule 2 was applied (since it is the only reduction rule that decreases k), and every reduced instance has two simplicial vertices that must be in any geodetic set by Observation 2. Otherwise, notice that any set of false twins in I contains at most five vertices. Hence, G has at most $|X| + 5 \cdot 2^{|X|} = 2^{\mathcal{O}(\text{vc})}$ vertices. $\square$

Next, we present an XP-algorithm parameterized by the vertex cover number. Together with Lemma 11, they imply Theorem 1 for GEODETIC SET.

Lemma 12. *GEODETIC SET admits an algorithm running in $n^{\mathcal{O}(\text{vc})}$ time.*

Proof. The algorithm first computes a minimum vertex cover X of G in $2^{\mathcal{O}(\text{vc})} \cdot n^{\mathcal{O}(1)}$ time using an FPT algorithm for the VERTEX COVER problem, e.g., the one in [11] or [27].

In polynomial time, we compute the set S of simplicial vertices of G . By Observation 2, any geodetic set of G contains all simplicial vertices of G . Now, notice that $X \cup S$ is a geodetic set of G . Indeed, any vertex v from I that is not simplicial has two non-adjacent neighbors x, y in X , and thus, v is covered by x and y (which are at distance 2 from each other).

Hence, to enumerate all possible minimum-size geodetic sets, it suffices to enumerate all subsets S' of vertices of size at most $|X|$ in $V(G) \setminus S$, and check whether $S \cup S'$ is a geodetic set. If one such set is indeed a geodetic set and has size at most k , we return Yes. Otherwise, we return No. The statement follows. $\square$

6. Conclusion

We have seen that both METRIC DIMENSION and GEODETIC SET have a non-trivial $2^{\Theta(vc^2)}$ running-time dependency (unless the ETH fails) in the vertex cover number parameterization. Both problems are FPT for related parameters, such as vertex integrity, treedepth, distance to (co-)cluster, distance to cograph, etc., as more generally, they are FPT for cliquewidth plus diameter [23,31]. For both problems, it was proven that the correct dependency in treedepth (and treewidth plus diameter) is in fact double-exponential [19], a fact that is also true for feedback vertex set number plus diameter for METRIC DIMENSION [19]. For distance to (co-)cluster, algorithms with double-exponential dependency were given for METRIC DIMENSION in [21]. For the parameter max leaf number ℓ , the algorithm for METRIC DIMENSION from [17] uses ILPs, with a dependency of the form $2^{\mathcal{O}(\ell^6 \log \ell)}$ (a similar algorithm for GEODETIC SET with dependency $2^{\mathcal{O}(f \log f)}$ exists for the feedback edge set number f [31]), which is not known to be tight. What is the correct dependency for all these parameters? In particular, it seems interesting to determine for which parameter(s) the jump from double-exponential to single-exponential dependency occurs.

The related problem STRONG METRIC DIMENSION is known to be NP-complete even on graphs of diameter two, as well as on graphs of constant pathwidth and constant feedback vertex set number [42]. For this problem, it is also known that the correct dependence on the vertex cover number is double-exponential [19]. It would be nice to determine whether similarly intriguing behaviors can be exhibited for related metric-based problems, such as STRONG GEODETIC SET, whose parameterized complexity was recently addressed in [16,35]. Perhaps our techniques are applicable to such related problems.

CRediT authorship contribution statement

Florent Foucaud: Investigation. **Esther Galby:** Investigation. **Liana Khazaliya:** Investigation. **Shaohua Li:** Investigation. **Fionn Mc Inerney:** Investigation. **Roohani Sharma:** Investigation. **Prafullkumar Tale:** Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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