Indian J. Discrete Math., Vol. 8, No.1 (2022) pp. 39–41.
Academy of Discrete Mathematics and Applications, India. *http://ijdm.adma.co.in*

Problems Related to a Conjecture on Location-Domination in Twin-Free Graphs

Florent Foucaud

Université Clermont-Auvergne, CNRS, Mines de Saint-Étienne, Clermont-Auvergne-INP, LIMOS, 63000 Clermont-Ferrand, France.
e-mail: florent.foucaud@uca.fr

Abstract. We summarize what is known about the beautiful conjecture that every twin-free graph without isolated vertices has a locating-dominating set of size at most half its order. This conjecture was stated by Garijo, González and Márquez in 2014.

Keywords: Locating-dominating set.

Mathematics Subject Classification (2000): 05C69.

1. The conjecture

A *locating-dominating set* of a graph G is a dominating set D of G that locates all the vertices in the sense that for any two distinct vertices v, w of $V(G) \setminus D$, we have $N(v) \cap D \neq N(w) \cap D$, that is, each vertex of $V(G) \setminus D$ is uniquely determined by its neighborhood in D. The concept of a locating-dominating set was introduced by the late Peter J. Slater [11, 12] and has been much studied since.

A graph without edges has the whole vertex set as its unique dominating set. Generally, any isolated vertex necessarily belongs to any dominating set. A nice and classic theorem of Ore shows that isolated vertices are the only obstruction to having a relatively small dominating set. Indeed, he showed in [10] that every graph without isolated vertices has a dominating set of size at most one-half its order.

This statement is not true for the location-domination: there are connected graphs with a very large location-domination number, indeed for stars and complete graphs the location-domination number is the order minus one [12]. Nevertheless, all known graphs with location-domination number more than half the order contain many *twins*, that is, pairs of vertices with the same closed or open neighborhood. Is it true that isolated vertices and twins are the only obstructions to having a relatively small locating-dominating set? Garijo, González, and Márquez [9] conjectured that this is indeed the case.[1]

Conjecture 1.1 (Garijo, González, and Márquez, 2014 [9]). Every twin-free graph without isolated vertices has a locating-dominating set of size at most half its order.

This conjecture, if true, can be seen as a nice extension of Ore's theorem. Indeed, twins, even though less trivial than isolated vertices, can be seen as redundant, behave trivially for most graph problems, and are easy to detect.

In the seminal paper [9], Conjecture 1.1 was proved for graphs without 4-cycles (which include trees) and for the class of graphs with independence number at least one-half the order (which includes bipartite graphs), graphs with upper domination number at least one-half the order or chromatic number at least three-quarters the order. In [4], the authors provide several constructions for twin-free graphs with location-domination number one-half their order and characterize the trees for which the bound is tight. Conjecture 1.1 was proved for split graphs and co-bipartite graphs in [4], for cubic graphs in [5] and for line graphs in [7]. In [3, Proposition 2] the conjecture is proved when G is a maximal outerplanar graph. The general bound of two-thirds the order always holds [4, 9].

2. Open problems

Conjecture 1.1 is still open for interesting graph classes such as subcubic graphs, (outer)planar graphs, interval or chordal graphs...

A related open problem discussed in [5] is to characterize those twin-free graphs for which the location-domination number is at most the matching number (in which case the conjectured bound holds). This is true for graphs with no 4-cycles [9]. It is unknown whether this holds for cubic graphs (even though matchings were used in the proof for cubic graphs [5]). No counterexample to this is known even for general graphs.

We also note that related versions of the conjecture are studied for directed graphs [1, 2, 8] and for locating-total dominating sets [6, 7]. We invite the reader to consult these papers for additional interesting problems.

References

[1] T. Bellitto, C. Brosse, B. Lévêque and A. Parreau, Locating dominating sets in local tournaments. Manuscript (2021). https://arxiv.org/abs/2109.03102

[1]The original statement is slightly different, see [5] for a discussion.

[2] N. Bousquet, Q. Deschamps, T. Lehtilä and A. Parreau, Locating-dominating sets: from graphs to oriented graphs. Manuscript (2021). https://arxiv.org/abs/2112.01910

[3] M. Claverol, A. García, G. Hernández, C. Hernando, M. Maureso, M. Mora and J. Tejel, Metric Dimension of Maximal Outerplanar Graphs. *Bull. Malays. Math. Sci. Soc.*, **44** (2021) 2603–2630.

[4] F. Foucaud, M. A. Henning, C. Löwenstein and T. Sasse, Locating-dominating sets in twin-free graphs. *Discrete Applied Math.*, **200** (2016) 52–58.

[5] F. Foucaud and M. A. Henning, Location-domination and matching in cubic graphs. *Discrete Math.*, **339** (2016) 1221–1231.

[6] F. Foucaud and M. A. Henning, Locating-total dominating sets in twin-free graphs: a conjecture. *Electron. J. Comb.*, **23** (2016) P3.

[7] F. Foucaud and M. A. Henning, Locating-dominating sets in line graphs. *Discrete Math.*, **340** (2017) 3140–3153.

[8] F. Foucaud, S. Heydarshahi and A. Parreau, Domination and location in twin-free digraphs. *Discrete Applied Math.*, **284** (2020) 42–52.

[9] D. Garijo, A. González and A. Márquez, The difference between the metric dimension and the determining number of a graph. *Applied Math. Computation*, **249** (2014) 487–501.

[10] O. Ore, *Theory of graphs. Amer. Math. Soc. Transl.*, (Amer. Math. Soc., Providence, RI) **38** (1962) 206–212.

[11] P. J. Slater, Dominating and location in acyclic graphs. *Networks*, **17** (1987) 55–64.

[12] P. J. Slater, Dominating and reference sets in graphs. *J. Math. Phys. Sci.*, **22** (1988) 445–455.