On identifying codes and Bondy's theorem on "induced subsets" *

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Identifying codes were defined to uniquely determine the vertices of a given graph. In this talk, we present several classifications of graphs corresponding to some extremal cases of the theory of identifying codes. In particular, we focus on (possibly infinite) graphs and digraphs having only their whole vertex set as an identifying code.

Given a graph G and a vertex x of G, we use $B_1(x)$ to denote the set of vertices at distance at most 1 from x ($x \in B_1(x)$). If G is directed, $B_1^-(x)$ denotes the set of incoming neighbours of x together with x. Given a graph (respectively digraph) G, an *identifying code* [3, 6] of G is a subset C of $V(G)$ such that for each vertex x of G, the set $B_1(x) \cap C$ (respectively $B_1^-(x) \cap C$) is both nonempty and unique.

The concept of identifying codes was introduced by Karpovsky et al. in [6]. It has been studied widely since then, mainly because of its numerous applications such as compact routing and fault-detection in communication networks or the location of threats in facilities.

It was proved in [4, 5] that any nontrivial undirected graph which is identifiable and whose vertices all have finite degree can be identified with a proper subset of its vertices. An example of such an infinite graph for which any identifying code consists of all the vertices, denoted A_∞, has been given in [4]. This graph has vertex set $\mathbb{Z} \times \{0, 1\}$; for $i \in \{0, 1\}$, the sets $\{(x, i) \mid x \in \mathbb{Z}\}$ form an infinite clique, and for all $x, y \in \mathbb{Z}$, the vertex $(x, 0)$ is joined to $(y, 1)$ if and only if $x < y$. In this talk, we give the following classification of all undirected infinite graphs having $V(G)$ as their only identifying code:

Theorem 1 *Let G be an infinite identifiable undirected graph. The only identifying code of G is $V(G)$ if and only if G can be constructed from some finite or infinite graph H having a perfect matching φ in the following way. For every edge $x\varphi(x)$ of φ, construct a copy of the graph A_∞ by letting x correspond to the sets $\Phi(x) = \mathbb{Z} \times \{0\}$, and $y = \varphi(x)$, to $\Phi(y) = \mathbb{Z} \times \{1\}$ in $V(A_\infty)$. Now, for all the other edges uv of H, join all the vertices of $\Phi(u)$ to all the vertices of $\Phi(v)$.*

It is interesting to note that the set of finite digraphs needing their whole vertex set to be identified is a more complicated family than in the undirected case. In this talk, we present the following classification of all such finite digraphs:

Theorem 2 *Let D be a finite identifiable digraph. The set $V(D)$ is the only identifying code of D if and only if D is the transitive closure of a forest of top-down oriented rooted trees.*

By translating the problem of identifying codes in digraphs into a problem of set systems, we use Theorem 2 to classify some extremal cases of a well celebrated theorem of Bondy:

Theorem 3 (Bondy, [1]) *For some positive integers n and k, let X be an $(n+k)$-set and $\mathcal{S}$, a family of n distinct subsets $S_1, ..., S_n$ of X. There exists a $(k+1)$-subset $X' = \{x_1, ..., x_{k+1}\}$ of X such that the sets $S_1 \setminus X', ..., S_n \setminus X'$ remain distinct (one being possibly the empty set).*

References

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