On identifying codes and Bondy's theorem on "induced subsets"

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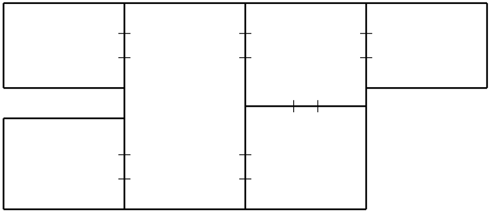
Outline

1. Introduction, definitions, examples
2. Finite and infinite undirected graphs
3. Finite digraphs
4. An application to Bondy's theorem

Locating a fire in a building

simple, undirected graph: models a building



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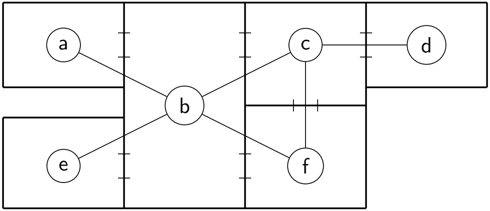

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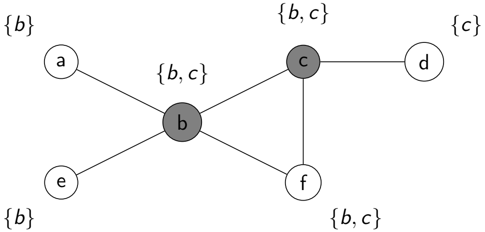
simple detectors: able to detect a fire in a neighbouring room



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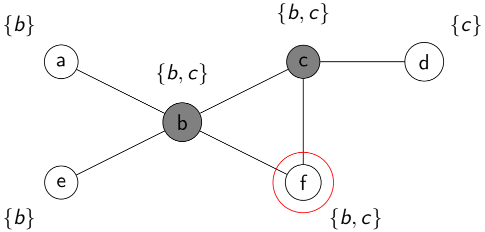

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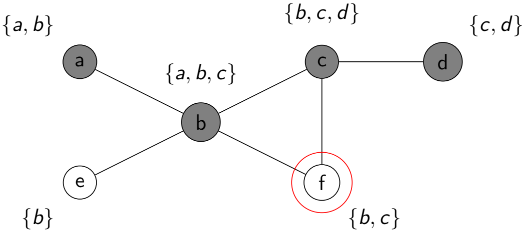

Identifying codes: definition

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Definition: identifying code of a graph G (Karpovsky et al. 1998)

subset C of V such that:

- C is a dominating set in G: for all $u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating code in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation

$\gamma^{\mathrm{ID}}(G)$: minimum cardinality of an identifying code of G

Identifiable graphs

Remark: not all graphs have an identifying code

u and v are *twins* if $N[u] = N[v]$.
A graph is *identifiable* iff it is *twin-free* (i.e. it has no twin vertices).

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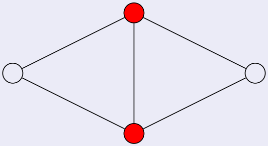

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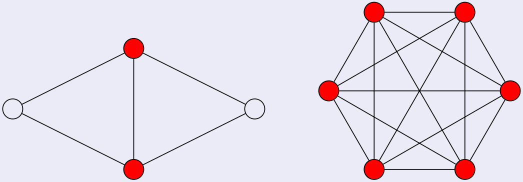

An upper bound

Theorem (Gravier, Moncel, 2007)

Let G be a finite identifiable graph with n vertices and at least one edge. Then $\gamma^{\text{ID}}(G) \leq n - 1$.

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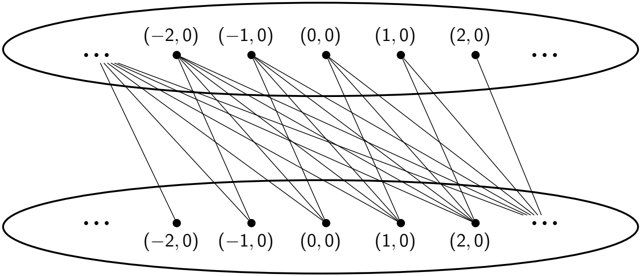
Corollary

The only finite graphs having their whole vertex set as a minimum identifying code are the stable sets $\overline{K_n}$.

The graph A_∞



Infinite clique on $\mathbb{Z}$

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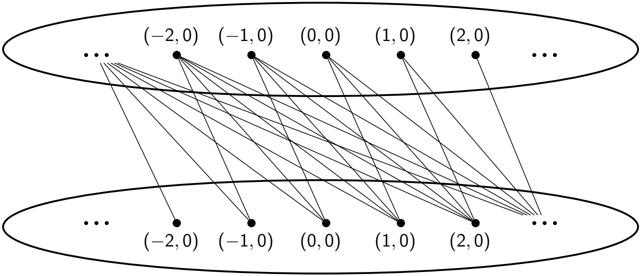

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A_∞ needs all its vertices in any identifying code.

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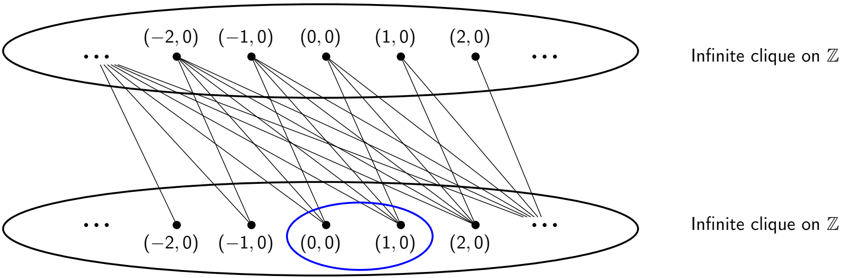

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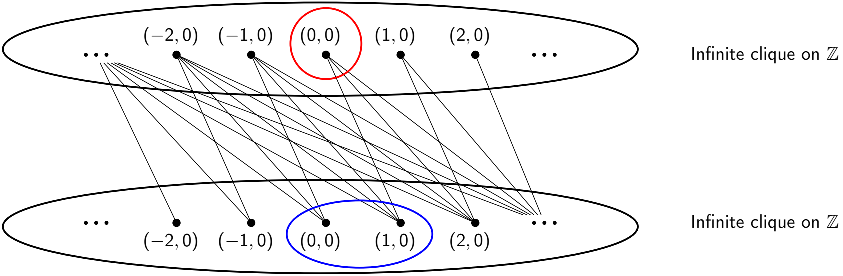

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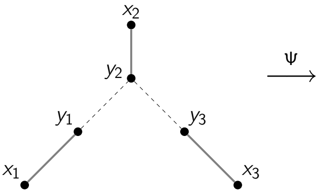
Constructing infinite graphs

Construction of $\Psi(H, \rho)$

H: finite or infinite simple graph with perfect matching
ρ: perfect matching of H

- Replace every edge $\{u, v\}$ of ρ by a copy of A_∞
- complete join along the other edges of H



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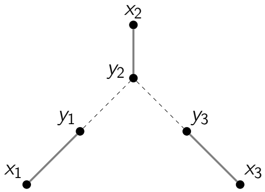

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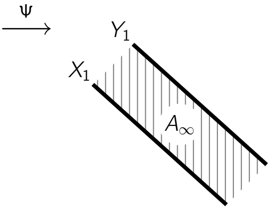

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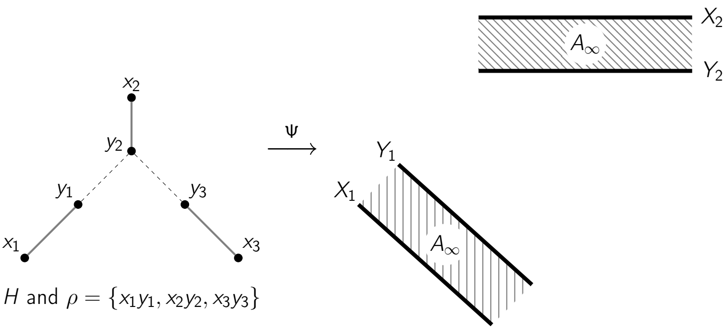

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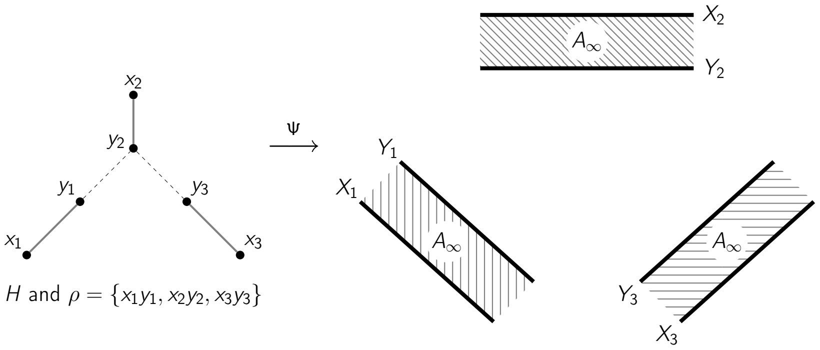

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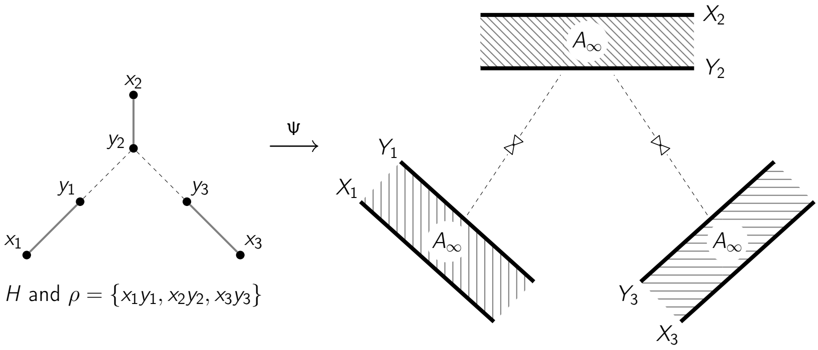

H and $\rho = \{x_1y_1, x_2y_2, x_3y_3\}$

The classification

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2010)

Let G be a connected infinite identifiable undirected graph. The only identifying code of G is $V(G)$ if and only if $G = \Psi(H, \rho)$ for some graph H with a perfect matching ρ.

Idcodes in digraphs

Let $N^-[u]$ be the set of *incoming neighbours* of u, plus u

Definition: identifying code of a digraph $D = (V, A)$

subset C of V such that:

- C is a dominating set in D: for all $u \in V$, $N^-[u] \cap C \neq \emptyset$, and
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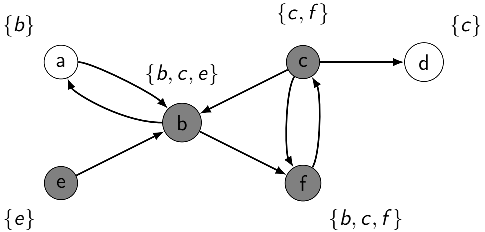

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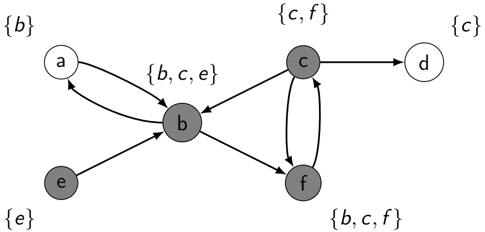


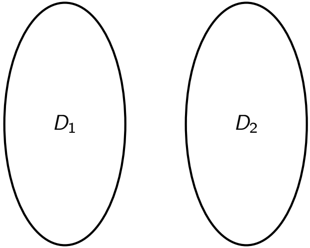
Definition

$\overrightarrow{\gamma^{\text{ID}}}(D)$: minimum size of an identifying code of D

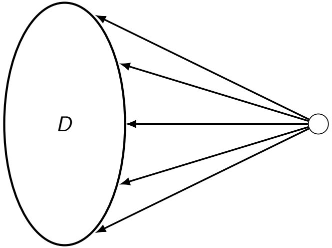
Which graphs need n vertices?

Two operations

- $D_1 \oplus D_2$: disjoint union of D_1 and D_2
- $\overrightarrow{\triangleleft}(D)$: D joined to K_1 by incoming arcs only



$D_1 \oplus D_2$



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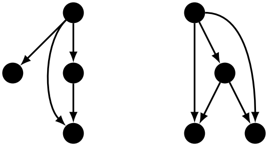

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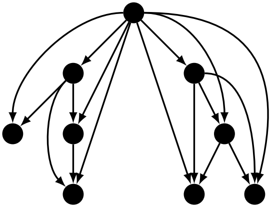

Which graphs need n vertices?

Two operations

- $D_1 \oplus D_2$: disjoint union of D_1 and D_2
- $\overrightarrow{\lhd}(D)$: D joined to K_1 by incoming arcs only

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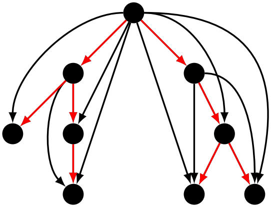
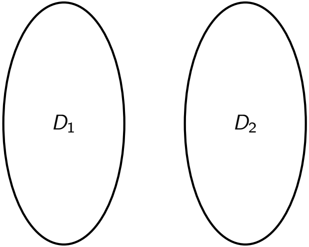


A characterization

Proposition

Let D be a digraph of $(K_1, \oplus, \overrightarrow{\triangleleft})$ on n vertices. $\overrightarrow{\gamma^{\text{ID}}}(D) = n$.



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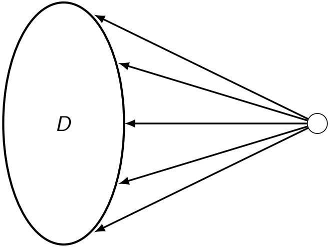

$\overrightarrow{\triangleleft}(D)$

A characterization

Theorem (F., Naserasr, Parreau, 2010)

Let D be an identifiable digraph on n vertices. $\overrightarrow{\gamma^{\text{ID}}}(G) = n$ iff $D \in (K_1, \oplus, \overrightarrow{\lhd})$.

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Let D be an identifiable digraph on n vertices. $\overrightarrow{\gamma^{\text{ID}}}(G) = n$ iff $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$.

A useful proposition

Let D be a digraph with $\overrightarrow{\gamma^{\text{ID}}}(G) = n - 1$, then there is a vertex x of D such that $\overrightarrow{\gamma^{\text{ID}}}(D - x) = n - 1$

Proof of the Theorem

- By contradiction: take a minimum counterexample, D
- By the proposition, there is a vertex x such that $\overrightarrow{\gamma^{\text{ID}}}(D - x) = |V(D - x)| - 1$. Hence $D - x \in (K_1, \oplus, \overrightarrow{\triangleleft})$.
- Show that by adding a vertex to $D - x$, we either stay in the family or decrease $\overrightarrow{\gamma^{\text{ID}}}$.

A theorem of Bondy

Theorem on "induced subsets" (Bondy, 1972)

Let $\mathcal{S} = \{S_1, S_2, \cdots S_n\}$ be a collection of distinct (possibly empty) subsets of an $(n+k)$-set X $(k \geq 0)$. Then there is a $(k+1)$-subset X' of X such that $S_1 - X', S_2 - X', \cdots S_n - X'$ are all distinct.

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Example with $k = 0$

$X = \{1,2,3,\mathbf{4}\}$ and $\mathcal{S} = \{\{1,\mathbf{4}\}, \{3\}, \{2,\mathbf{4}\}, \{1,2,\mathbf{4}\}\}$

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Example with $k = 1$

$X = \{1,2,\mathbf{3},\mathbf{4},5\}$ and $\mathcal{S} = \{\{1,\mathbf{4},5\}, \{\mathbf{3}\}, \{2,\mathbf{4},5\}, \{1,2,\mathbf{4},5\}\}$

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The result is best possible

$X = \{1,2,3,4\}$ and $\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$

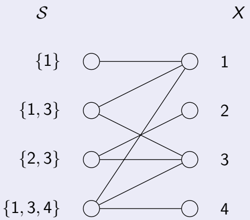
Bipartite representation

Bipartite representation

We can build a bipartite graph $B = (\mathcal{S} + X, E)$ where S_i connected to x iff $x \in S_i$.
Bondy's theorem states that there exists a code $C \subseteq X$ which *separates* $\mathcal{S}$ of size at most $|X| - 1$ in B.

Example

$X = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{\{1\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}\}$



Bipartite separating codes and identifying codes

Remark

Let B be the bipartite graph representing $(\mathcal{S}, X)$. If B has a matching from $\mathcal{S}$ to X, B is the neighbourhood graph of a digraph D.
$\Rightarrow$ A code separating $\mathcal{S}$ with X in B is a separating code of D.

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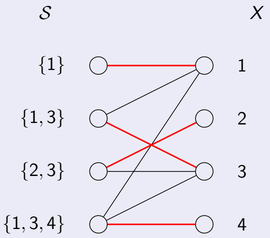
$X = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{\{1\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}\}$



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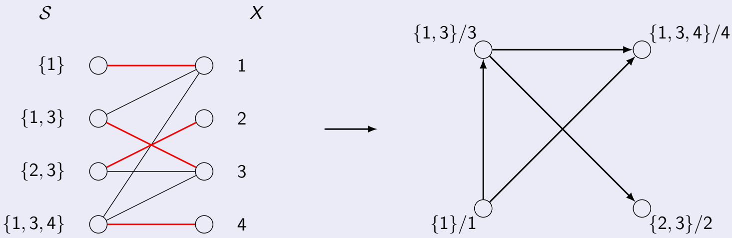

Application to Bondy's setting

Corollary (F., Naserasr, Parreau, 2010)

In Bondy's theorem, if for any good choice of x we have $S_i - x = \emptyset$ for some S_i, then B is the neighbourhood graph of a digraph in $(K_1, \oplus, \overrightarrow{\triangleleft})$.

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Proof (1)

<u>If $|X| > |\mathcal{S}|$</u> ($|X| = n + k$, $k > 0$):

by Bondy's theorem we can remove $k + 1 \geq 2$ elements of X.

At most one of them can create $\emptyset$, so we choose another one!

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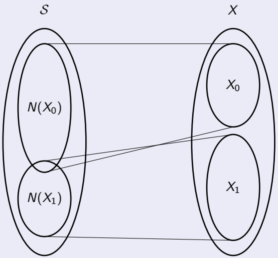
Proof (2)

<u>If $|X| = |\mathcal{S}|$</u>

- If B has a perfect matching: use our theorem.
- Otherwise, by Hall's theorem, there is a subset X_1 of X s.t. $|X_1| > |N(X_1)|$.



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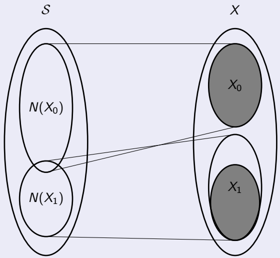

Conclusion

Future work

- Classify infinite digraphs D with $V(D)$ as their only identifying code
- What about graphs having $V(D) - x$ as an only identifying code?