On some extremal problems on identifying codes in (di)graphs

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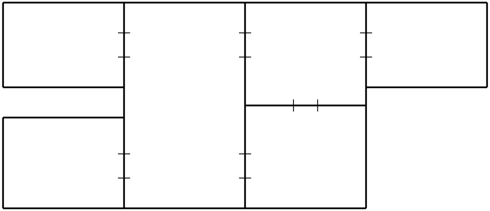
Locating a fire in a building

simple, undirected graph : models a building



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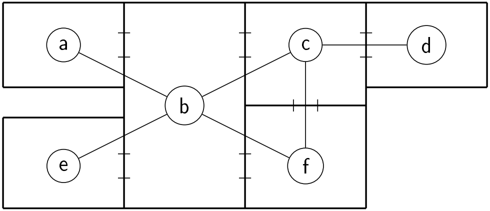

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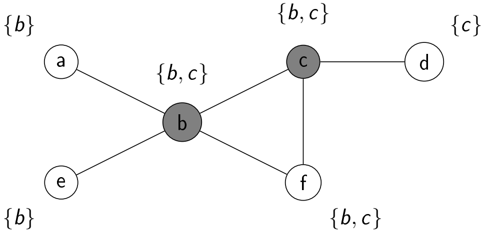

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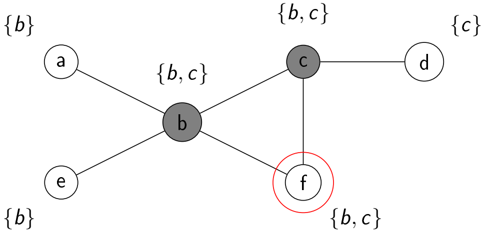

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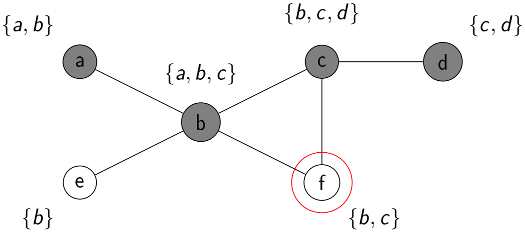

Identifying codes : definition

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Definition : identifying code of a graph G (Karpovsky et al. 1998)

subset C of V such that :

- C is a dominating set in G : for all $u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating code in G : $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

Identifiable graphs

Remark : not all graphs have an identifying code

u and v are *twins* if $N[u] = N[v]$.
A graph is *identifiable* iff it is *twin-free* (i.e. it has no twin vertices).

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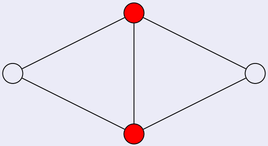

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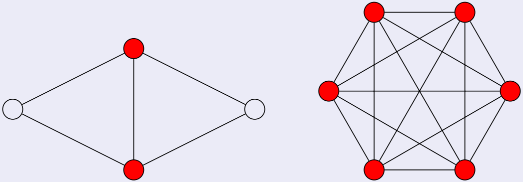

Two lower bounds

Thm (Karpovsky et al. 98)

Let G be an identifiable graph with n vertices. Then $\gamma^{\text{ID}}(G) \geq \lceil \log_2(n+1) \rceil$.

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Thm (Karpovsky et al. 98)

Let G be an identifiable graph with n vertices and maximum degree Δ. Then $\gamma^{\text{ID}}(G) \geq \frac{2n}{\Delta + 2}$.

An upper bound

Thm (Gravier, Moncel 2007)

Let G be a twin-free graph with $n \geq 3$ vertices and at least one edge. Then $\gamma^{\text{ID}}(G) \leq n - 1$.

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Thm (Charon, Hudry, Lobstein, 2007 + Skaggs, 2007)

For all $n \geq 3$, there exist twin-free graphs with n vertices and $\gamma^{\text{ID}}(G) = n-1$.

Upper bound - small examples

Recall the definition

- C is a dominating set in G : for all $u \in V$, $N[u] \cap C \neq \emptyset$, and
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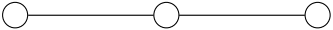

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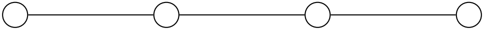

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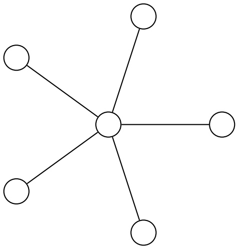

Upper bound - complete graph minus max. matching

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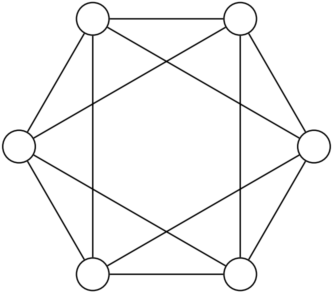


A conjecture

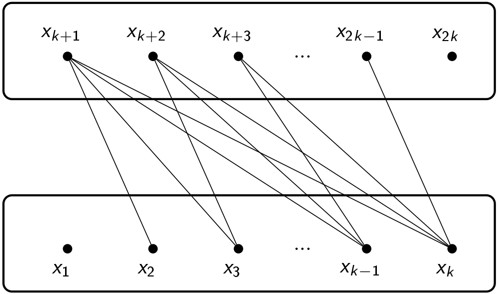
Conjecture (Charon, Hudry, Lobstein, 2008)

$\gamma^{\text{ID}}(G) = n - 1$ iff $G \in \{P_4, K_n \setminus M, K_{1,n-1}\}$.

A class of graphs called $\mathcal{A}$

Definition : graph A_k

$V(A_k) = \{x_1, ..., x_{2k}\}$.
x_i connected to x_j iff $|j - i| \leq k - 1$



Clique on $\{x_{k+1}, ..., x_{2k}\}$

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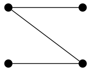
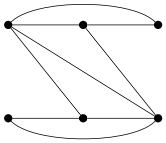
A class of graphs called $\mathcal{A}$ - examples

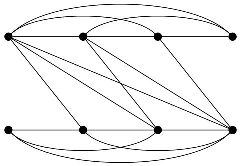
$A_1 = \overline{K_2}$



$A_2 = P_4$



$A_3 = P_6^2$

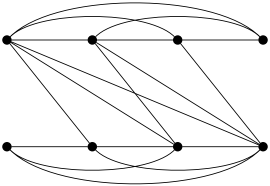


$A_4 = P_8^3$

Properties

Proposition

Let $k \geq 2$, $n = 2k$. $\gamma^{\text{ID}}(A_k) = n - 1$.



Remark

In every minimum code C of A_k, there exists a vertex x such that $C = N[x]$.

$(\mathcal{A}, \bowtie)$

Join operation

$G_1 \bowtie G_2$: disjoint copies of G_1 and G_2 + all possible edges between G_1 and G_2

Definition

Let $(\mathcal{A}, \bowtie)$ be the closure of graphs of $\mathcal{A}$ with respect to $\bowtie$.

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Proposition

Let G be a graph of $(\mathcal{A}, \bowtie) \setminus \{\overline{K_2}\}$ with n vertices. $\gamma^{\text{ID}}(G) = n - 1$.

$(\mathcal{A}, \bowtie) \bowtie K_1$

Proposition

Let G be a graph of $(\mathcal{A}, \bowtie)$ with $n-1$ vertices. $\gamma^{\text{ID}}(G \bowtie K_1) = n-1$.

A characterization

Thm (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2010)

Let G be a twin-free graph on n vertices. $\gamma^{\text{ID}}(G) = n - 1$ iff $G \in \mathcal{S} \cup (\mathcal{A}, \bowtie) \cup (\mathcal{A}, \bowtie) \bowtie K_1$ and $G \neq \overline{K_2}$.

A useful Proposition

Proposition

Let G be a twin-free graph and $S \subseteq V$ such that $G - S$ is twin-free. Then $\gamma^{\text{ID}}(G) \leq \gamma^{\text{ID}}(G - S) + |S|$.

Corollary

Let G be a graph with $\gamma^{\text{ID}}(G) = |V(G)| - 1$, then there is a vertex x of G such that $\gamma^{\text{ID}}(G - x) = |V(G - x)| - 1$

Proof ideas

Proof

- By contradiction : take a minimum counterexample, G
- By the proposition, there is a vertex x such that $\gamma^{\text{ID}}(G-x) = |V(G-x)| - 1$. Hence $G - x \in \mathcal{S} \cup (\mathcal{A}, \bowtie) \cup (\mathcal{A}, \bowtie) \bowtie K_1$ and $G \neq \overline{K_2}$.
- For the three cases, show that by adding a vertex to $G - x$, we either stay in the family or decrease γ^{ID}.

Upper bound and Δ : a corollary and a question

Corollary

Let G be an identifiable graph on n vertices and maximum degree $\Delta \leq n-3$. Then $\gamma^{\text{ID}}(G) \leq n-2$.

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Question

What is the best bound for γ^{ID} using n and Δ ?

Proposition 1

Let G be an identifiable graph, and x a vertex of G. There exists a vertex y, $d(x,y) \leq 1$, and $V - y$ is an identifying code of G.

Upper bound and Δ - two propositions

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Let G be an identifiable graph, and x a vertex of G. There exists a vertex y, $d(x,y) \leq 1$, and $V - y$ is an identifying code of G.

Proposition 2

Let G be an identifiable graph, and I a 4-independent set of G (all distances ≥ 4). If for all $x \in I$, $V - x$ is an identifying code of G, $V - I$ is also one.

A first bound

Corollary (F., Klasing, Kosowski, Raspaud, 2009)

Let G be an identifiable graph of maximum degree Δ. $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^5)}$. If G is Δ-regular, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$.

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Proof

- Consider a maximal 6-independant set I : distance between two vertices is at least 6 and $|I| \geq \frac{n}{\Theta(\Delta^5)}$
- For every $x \in I$, let $f(x)$ be the vertex found in Prop. 1.
- $V - f(I)$ is an identifying code of size at most $n - |I|$ by Prop. 2.

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Question

Is this bound sharp ?

$\Delta = 2$

Thm (Gravier, Moncel 2006)

Let G be a path or a cycle on n vertices. Then $\gamma^{\text{ID}}(G) \leq \frac{n}{2} + \frac{3}{2}$.

Connected cliques

- Take any Δ-regular graph H with m vertices
- replace any vertex of H by a clique of Δ vertices

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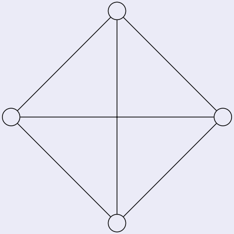
Example : $H = K_4$



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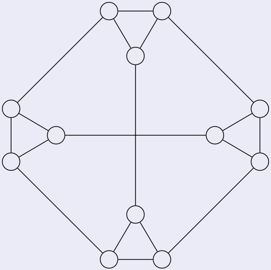

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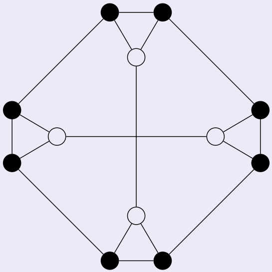

For every clique, at least $\Delta - 1$ vertices in the code
$\Rightarrow \gamma^{\text{ID}}(G) = m \cdot (\Delta - 1) = n - \frac{n}{\Delta}$

Triangle-free graphs

Thm (F., Klasing, Kosowski, Raspaud, 2009)

Let G be an identifiable triangle-free graph G with $n \geq 3$ vertices, no isolated vertices, and maximum degree $\Delta \geq 2$. Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{3\Delta+3}$. If G has minimum degree 3, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{2\Delta+2}$.

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Proof

- Consider a maximal independent set I : $|S| \geq \frac{n}{\Delta+1}$
- $C = V \backslash I$
- Some vertices may not be identified correctly
- $\rightarrow$ modify C locally. It is possible to add not too much vertices to C.

Is this bound sharp ?

Proposition

Let $K_{m,m}$ be the complete bipartite graph with $n = 2m$ vertices.
$\gamma^{\text{ID}}(K_{m,m}) = 2m - 2 = n - \frac{n}{\Delta}$.

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Thm (Bertrand et al. 05)

Let T_k^h be the k-ary tree with h levels and n vertices. $\gamma^{\text{ID}}(T_k^h) = \left\lceil \frac{k^2 n}{k^2 + k + 1} \right\rceil = n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}}$.

A conjecture

Conjecture (2009)

Let G be an identifiable connected graph of maximum degree $\Delta \geq 2$. Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + O(1)$.

Idcodes in digraphs

Let $N^-[u]$ be the set of *incoming neighbours* of u, plus u

Definition : identifying code of a digraph $D = (V, A)$

subset C of V such that :

- C is a dominating set in D : for all $u \in V$, $N^-[u] \cap C \neq \emptyset$, and
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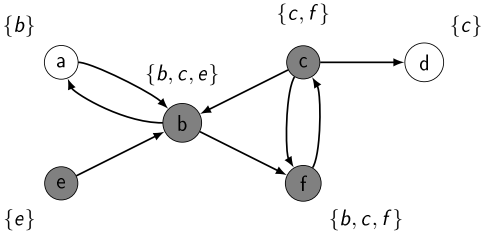

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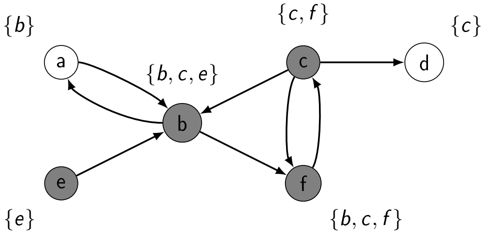


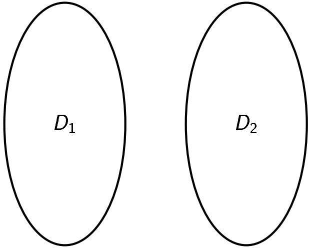
Definition

$\overrightarrow{\gamma^{\text{ID}}}(D)$: minimum size of an identifying code of D

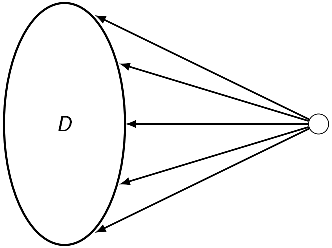
Which graphs need n vertices ?

Two operations

- $D_1 \oplus D_2$: disjoint union of D_1 and D_2
- $\overrightarrow{\triangleleft}(D)$: D joined to K_1 by incoming arcs only



$D_1 \oplus D_2$



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Definition

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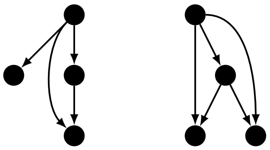

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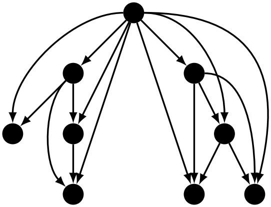

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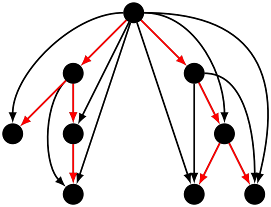
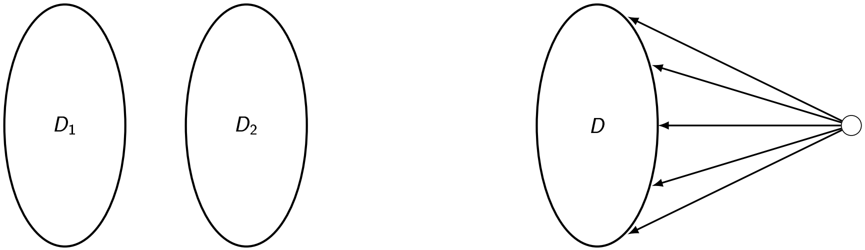


A characterization

Proposition

Let D be a digraph of $(K_1, \oplus, \overrightarrow{\triangleleft})$ on n vertices. $\overrightarrow{\gamma^{\text{ID}}}(D) = n$.



$D_1 \oplus D_2$

$\overrightarrow{\triangleleft}(D)$

A characterization

Theorem (F., Naserasr, Parreau, 2010)

Let D be an identifiable digraph on n vertices. $\overrightarrow{\gamma^{\text{ID}}}(G) = n$ iff $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$.

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Let D be an identifiable digraph on n vertices. $\overrightarrow{\gamma^{\text{ID}}}(G) = n$ iff $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$.

A useful proposition

Let D be a digraph with $\overrightarrow{\gamma^{\text{ID}}}(G) = n-1$, then there is a vertex x of D such that $\overrightarrow{\gamma^{\text{ID}}}(D-x) = n-1$

Proof of the Theorem

- By contradiction : take a minimum counterexample, D
- By the proposition, there is a vertex x such that $\overrightarrow{\gamma^{\text{ID}}}(D-x) = |V(D-x)| - 1$. Hence $D - x \in (K_1, \oplus, \overrightarrow{\triangleleft})$.
- Show that by adding a vertex to $D-x$, we either stay in the family or decrease $\overrightarrow{\gamma^{\text{ID}}}$.

A theorem of Bondy

Theorem on "induced subsets" (Bondy, 1972)

Let $\mathcal{S} = \{S_1, S_2, \cdots S_n\}$ be a collection of distinct (possibly empty) subsets of an $(n+k)$-set X $(k \geq 0)$. Then there is a $(k+1)$-subset X' of X such that $S_1 - X', S_2 - X', \cdots S_n - X'$ are all distinct.

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$X = \{1,2,3,4\}$ and $\mathcal{S} = \{\{1,4\}, \{3\}, \{2,4\}, \{1,2,4\}\}$

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The result is best possible

$X = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$

A theorem of Bondy - proof

Proof

Note : if $S_1, S_2 \subseteq X$ and $S_1 - x = S_2 - x$, then $S_1 \Delta S_2 = \{x\}$.
By contradiction :
Construct a graph $H = (\mathcal{S}, E)$ where for each $x \in X$, choose one unique (i, j) s.t. $S_i \Delta S_j = \{x\}$, and connect S_i to S_j.
Claim : H has no cycle - a contradiction !

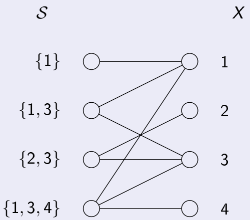
Bipartite representation

Bipartite representation

We can build a bipartite graph $B = (\mathcal{S} + X, E)$ where S_i connected to x iff $x \in S_i$. Bondy's theorem states that there exists a code $C \subseteq X$ which *separates* $\mathcal{S}$ of size at most $|X| - 1$ in B.

Example

$X = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{\{1\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}\}$



Bipartite separating codes and identifying codes

Remark

Let B be the bipartite graph representing $(\mathcal{S}, X)$. If B has a matching from $\mathcal{S}$ to X, B is the neighbourhood graph of a digraph D.
$\Rightarrow$ A code separating $\mathcal{S}$ with X in B is a separating code of D.

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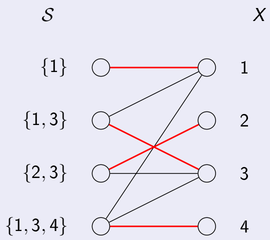

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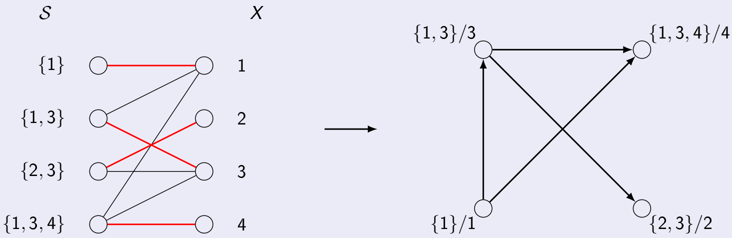


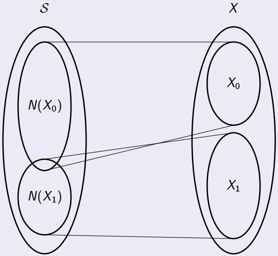
Application to Bondy's setting

Corollary (F., Naserasr, Parreau, 2010)

In Bondy's theorem (with $|X| = |\mathcal{S}|$ and non-empty sets), if for any good choice of x we have $S_i - x = \emptyset$ for some S_i,
then B is the neighbourhood graph of a digraph in $(K_1, \oplus, \overrightarrow{\triangleleft})$.

Proof

- If B has a perfect matching : use our theorem.
- Otherwise, by Hall's theorem, there is a subset X_1 of X s.t. $|X_1| > |N(X_1)|$.



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