On some problems in graph theory: from colourings to vertex identification

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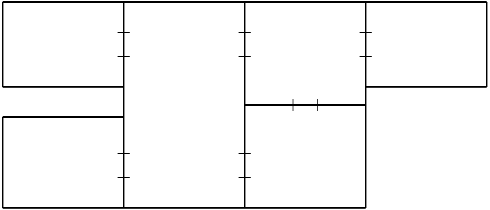
Calicut University, Kerala, India - September 4th, 2010

Outline

1. Graph theory - an introduction (definitions, colouring maps, the 4CT) [board]
2. Networks as graphs, the domination problem [board]
3. Complexity of graph theory problems - the “P versus NP” problem [board]
4. Identifying codes - definitions, complexity, extremal problems [slides]

Locating an intruder in a building

simple, undirected graph: models a building



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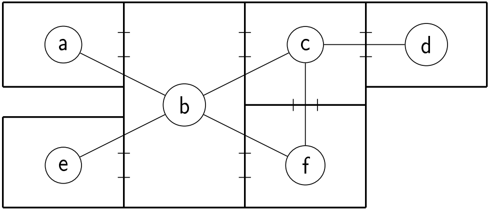
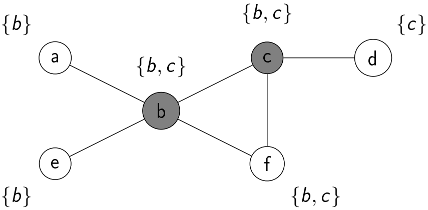


Locating an intruder in a building

simple detectors: able to detect an intruder in a neighbouring room

goal: locate an eventual intruder



		b	c	
a		•		
b		•	•	
c		•	•	
d			•	
e		•		
f		•	•	

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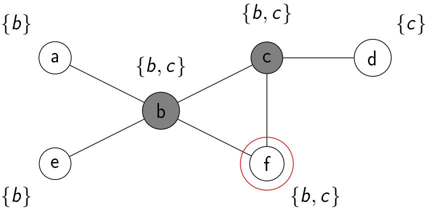
intruder in room f



		b	c	
a		●		
b		●	●	
c		●	●	
d			●	
e		●		
f		●	●	

Locating an intruder in a building

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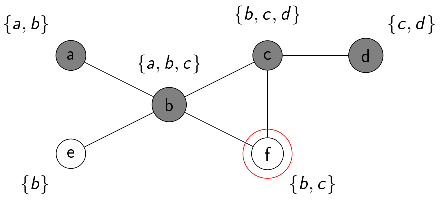
the *identifying sets* of all vertices must be distinct



	a	b	c	d
a	•	•		
b	•	•	•	
c		•	•	•
d			•	•
e		•		
f		•	•	

Identifying codes: definition

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$.

Definition: identifying code of G (Karpovsky et al. 1998)

subset C of V such that:

- C is a dominating set in G: for all $u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating set in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$

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Notation

$\gamma^{\mathrm{ID}}(G)$: minimum cardinality of an identifying code of G

Identifiable graphs

Remark: not all graphs have an identifying code

u and v are *twins* if $N[u] = N[v]$.
A graph is *identifiable* iff it is *twin-free* (i.e. it has no twin vertices).

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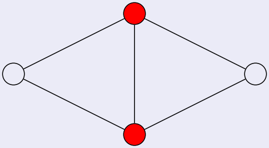
Non-identifiable graphs



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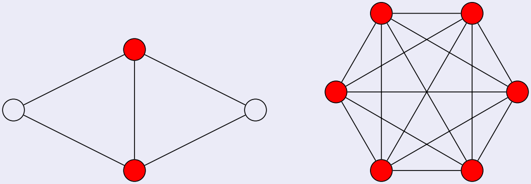

Hardness

Decision problem ID-CODE

Input: identifiable graph G and an integer k
Question: Is $\gamma^{\text{ID}}(G) \leq k$?

Thm (Cohen et al. 01, Auger et al. 09)

ID-CODE is NP-complete even in bipartite and planar graphs.

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Optimization problem MIN ID-CODE

Input: identifiable graph G
Output: $\gamma^{\text{ID}}(G)$

Thm (Trachtenberg et al. 06, Suomela 07, Gravier et al. 08)

MIN ID-CODE can be approximated within a logarithmic factor, but not within a constant factor.

An upper bound

Thm (Gravier, Moncel 2007)

Let G be a twin-free graph with $n \geq 3$ vertices and at least one edge. Then $\gamma^{\mathrm{ID}}(G) \leq n - 1$.

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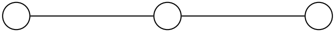
Thm (Charon, Hudry, Lobstein, 2007 + Skaggs, 2007)

For all $n \geq 3$, there exist twin-free graphs with n vertices and $\gamma^{\text{ID}}(G) = n - 1$.

Upper bound - small examples

Recall the definition

- C is a dominating set in G: for all $u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating set in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$



Upper bound - stars

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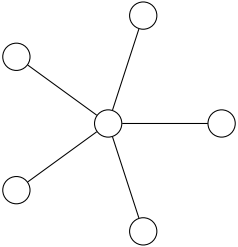

Upper bound - complete graph minus max. matching

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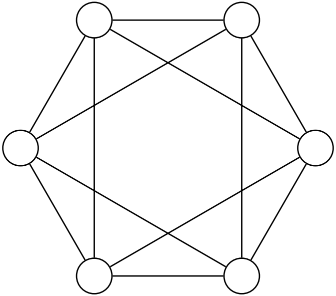


A conjecture

Conjecture (Charon, Hudry, Lobstein, 2008)

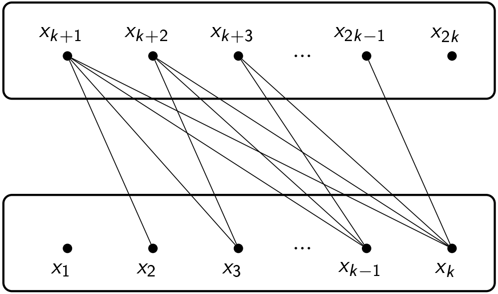
$\gamma^{\mathrm{ID}}(G) = n - 1$ iff $G \in \{P_4, K_n \setminus M, K_{1,n-1}\}$.

A class of graphs called $\mathcal{A}$

G^r: graph where x, y are adjacent iff $d(x,y) \leq r$ in G

Definition: graph A_k

$V(A_k) = \{x_1, ..., x_{2k}\}$.
x_i connected to x_j iff $|j - i| \leq k - 1$
$A_1 = \overline{K_2}$; for $k \geq 2$, $A_k = P_{2k}^{k-1}$



Clique on $\{x_{k+1}, ..., x_{2k}\}$

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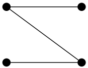
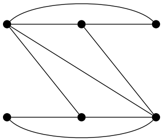
A class of graphs called $\mathcal{A}$ - examples

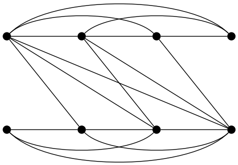
$A_1 = \overline{K_2}$



$A_2 = P_4$



$A_3 = P_6^2$

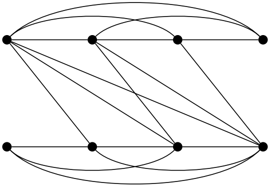


$A_4 = P_8^3$

Properties

Proposition

Let $k \geq 2$, $n = 2k$. $\gamma^{\text{ID}}(A_k) = n - 1$.



Remark

In every minimum code C of A_k, there exists a vertex x such that $C = N[x]$.

$(\mathcal{A}, \bowtie)$

Join operation

$G_1 \bowtie G_2$: disjoint copies of G_1 and G_2 + all possible edges between G_1 and G_2

Definition

Let $(\mathcal{A}, \bowtie)$ be the closure of graphs of $\mathcal{A}$ with respect to $\bowtie$.

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Let $(\mathcal{A}, \bowtie)$ be the closure of graphs of $\mathcal{A}$ with respect to $\bowtie$.

Proposition

Let G be a graph of $(\mathcal{A}, \bowtie) \setminus \{\overline{K_2}\}$ with n vertices. $\gamma^{\mathrm{ID}}(G) = n - 1$.

$(\mathcal{A}, \bowtie) \bowtie K_1$

Proposition

Let G be a graph of $(\mathcal{A}, \bowtie)$ with $n-1$ vertices. $\gamma^{\text{ID}}(G \bowtie K_1) = n-1$.

A characterization

Thm (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2010)

Let G be a twin-free graph on n vertices. $\gamma^{\text{ID}}(G) = n - 1$ iff $G \in \mathcal{S} \cup (\mathcal{A}, \bowtie) \cup (\mathcal{A}, \bowtie) \bowtie K_1$ and $G \neq \overline{K_2}$.

A useful Proposition

Proposition

Let G be a twin-free graph and $S \subseteq V$ such that $G - S$ is twin-free. Then $\gamma^{\text{ID}}(G) \leq \gamma^{\text{ID}}(G - S) + |S|$.

Corollary

Let G be a graph with $\gamma^{\text{ID}}(G) = |V(G)| - 1$, then there is a vertex x of G such that $\gamma^{\text{ID}}(G - x) = |V(G - x)| - 1$

Proof ideas

Proof

- By contradiction: take a minimum counterexample, G
- By the proposition, there is a vertex x such that $\gamma^{\mathrm{ID}}(G-x) = |V(G-x)| - 1$. Hence $G - x \in \mathcal{S} \cup (\mathcal{A}, \bowtie) \cup (\mathcal{A}, \bowtie) \bowtie K_1$ and $G \neq \overline{K_2}$.
- For the three cases, show that by adding a vertex to $G-x$, we either stay in the family or decrease γ^{ID}.

What about r-id codes? - Definition

Definition: r-identifying codes

subset C of V such that:

- C is an r-dominating set in G: for all $u \in V$, $B_r(u) \cap C \neq \emptyset$, and
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Remark

- C is an r-identifying code of G iff C is a 1-identifying code of G^r
- $\rightarrow \gamma_r^{\text{ID}}(G) = \gamma^{\text{ID}}(G^r)$
- $\rightarrow \{G \mid \gamma_r^{\text{ID}}(G) = n-1\} = \{G \mid G^r \in (\mathcal{S} \cup (\mathcal{A}, \bowtie) \cup (\mathcal{A}, \bowtie) \bowtie K_1) \setminus \overline{K_2}\}$

What about r-id codes? - Roots of A_k

Question

Let $r \geq 2$ and $k \geq 2$. What are the graphs G such that $G^r = A_k = P_{2k}^{k-1}$?

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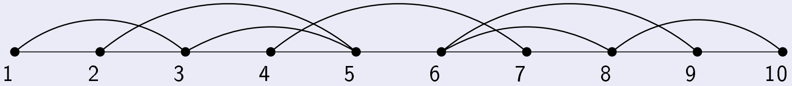
Partial answer

- All P_{2k}^s such that $s \cdot r = k - 1$. Example: $(P_{10}^2)^2 = P_{10}^4$ hence $\gamma_2^{\mathrm{ID}}(P_{10}^2) = 9$.
- Those graphs minus some edges...
- And other ones! Example: G such that $G^2 = P_{10}^4$ (so $\gamma_2^{\mathrm{ID}}(G) = 9$)



Idcodes in digraphs

Let $N^-[u]$ be the set of *incoming neighbours* of u, plus u

Definition: identifying code of a digraph $D=(V,A)$

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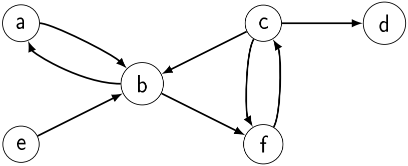

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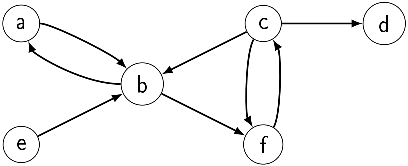

Definition

Let $\overrightarrow{\gamma^{\text{ID}}}(D)$ be the minimum size of an identifying code of D

Which graphs need n vertices?

Two operations

- $D_1 \oplus D_2$: disjoint union of D_1 and D_2
- $\overrightarrow{\triangleleft}(D)$: D joined to K_1 by incoming arcs only

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Remark

Every element of $(K_1, \oplus, \overrightarrow{\triangleleft})$ is the transitive closure of a forest of rooted oriented trees.

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Proposition

Let D be a digraph of $(K_1, \oplus, \overrightarrow{\triangleleft})$ on n vertices. $\overrightarrow{\gamma}^{\mathrm{ID}}(D) = n$.

A characterization

Thm (F., Naserasr, 2010)

Let D be a twin-free digraph on n vertices. $\overrightarrow{\gamma^{\text{ID}}}(G) = n$ iff $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$.

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A useful proposition (digraphs)

Let D be a digraph with $\overrightarrow{\gamma^{\text{ID}}}(G) = |V(D)| - 1$, then there is a vertex x of D such that $\overrightarrow{\gamma^{\text{ID}}}(D - x) = |V(D - x)| - 1$

Proof

- By contradiction: take a minimum counterexample, D
- By the proposition, there is a vertex x such that $\overrightarrow{\gamma^{\text{ID}}}(D - x) = |V(D - x)| - 1$. Hence $D - x \in (K_1, \oplus, \overrightarrow{\triangleleft})$.
- Show that by adding a vertex to $D - x$, we either stay in the family or decrease $\overrightarrow{\gamma^{\text{ID}}}$.

A theorem of Bondy

Theorem on "induced subsets" (Bondy, 1972)

Let $\mathcal{S} = \{S_1, S_2, \cdots S_n\}$ be a collection of distinct (possibly empty) subsets of an $n+k$-set X ($k \geq 0$). Then there is a $(k+1)$-subset X' of X such that $S_1 - X', S_2 - X', \cdots S_n - X'$ are all distinct.

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Example with $k = 0$

$X = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{\{1, 4\}, \{3\}, \{2, 4\}, \{1, 2, 4\}\}$

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Example with $k = 1$

$X = \{1, 2, 3, 4, 5\}$ and $\mathcal{S} = \{\{1, 4, 5\}, \{3\}, \{2, 4, 5\}, \{1, 2, 4, 5\}\}$

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Example with $k = 1$

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False for $k = -1$

$X = \{1,2,3,4\}$ and $\mathcal{S} = \{\emptyset,\{1\},\{2\},\{3\},\{4\}\}$

A theorem of Bondy - proof

Proof

Note: if $S_1, S_2 \subseteq X$ and $S_1 - x = S_2 - x$, then $S_1 \Delta S_2 = \{x\}$.
By contradiction:
Construct a graph $H = (\mathcal{S}, E)$ where for each $x \in X$, choose one unique (i,j) s.t. $S_i \Delta S_j = \{x\}$, and connect S_i to S_j.
Claim: H has no cycle - a contradiction!

Saying the same thing in another language

Bipartite representation

We can build a bipartite graph $B = (\mathcal{S} + X, E)$ where S_i connected to x iff $x \in S_i$.
Bondy's theorem states that there exists a *discriminating code* (see Charon, Cohen, Lobstein, Hudry, 2006) of $\mathcal{S}$ using X of size at most $|X| - 1$ in B.

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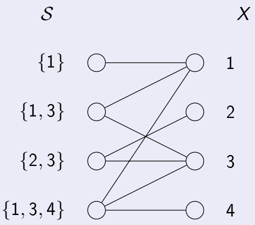

Discriminating codes and identifying codes

Remark

Let B be the bipartite graph representing $\mathcal{S}, X$ If B has a matching from $\mathcal{S}$ to X, B is the neighbourhood graph of a digraph D.
A discriminating code in B is a separating set of D!

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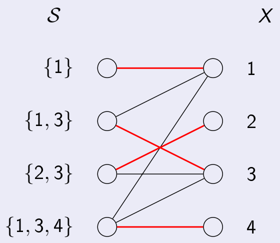

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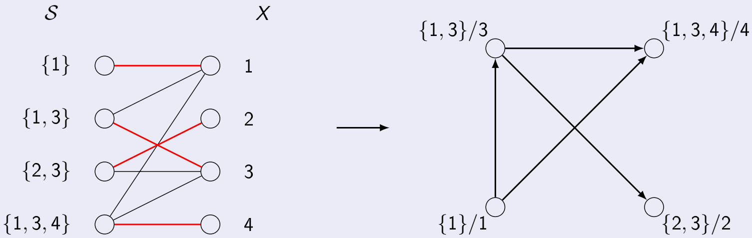
Example

$X = \{1, 2, 3, 4\}$ and $\mathcal{S} = \{\{1\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}\}$



Application to Bondy's setting

Corollary (F., Naserasr, 2010)

In Bondy's theorem, if we have $S_i - x = \emptyset$ for some S_i and for any good choice of x, then B is the neighbourhood graph of a digraph in $(K_1, \oplus, \overrightarrow{\triangleleft})$.

In other words

This happens iff for every $S_i, S_j \in \mathcal{S}$, $S_i \cap S_j \neq \emptyset \Rightarrow S_i \subseteq S_j$ or $S_j \subseteq S_i$.

Application to Bondy's setting - Hall's theorem

Marriage theorem (Hall, 1935)

Let $B = (X + Y, E)$ be a bipartite graph. B has a matching from X to Y iff for all $X' \subseteq X$, $|X'| \leq |N(X')|$.

Application to Bondy's setting - a proof

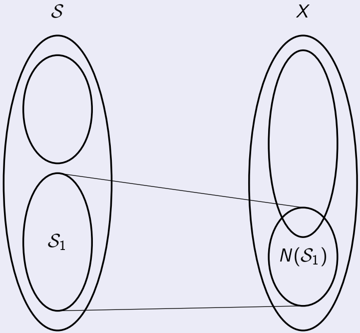
Proof (1)

If $|X| > |\mathcal{S}|$ ($|X| = n + k$, $k \geq 0$): by Bondy's theorem we can remove $k + 1$ elements of X.
At most one can create an $\emptyset$, so we choose another one of the $k + 1$.

($\Leftarrow$) By our theorem: $\overrightarrow{\gamma^{\text{ID}}} = n \Rightarrow$ separating set of size $\geq n - 1$

Proof (2) ($\Rightarrow$)

- If B has a perfect matching: use our theorem.
- Otherwise, by Hall's theorem, there is a subset $\mathcal{S}_1$ of $\mathcal{S}$ s.t. $|\mathcal{S}_1| > |N(\mathcal{S}_1)|$.



Upper bound and Δ: a corollary and a question

Corollary

Let G be a twin-free graph on n vertices and maximum degree $\Delta \leq n-3$. Then $\gamma^{\mathrm{ID}}(G) \leq n-2$.

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Question

Is γ^{ID} bounded by a function of n and Δ?

Upper bound and Δ - two propositions

Proposition 1

Let G be a twin-free graph, and x a vertex of G. There exists a vertex y, $d(x,y) \leq 1$, and $V - y$ is an identifying code of G.

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Proposition 2

Let G be a twin-free graph, and I a 4-independent set of G (all distances ≥ 4). If for all $x \in I$, $V - x$ is an identifying code of G, $V - I$ is also one.

A first bound

Corollary (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a twin-free graph of maximum degree Δ. $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^5)}$.

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Proof

- Consider a maximal 6-independant set I: distance between two vertices is at least 6 and $|I| \geq \frac{n}{\Theta(\Delta^5)}$
- For every $x \in I$, let $f(x)$ be the vertex found in Prop. 1.
- $V - f(I)$ is an identifying code of size at most $n - |I|$ by Prop. 2.

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Remark

Can be improved to $n - \frac{n}{\Theta(\Delta^4)}$ with a bit of modifications.

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Is this bound sharp?

Connected cliques

- Take any Δ-regular graph H with m vertices
- replace any vertex of H by a clique of Δ vertices

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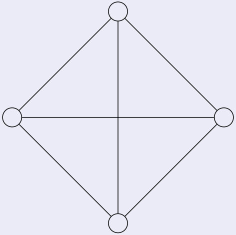

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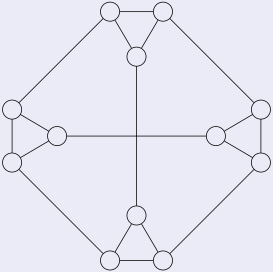

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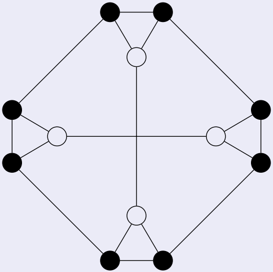

For every clique, at least $\Delta - 1$ vertices in the code
$\Rightarrow \gamma^{\mathrm{ID}}(G) = m \cdot (\Delta - 1) = n - \frac{n}{\Delta}$

Triangle-free graphs

Thm (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a twin-free connected triangle-free graph G with $n \geq 3$ vertices and maximum degree Δ. Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{3\Delta+3}$. If G has minimum degree 3, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{2\Delta+2}$.

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Proof

- Consider a maximal independent set I: $|S| \geq \frac{n}{\Delta+1}$
- $C = V \backslash I$
- Some vertices may not be identified correctly
- $\rightarrow$ modify C locally. It is possible to add not too much vertices to C.

Is this bound sharp?

Proposition

Let $K_{m,m}$ be the complete bipartite graph with $n = 2m$ vertices.
$\gamma^{\text{ID}}(K_{m,m}) = 2m - 2 = n - \frac{n}{\Delta}$.

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Thm (Bertrand et al. 05)

Let T_k^h be the k-ary tree with h levels and n vertices.$\gamma^{\text{ID}}(T_k^h) = \left\lceil \frac{k^2 n}{k^2 + k + 1} \right\rceil = n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}}$.

A conjecture

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a connected twin-free graph of maximum degree Δ. Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+1}$.

Graphs of girth at least 5

Thm (F., Klasing, Kosowski, Raspaud, 2009)

Let G be a twin-free graph with n vertices, of minimum degree $\delta \geq 2$ and girth $g \geq 5$. Then $\gamma^{\text{ID}}(G) \leq \frac{7n}{8} + 1$.

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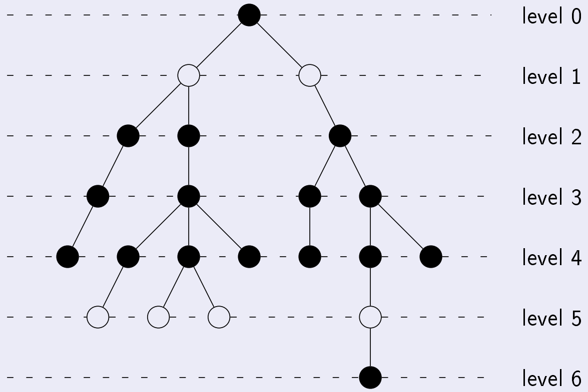
Proof

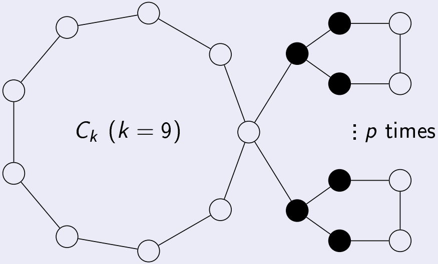
- Construct a DFS spanning tree T of G
- Partition the vertices into 4 classes V_0, V_1, V_2, V_3 depending on their level in T
- Take $C = V \setminus V_i$ as a code, $|V_i| \geq \frac{n}{4}$: $|V_i| \leq \frac{3n}{4}$
- C must be modified locally; the size of C might increase

Graphs of girth at least 5



Graphs of girth at least 5 - bad example

$G_{k,p} : \delta = 2, \Delta = p + 2, n = (5p + 1)k$



$\gamma^{\mathrm{ID}}(G_{k,p}) = 3pk = \frac{3}{5}(n - k) \to \frac{3n}{5}$

A general lower bound

Thm (Karpovsky et al. 98)

Let G be a twin-free graph with n vertices. Then $\gamma^{\text{ID}}(G) \geq \lceil \log_2(n+1) \rceil$.

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Characterization

The graphs reaching this bound have been characterized (Moncel 06)

Lower bound and maximum degree

Thm (Karpovsky et al. 98)

Let G be a twin-free graph with n vertices and maximum degree Δ. Then $\gamma^{\mathrm{ID}}(G) \geq \dfrac{2n}{\Delta + 2}$.

Graphs reaching the lower bound

Characterization (F., Klasing, Kosowski, Raspaud, 2009)

- n vertices
- independent set C of size $\frac{2n}{\Delta+2}$ (id. code)
- every vertex of C has exactly Δ neighbours
- $\frac{\Delta n}{\Delta+2}$ vertices connected to exactly 2 code vertices each

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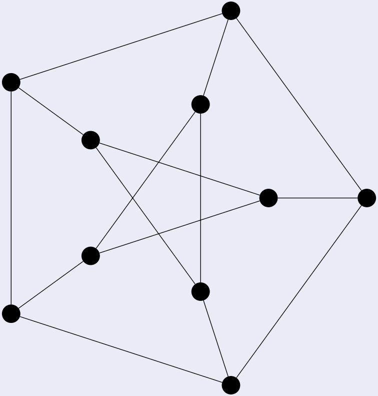
Construction

- Take a simple Δ-regular graph D (code)
- Put a new vertex on each edge of D
- Eventually add edges between the new vertices

Graphs reaching the lower bound - example

Example: D=Petersen graph, $\Delta = 3$, $n = 10$



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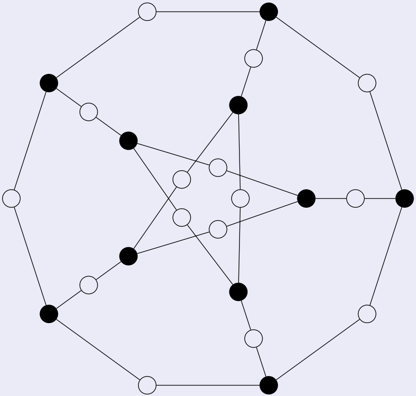

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