Bounding the identifying code number of a graph using its degree parameters

(a probabilistic approach)

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GT probas, LaBRI - September 16th, 2011

joint work with **Guillem Perarnau** (UPC, Barcelona)

Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Definition - Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of V such that:

- C is a dominating set in G: $\forall u \in V$, $N[u] \cap C \neq \emptyset$, and
- C is a separating code in G: $\forall u \neq v$ of V, $N[u] \cap C \neq N[v] \cap C$
  Equivalently: $(N[u] \Delta N[v]) \cap C \neq \emptyset$ (covering symmetric differences)

Notation - Identifying code number

$\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

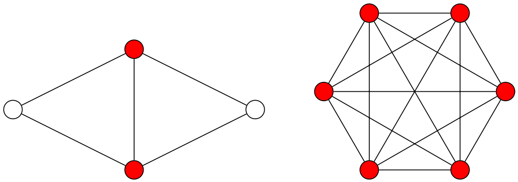
Let $N[u]$ be the set of vertices v s.t. $d(u, v) \leq 1$

Remark

Not all graphs have an identifying code!

Twins $=$ pair u, v such that $N[u] = N[v]$.

A graph is identifiable iff it is twin-free (i.e. it has no twins).



Degree parameters of a graph

Graph $G = (V, E)$, vertex $v \in V$.

- degree of v: number of edges it is incident to
- minimum degree δ of G: min. degree of a vertex in G
- maximum degree d of G: max. degree of a vertex in G
- d-regular graph: all vertices have degree d

Theorem (Karpovsky, Chakrabarty, Levitin, 1998 + Gravier, Moncel, 2007)

Let G be an identifiable graph with at least one edge, then

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G) \leq n-1$$

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

Let G be an identifiable graph with maximum degree d, then

$$\frac{2n}{d+2} \leq \gamma^{\text{ID}}(G)$$

Conjecture (F., Klasing, Kosowski, Raspaud, 2009+)

There exists a constant c, such that for every connected nontrivial identifiable graph G of max. degree d,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + c$$

This would be tight. True for $d = 2$ and $d = n-1$.

Technique initiated, among others, by Pál Erdős
used mainly in combinatorics (Ramsey theory, graph theory, ...)

1. Define a suitable probability space
2. Select some object from this space using randomness
3. Prove that with nonzero probability, certain "good" conditions hold
4. Conclusion: there always exists a "good" object

Classic reference: Noga Alon and Joel Spencer, *The probabilistic method*

Corollaries

$NF(G)$: proportion of non forced vertices of G

$$NF(G) = \frac{\#\text{non-forced vertices in G}}{\#\text{vertices in G}}$$

Theorem (F., Perarnau, 2011+)

There exists an integer d_0 such that for each identifiable graph G on n vertices having maximum degree $d \geq d_0$ and no isolated vertices,

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{85d}$$

Corollary

- In general, $NF(G) \geq \frac{1}{d+1}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d^3)}$
- If G is d-regular, $NF(G) = 1$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{85d}$.
- If G has clique number bounded by k, $NF(G) \geq \frac{1}{c(k)}$ and $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(d)}$.

Let G be a d-regular graph.

$$\gamma^{\text{ID}}(G) \geq \frac{2n}{d+2} \qquad \text{Karpovsky et al. (1998)}$$

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{d} + c \qquad \text{Conjecture (2009)}$$

Where are most of the d-regular graphs?

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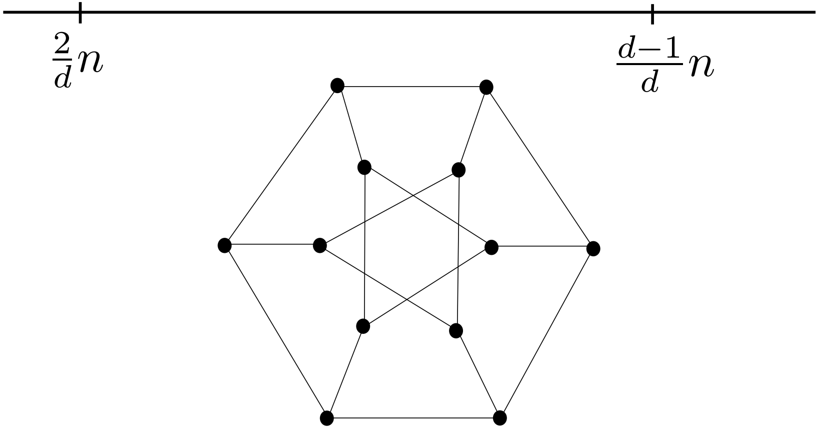

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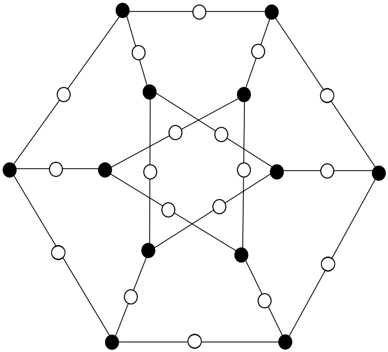
$\frac{2}{d}n$ $\frac{d-1}{d}n$



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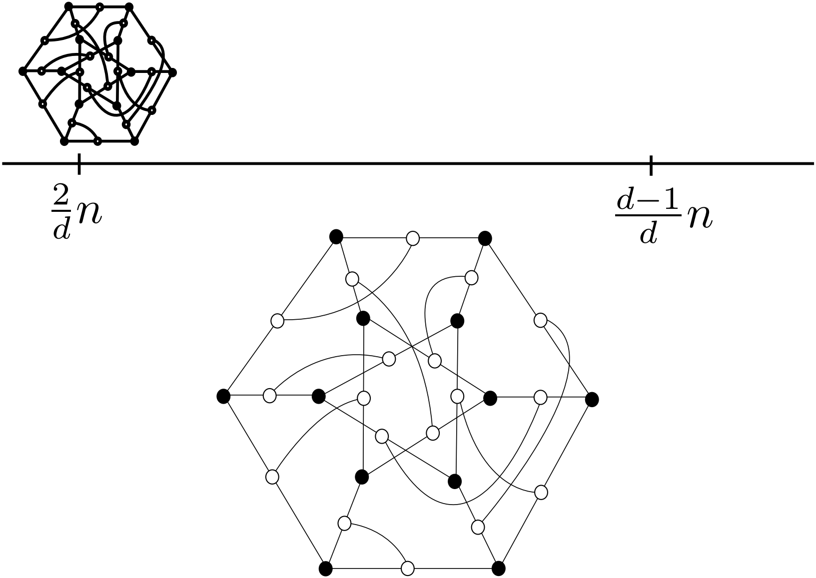

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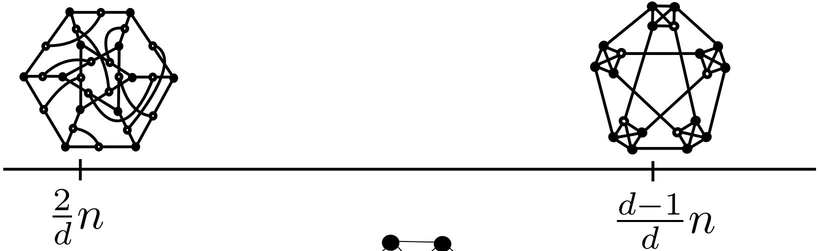

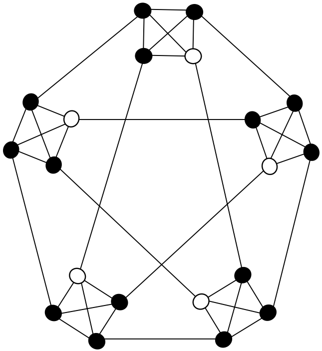

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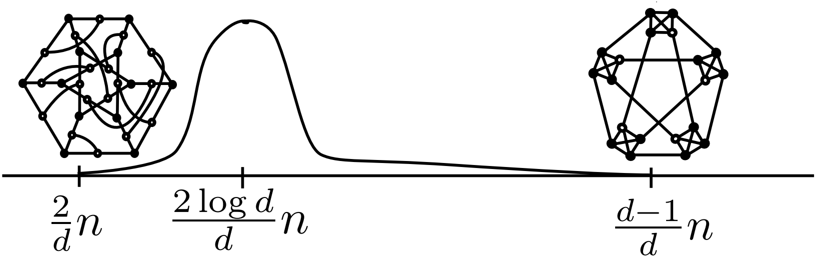

Theorem (F., Perarnau, 2011+)

Let G be a random d-regular graph. Then a.a.s.

$$(1+o_d(1))\frac{\log d}{d}n \le \gamma^{\text{ID}}(G) \le (1+o_d(1))\frac{2\log d}{d}n$$

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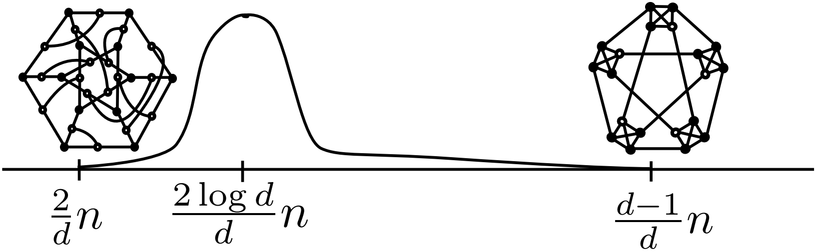

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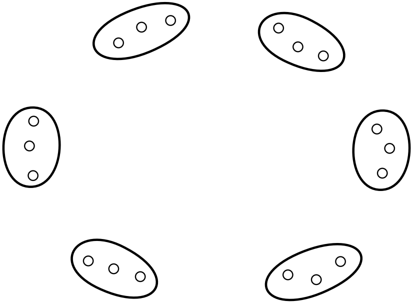
$$\frac{\log d - 2\log\log d}{d}n \leq \gamma^{\text{ID}}(G) \leq \frac{\log d + \log\log d + O_d(1)}{d}n$$

The pairing model (a.k.a. configuration model) - Bollobás, 1980

Probability space $\mathcal{G}^*_{n,d}$ of d-regular multigraphs on n vertices.



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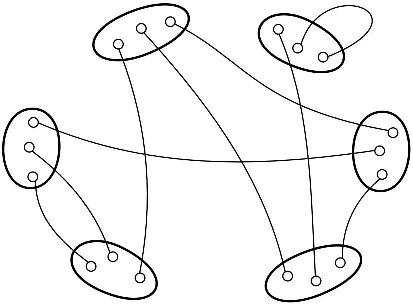

- Take nd vertices grouped in n buckets of size d
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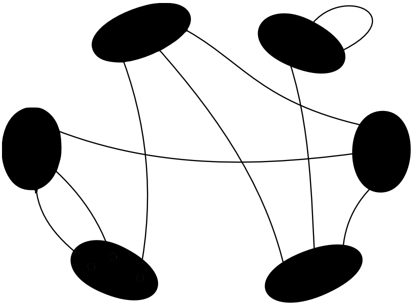

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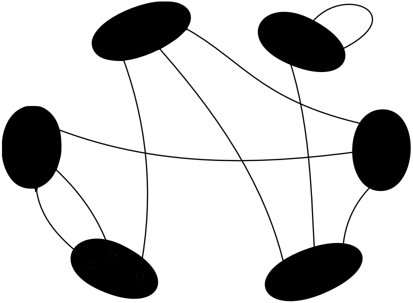

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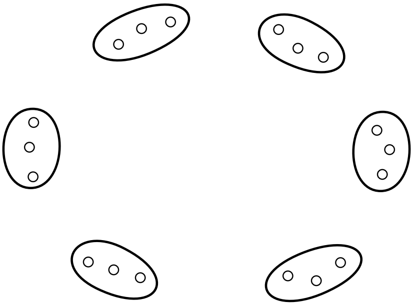

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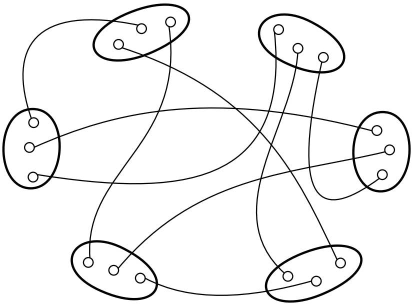

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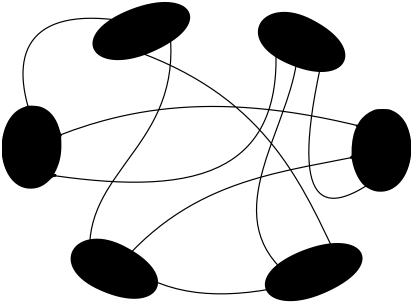

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Probability space $\mathcal{G}^*_{n,d}$ of d-regular (labelled) multigraphs on n vertices.

Proposition (Bollobás, 1980 - Wormald, 1981)

Let $G \in \mathcal{G}^*_{n,d}$. Then $Pr(G \text{ is simple}) \longrightarrow e^{\frac{1-d^2}{4}} > 0$

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Notation - Simple random regular graphs

Let $\mathcal{G}_{n,d} = \mathcal{G}^*_{n,d} \mid$ the graph is simple.

$\mathcal{G}^*_{n,d}$: non-uniform distribution. $\mathcal{G}_{n,d} = \mathcal{G}$: uniform distribution

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Theorem: any property which holds a.a.s. for $\mathcal{G}^*_{n,d}$, also does for $\mathcal{G}_{n,d}$.

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Other models are known, but do not provide a uniform distribution

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$L_d(n)$:# labelled d-regular graphs on n vertices, $U_d(n)$:# UNlabelled d-regular graphs.

Bollobás, 1982: $U_d(n) \sim \frac{L_d(n)}{n} \sim \frac{(rn)!e^{(1-d^2)/4}}{(\frac{rn}{2})!2^{rn/2}(r!^n n!)}$

(Note: no exact formula is known!)

Corollary: any property which holds a.a.s. for labelled d-regular graphs also does for unlabelled ones.

Proposition (Bollobás, 1980 - Wormald, 1981)

$\mathbb{E}$(number of k-cycles in $\mathcal{G}^*_{n,d}$) $\longrightarrow \frac{(d-1)^k}{2k}$

In fact, stronger result: the distribution of the numbers of k-cycles for fixed $k \in \{2, \ldots\}$ all jointly tend to independent Poisson variables of parameter $\lambda_k = \frac{(d-1)^k}{2k}$.

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Proposition

Let $G \in \mathcal{G}^*_{n,d}$, then a.a.s. G is identifiable (no twins).

Graphs with girth at least 5

Proposition (F., Perarnau, 2011+)

Let G be a twin-free graph on n vertices having girth at least 5. Let D be a 2-dominating set of G. If the subgraph induced by D, $G[D]$, has no isolated edge, D is an identifying code of G.

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Theorem (F., Perarnau, 2011+)

Let G be a d-regular graph with girth at least 5. Then

$$\gamma^{\text{ID}}(G) \leq \frac{\log d + \log\log d + O_d(1)}{d} n$$

Sketch of the proof: construct 2-dominating set D

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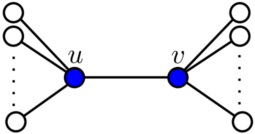
- $\mathcal{C} = S \cup \{v : X_v = 1\}$, $p = \frac{\log d + \log\log d}{d}$

  $$\mathbb{E}(|D|) = \mathbb{E}(|S|) + X(S) \leq \frac{\log d + \log\log d}{d} n + \frac{1 + \log d + \log\log d}{d \log d} n$$

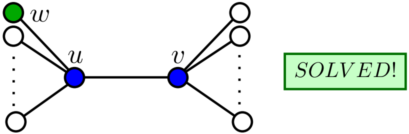
  $$\mathbb{E}(|D|) \leq \frac{\log d + \log\log d + O_d(1)}{d} n$$

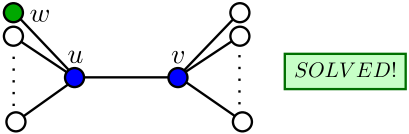
Sketch of the proof: identifying code



PROBLEM!

Sketch of the proof: identifying code





$$Y_{uv} = \begin{cases} 1 & \text{if } \text{[graph]} \\ 0 & \text{otherwise} \end{cases}$$

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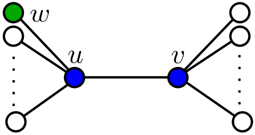

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$$\mathcal{C} = S \cup \{v : X_v = 1\} \cup \{w : w \in N(u),\ Y_{uv} = 1\},\ p = \frac{\log d + \log\log d}{d}$$

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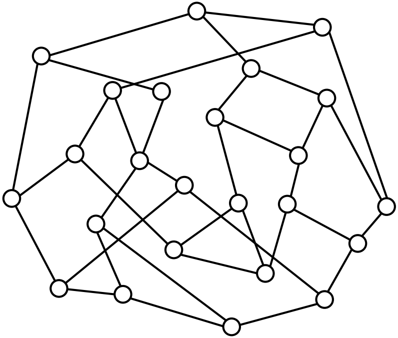

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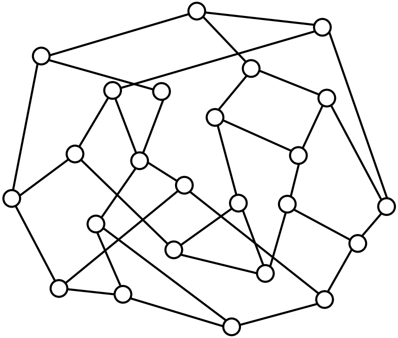

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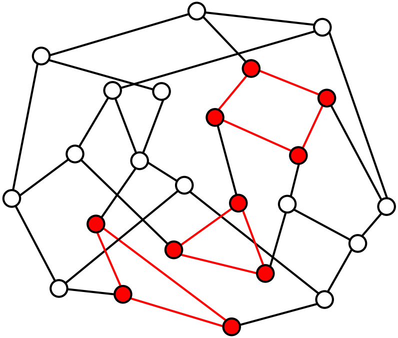

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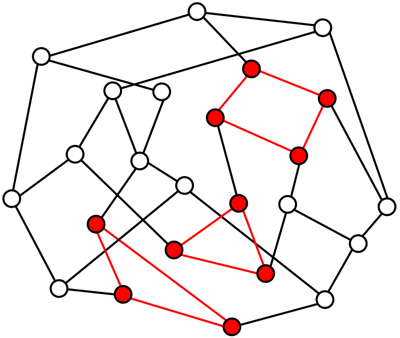

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