Identifying vertices of a graph using paths

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The test cover problem (TCP)

Definition - Test cover problem (mentioned in Garey, Johnson, 1979)

INPUT: a set system (or hypergraph) $(X, \mathcal{S})$
PROBLEM: find the minimum subset $\mathcal{T} \subseteq \mathcal{S}$ such that each element $x \in X$ belongs to a different set of sets in $\mathcal{T}$.

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Equivalently: for any pair x, y of elements of X, there is a set in $\mathcal{T}$ that contains **exactly** one of x, y.

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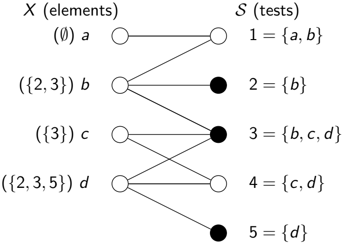

example: $\mathcal{T} = \{2, 3, 5\}$

The identification problem (IDP)

Definition - Identification problem

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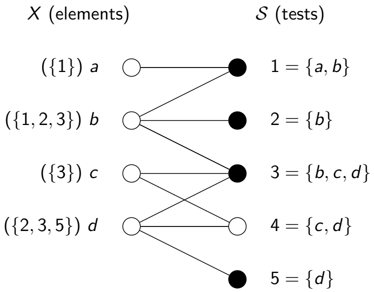

example: $\mathcal{T} = \{1, 2, 3, 5\}$

Motivation

- Fault analysis: tests are fault-detectors
- medical diagnostics: tests are tests for diseases
- biological identification: tests are attributes

General bounds

Theorem (Folklore)

Given a set system $(X, \mathcal{S})$, a solution to the TCP has size at least $\log_2(|X|)$. A solution to the IDP has size at least $\log_2(|X|+1)$. These bounds are tight.

Proof: Must assign to each element of X, a distinct subset of $\mathcal{T}$. Hence $|X| \leq 2^{|\mathcal{T}|}$ (TCP) and $|X| \leq 2^{|\mathcal{T}|} - 1$ (IDP). □

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Theorem (Bondy's theorem, 1972)

Given a set system $(X, \mathcal{S})$, a minimal solution to the TCP has size at most $|X| - 1$. A minimal solution to the IDP has size at most $|X|$. These bounds are tight.

Proof: TCP: nice graph-theoretic argument.
IDP: sizes of solutions to TCP and IDP differ by at most 1! □

TCP-k and IDP-k

Definition - k-bounded Test Cover Problem and Identification Problem

INPUT: a set system $(X, \mathcal{S})$ such that
each test has size at most k
PROBLEM: find the minimum subset $\mathcal{T} \subseteq \mathcal{S}$ such that each element $x \in X$ belongs to a different (nonempty) set of sets in $\mathcal{T}$.

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Theorem (Moret and Shapiro, 1985)

Given a k-bounded set system $(X, \mathcal{S})$, a solution to the TCP or IDP has size at least $\frac{2|X|}{k+1}$. This bound is tight.

Proof: i_1: elements belonging to 1 test of $\mathcal{T}$; i_2: elements in at least 2 tests

$i_1 \leq |\mathcal{T}|$, $i_2 \leq \frac{|\mathcal{T}|k - i_1}{2}$

$|X| = i_1 + i_2 \leq |\mathcal{T}| + \frac{|\mathcal{T}|k - i_1}{2} = \frac{|\mathcal{T}|(k+1)}{2}$ □

Complexity results

Theorem (Garey, Johnson, 1979)

TCP is NP-complete.

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TCP is $O(\log(|X|))$-approximable, but NP-hard to approximate within $o(log(|X|))$. TCP-k is $O(\log(k))$-approximable.

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Remark: The same holds for IDP and IDP-k.

Special cases of IDP

Rich literature (250+ publications) on variants arising from **graph theory**:

Definition - Identifying codes (Karpovsky, Chakrabarty, Levitin, 1998)

Given a graph G, it is the IDP problem where $X = V(G)$ and $\mathcal{S}$ is the set of **closed neighbourhoods** in G (a vertex identifies its neighbours and itself).

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Motivation: fault-detection in computer networks or location of threats in facilities

The path identifying cover problem

Definition - Identifying path cover (F., Kovše)

Given a graph G, it is the IDP problem where $X = V(G)$ and $\mathcal{S}$ is the set of **paths** in G: we look for a set $\mathcal{P}$ of paths such that:

- each vertex belongs to some path in $\mathcal{P}$, and
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Notation - $p^{\mathrm{ID}}(G)$

minimum number of paths needed in an identifying path cover.

Example: paths and cycles

To distinguish two adjacent vertices: need a path stopping at one of them.

P_n, C_n: the path and cycle on n vertices.

Theorem (F., Kovše)

We have:

- $p^{\mathrm{ID}}(P_n) = \lceil \frac{n+1}{2} \rceil$
- $p^{\mathrm{ID}}(C_3) = 2$, $p^{\mathrm{ID}}(C_4) = 3$
- for $n \geq 5$, $p^{\mathrm{ID}}(C_n) = \lceil \frac{n}{2} \rceil$

Example: stars

The optimum is roughly to cover three leaves using two paths.

$K_{1,n-1}$: star on n vertices.

Theorem (F., Kovše)

We have $p^{\text{ID}}(K_{1,n-1}) = \lceil \frac{2(n-1)}{3} \rceil$.

Trees

Same idea can be used for trees.
T: tree with ℓ leaves, t degree 2 vertices.

Identify all vertices that are not of degree 2 using $\lceil \frac{2(\ell-1)}{3} \rceil$ paths.
Intuition: first contract the tree to a star, then de-contract it level by level; at each step, re-route paths accordingly.

Degree 2 vertices can be identified using at most $\lceil \frac{t}{2} \rceil$ additional paths.

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Corollary

Let G be a **connected** graph. Then $p^{\text{ID}}(G) \leq \lceil \frac{2n}{3} \rceil$.

Proof: Take spanning tree T of G. An identifying path cover of T is also one for G! In the worst case, T has many leaves (up to $n-1$).

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This an improvement over IDP, where for a connected instance:

$$\lceil \log_2(n+1) \rceil \leq p^{\text{ID}}(G) \leq n$$

(see e.g. Foucaud, Naserasr, Parreau 2012 for the special case of identifying codes in digraphs).

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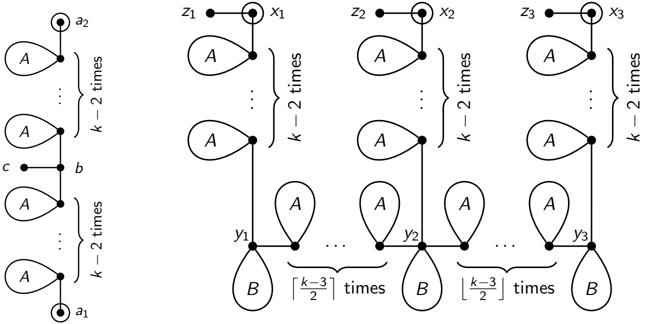
Proof: L-reduction from VERTEX COVER in cubic graphs. A and B are local sub-gadgets.



Open problems

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- Is there a polynomial-time algorithm for ID. PATH COVER in trees?

- What is the complexity of the general ID. PATH COVER problem?

- What are the graphs that admit a k-path identifying cover (i.e. all paths have **exactly** k vertices)?

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Thank you / Nandri / Shukriya / Merci!