Combinatorial and algorithmic aspects of identifying codes in graphs

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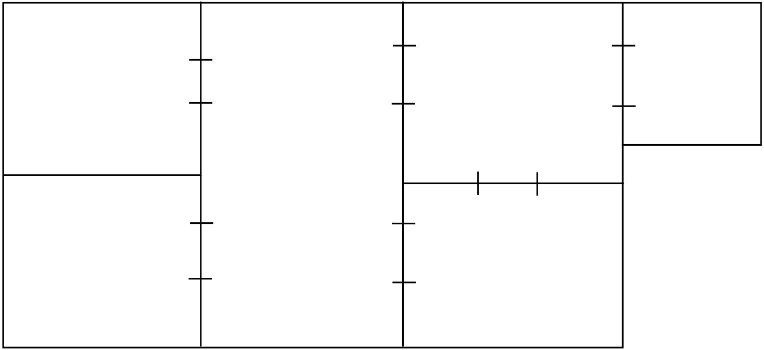
Bordeaux

December 10th, 2012

Identifying the rooms of a building



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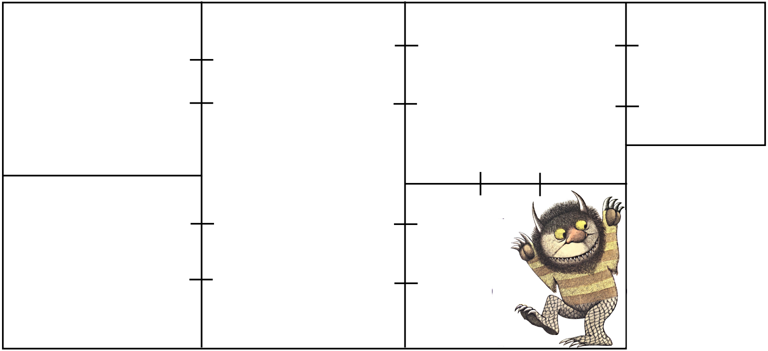

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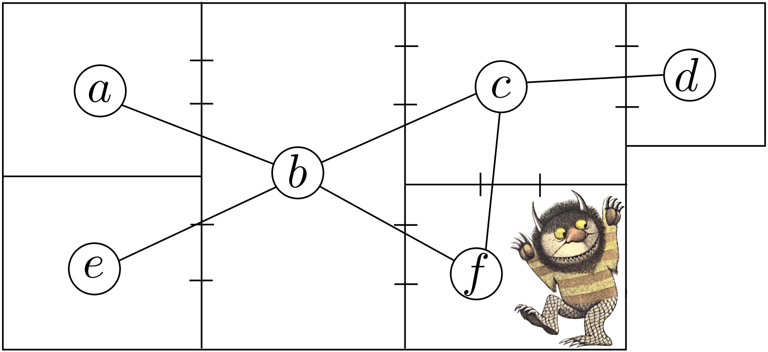

Graph $G = (V, E)$. V: vertices (rooms), $E \subseteq V \times V$: edges (doors)

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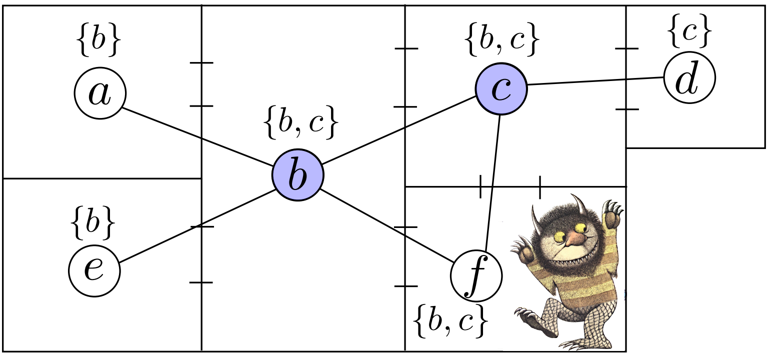

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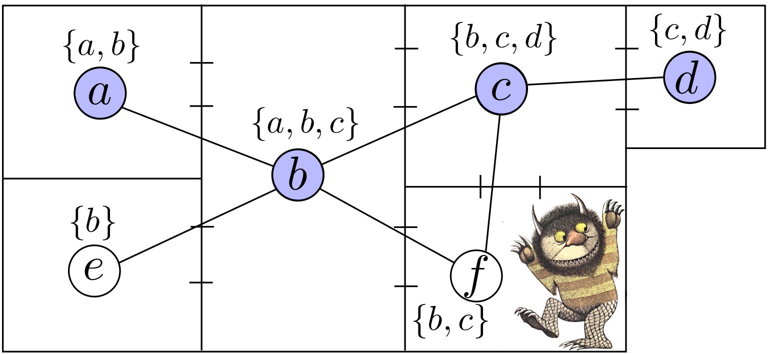

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Identifying codes

G: undirected graph
$N[u]$: set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that:

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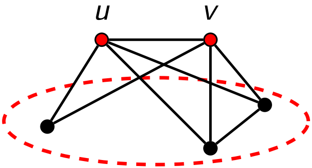
$\gamma^{\text{ID}}(G)$: minimum size of an identifying code in G

Identifiable graphs

Remark

Not all graphs have an identifying code!

Twins = pair u, v such that $N[u] = N[v]$.



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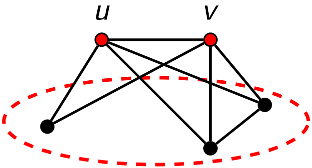

Proposition

A graph is **identifiable** if and only if it is **twin-free** (i.e. has no twins).

Bounds on $\gamma^{\text{ID}}(G)$

n: number of vertices

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

G identifiable graph on n vertices:

$$\lceil \log_2(n+1) \rceil \leq \gamma^{\text{ID}}(G)$$

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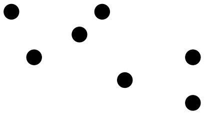
$\gamma^{\text{ID}}(G) = n \Leftrightarrow G$ has no edges



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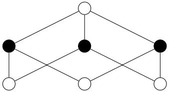

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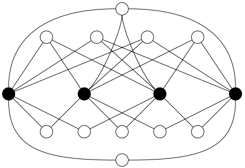

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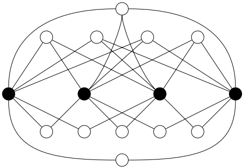

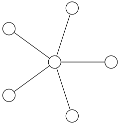

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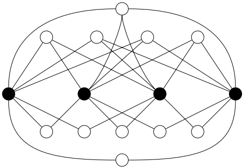

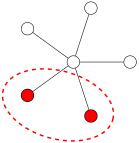

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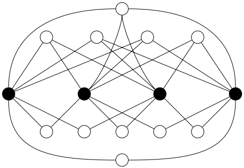

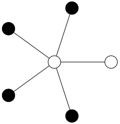

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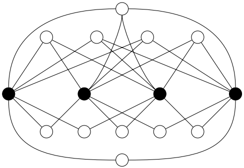

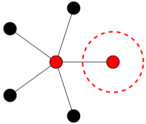

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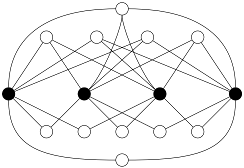

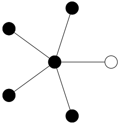

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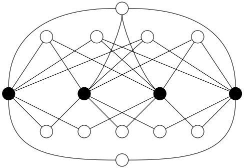

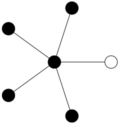

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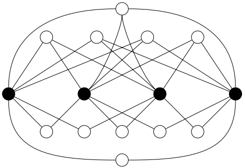

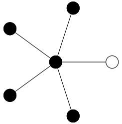

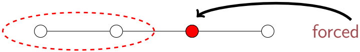

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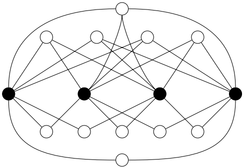

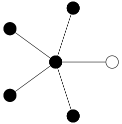

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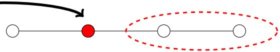

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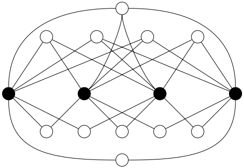

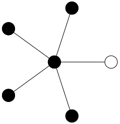

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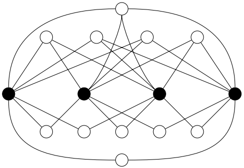
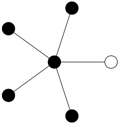




A question

Theorem (Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

G identifiable graph on n vertices with at least one edge:

$$\gamma^{\text{ID}}(G) \leq n - 1$$

Question

What are the graphs G with n vertices and $\gamma^{\text{ID}}(G) = n - 1$?

Part 1
Graphs with large identifying code number

Part 2
Identifying code number and maximum degree

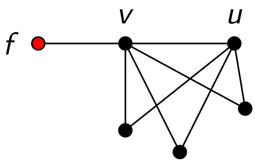
Part 3
Algorithmic hardness of the identifying code problem

Forced vertices

u, v such that $N[v] \ominus N[u] = \{f\}$:

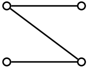
f belongs to **any identifying code**

$\rightarrow f$ forced by u, v.

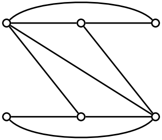


Graphs with many forced vertices

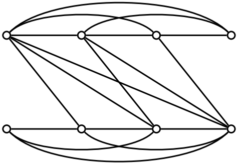
Special path powers: $A_k = P_{2k}^{k-1}$



$A_2 = P_4$



$A_3 = P_6^2$



$A_4 = P_8^3$

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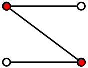
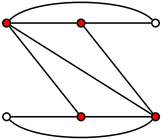
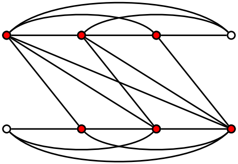


$A_2 = P_4$



$A_3 = P_6^2$



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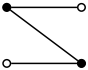

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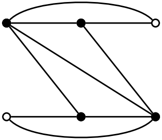

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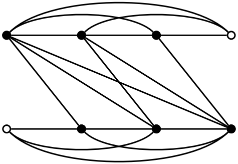

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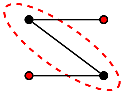

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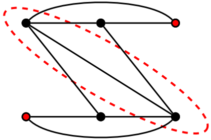

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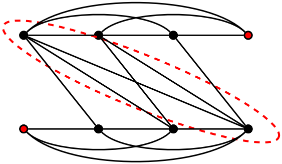

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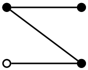

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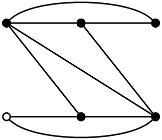

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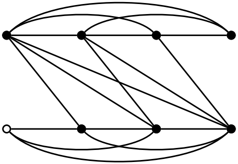
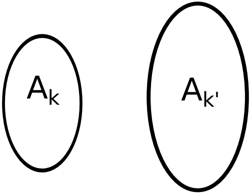


$A_4 = P_8^3$

Proposition

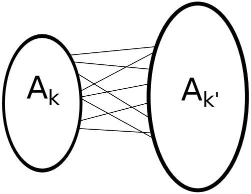
$$\gamma^{\text{ID}}(A_k) = n - 1$$

Constructions using joins



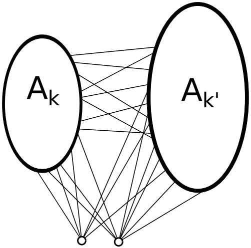
Two graphs A_k and $A_{k'}$

Constructions using joins



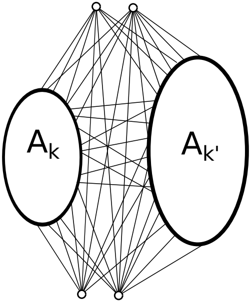
Join: add all edges between them

Constructions using joins



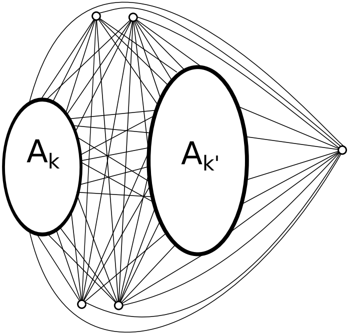
Join the new graph to two non-adjacent vertices ($\overline{K_2}$)

Constructions using joins



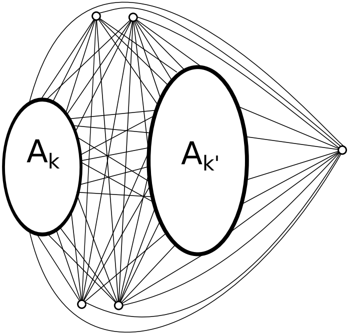
Join the new graph to two non-adjacent vertices, again

Constructions using joins



Finally, add a **universal vertex**

Constructions using joins



Finally, add a **universal vertex**

Proposition

At each step, the constructed graph has $\gamma^{\mathrm{ID}} = n - 1$

A characterization

(1) stars
(2) $A_k = P_{2k}^{k-1}$
(3) joins between 0 or more members of (2) and 0 or more copies of $\overline{K_2}$
(4) (2) or (3) with a universal vertex

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

G connected identifiable graph, n vertices:

$$\gamma^{\text{ID}}(G) = n - 1 \Leftrightarrow G \in (1), (2), (3) \text{ or } (4)$$

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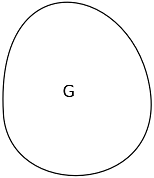

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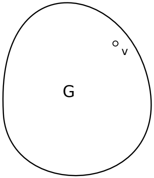

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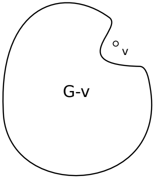

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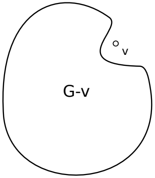

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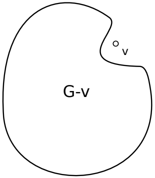

- Put v back $\Rightarrow$ **contradiction:** no counterexample exists!

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Observation

All these graphs have maximum degree $n-1$ or $n-2$

Part 1
Graphs with large identifying code number

Part 2
Identifying code number and maximum degree

Part 3
Algorithmic hardness of the identifying code problem

A lower bound using the maximum degree

maximum degree of G: maximum number of neighbours of a vertex in G

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

G identifiable graph, n vertices, maximum degree Δ:

$$\frac{2n}{\Delta+2} \leq \gamma^{\text{ID}}(G)$$

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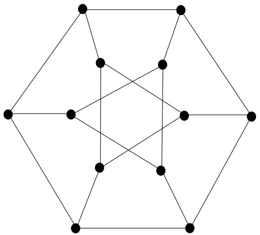
Theorem (F., Klasing, Kosowski, 2009)

Equality if and only if G can be constructed as follows:

- Take Δ-regular graph H



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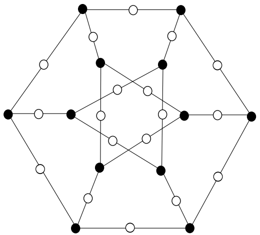

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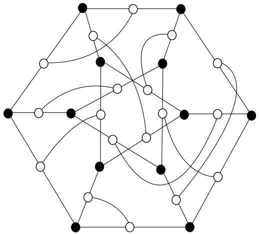
Theorem (F., Klasing, Kosowski, 2009)

Equality if and only if G can be constructed as follows:

- Take Δ-regular graph H
- Subdivide each edge once
- Possibly add some edges



The influence of the maximum degree

Question

What is a good **upper bound** on γ^{ID} using the maximum degree?

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Proposition

There exist graphs with n vertices, max. degree Δ and $\gamma^{\text{ID}}(G) = n - \frac{n}{\Delta}$.

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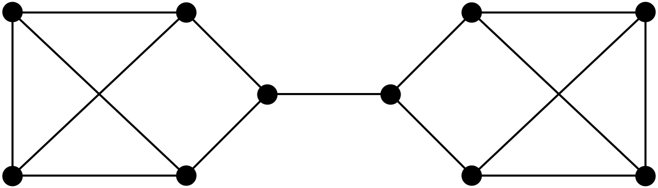

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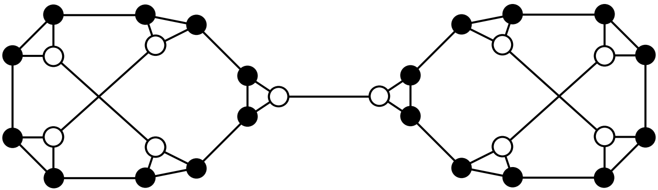

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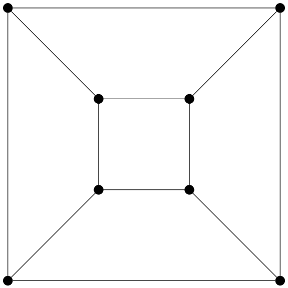

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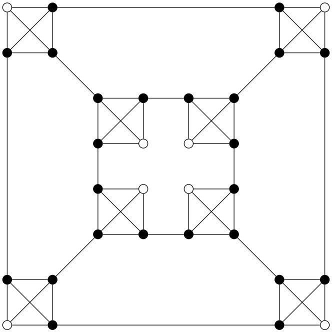

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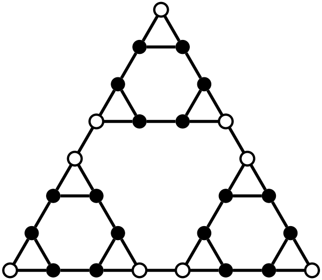
There exist graphs with n vertices, max. degree Δ and $\gamma^{\text{ID}}(G) = n - \frac{n}{\Delta}$.

Also: Sierpiński graphs

(Gravier, Kovše, Mollard, Moncel, Parreau, 2011)



A conjecture

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

G connected identifiable graph, n vertices, max. degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + c \text{ for some constant } c$$

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Question

Can we prove that $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta)}$?

Triangle-free graphs

Theorem (F., Klasing, Kosowski, Raspaud, 2009)

G identifiable triangle-free graph, n vertices, max. degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta + \frac{3\Delta}{\ln \Delta - 1}} = n - \frac{n}{\Delta(1 + o_\Delta(1))}$$

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Proof idea: Constructive.

Triangle-free graphs have **large** independent sets

(see e.g. Shearer: $\alpha(G) \geq \frac{\ln\Delta}{\Delta} n$)

$\rightarrow$ Locally modify such an independent set:

its complement is a "small" id. code.

Triangle-free graphs

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Remark

Same technique applies to families of triangle-free graphs with large independent sets.
$\rightarrow$ bipartite graphs: $\alpha(G) \geq \frac{n}{2} \Rightarrow \gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+9}$

The probabilistic method

1. Define a suitable probability space
2. Select some object from this space using a random process
   $\rightarrow$ select random set
3. Prove that with nonzero probability, certain "good" conditions hold
   $\rightarrow$ selected set is small id. code
4. Conclusion: there always exists a "good" object
   $\rightarrow$ small id. code

Upper bounds for $\gamma^{\text{ID}}(G)$

Theorem (F., Perarnau, 2011)

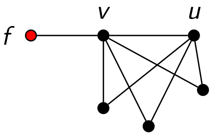
G identifiable graph, n vertices, maximum degree Δ, no isolated vertices:

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{105\Delta}$$

Notation

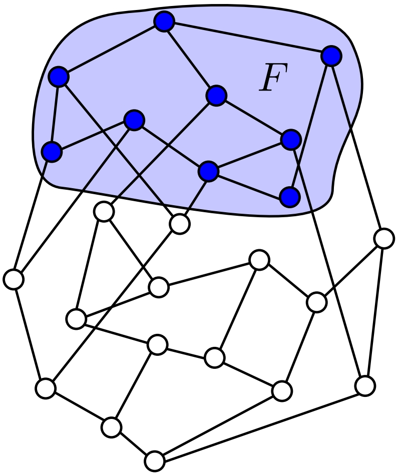
$NF(G)$: proportion of non forced vertices of G

$$NF(G) = \frac{\#\text{non forced vertices in G}}{\#\text{vertices in G}}$$



Proof

1) F: forced vertices. Select "big" random set S from $V(G) \setminus F$



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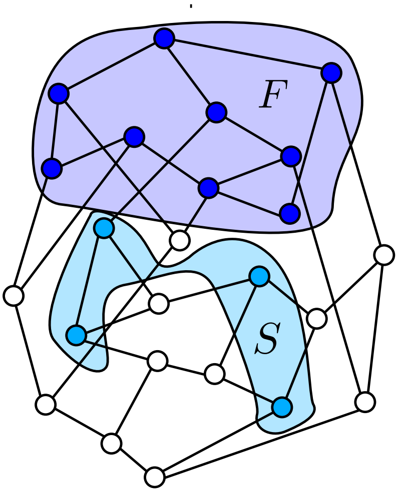


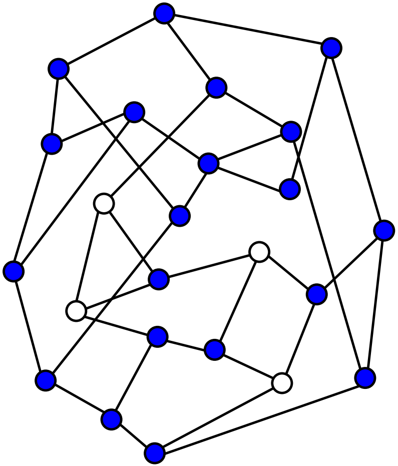
for each vertex v from $V(G) \setminus F$
$\rightarrow v \in S$ with probability p.

Want: $p = \Theta\left(\frac{1}{\Delta}\right)$

$\mathbb{E}(|S|) = p \cdot nNF(G) = \frac{nNF(G)}{\Theta(\Delta)}$

Proof

1) F: forced vertices. Select "big" random set S from $V(G) \setminus F$
Goal: $\mathcal{C} = V(G) \setminus S$ small identifying code



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$$\mathbb{E}(|\mathcal{C}|) = n - \frac{nNF(G)}{\Theta(\Delta)}$$

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1) F: forced vertices. Select "big" random set S from $V(G) \setminus F$
Goal: $\mathcal{C} = V(G) \setminus S$ small identifying code

2) Use Lovász Local Lemma: define bad events
$\rightarrow$ if none occurs, $V(G) \setminus S$ is an identifying code

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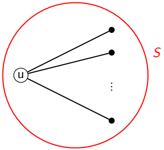
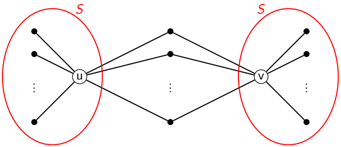
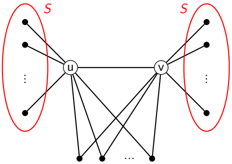


each vertex u
$\rightarrow$ 1 event for domination





each pair u, v at dist. at most 2
$\rightarrow$ 1 event for separation

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3) Compute probability $Pr_E(p, \Delta)$ of each bad event E

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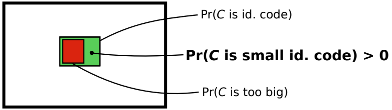
5) Solution: Chernoff bound $\rightarrow$ w.h.p. $|S|$ is close to expected size



Bounding the number of forced vertices

$NF(G)$: proportion of non forced vertices of G

Theorem (F., Perarnau, 2011)

G identifiable graph on n vertices having maximum degree Δ and no isolated vertices:

$$\gamma^{\text{ID}}(G) \leq n - \frac{n \cdot NF(G)^2}{105\Delta}$$

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What can be said about $NF(G)$?

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What can be said about $NF(G)$?

$$G \text{ regular} \Rightarrow NF(G) = 1$$

Corollary

G regular: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{105\Delta}$

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G: identifiable graph having no isolated vertices. Let x be a vertex of G. There exists a non forced vertex in $N[x]$.
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Corollary

$$\frac{1}{\Delta+1} \leq NF(G) \leq 1 \text{ and } \gamma^{\text{ID}}(G) \leq n - \frac{n}{105(\Delta+1)^3}$$

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clique number of G: max. size of a complete subgraph in G

Proposition (F., Perarnau, 2011)

Let G be a graph of clique number at most k. There exists a (huge) function c such that:

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$$\gamma^{\text{ID}}(G) \leq n - \frac{n}{105c(k)^2\Delta} = n - \frac{n}{\Theta(\Delta)}$$

Summary

Conjecture (F., Klasing, Kosowski, Raspaud, 2009)

G connected identifiable graph, n vertices, max. degree Δ. Then

$$\gamma^{\text{ID}}(G) \leq n - \tfrac{n}{\Delta} + c \text{ for some constant } c$$

Theorem

in general: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^3)}$

triangle-free: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta(1+o_\Delta(1))}$

bipartite: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta+9}$

regular: $\gamma^{\text{ID}}(G) \leq n - \frac{n}{105\Delta}$

clique number k: $n - \frac{n}{105c(k)^2\Delta}$

Part 1
Graphs with large identifying code number

Part 2
Identifying code number and maximum degree

Part 3
Algorithmic hardness of the identifying code problem

Computational problems

Definition - Computational problem

- Set of **inputs**
- Given an input, **task** to be solved by an algorithm

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Polynomial-time for:

- trees (Auger, 2010)
- bounded treewidth (Moncel, 2005)

NP-complete for:

- planar subcubic graphs (Auger et al. 2010)
- planar bipartite unit disk graphs (Müller, Sereni, 2009)
- etc.

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$O(\log(n))$-approximation algorithm (n: order of input graph)

No $o(\log(n))$-approximation algorithm, unless $P = NP$
(Berger-Wolf et al. 2006 / Suomela, 2007)

Question

Question

What is the complexity of IDCODE and MIN IDCODE for various standard graph classes?
$\rightarrow$ restriction of the input set

Polynomial-time reductions

Definition - Reduction

Two computational problems A, B
Polynomial-time computable function $r : A \rightarrow B$ such that:

B efficiently solvable $\Rightarrow$ A efficiently solvable.

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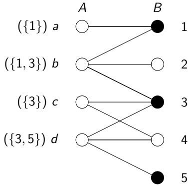
Proposition

If A is hard, then B is hard.

Discriminating code

Definition - Discriminating code of a bipartite graph $G(A, B)$

Subset $\mathcal{C} \subseteq B$ which dominates and separates vertices of A.



example: $\mathcal{C} = \{1, 3, 5\}$

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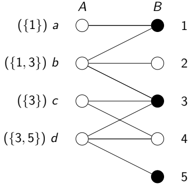


example: $\mathcal{C} = \{1, 3, 5\}$

Definition - MIN DISCR CODE

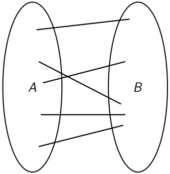
INPUT: bipartite graph G
TASK: find smallest possible discriminating code of G

No $o(\log(n))$-approximation algorithm, unless $P = NP$

(De Bontridder et al. 2003)

New and non-approximability reductions

Reduction: MIN DISCR CODE to MIN IDCODE for bipartite graphs.



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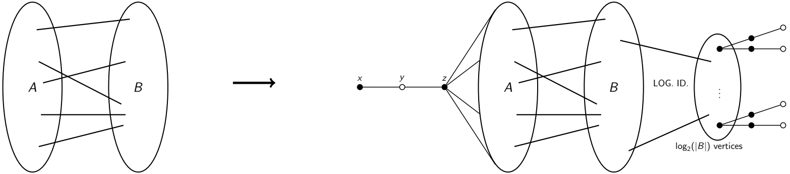

Theorem (F., 2012)

- $G(A, B)$ has discr. code of size k if and only if G' has an identifying code of size $k + 3\lceil \log_2(|B| + 1)\rceil + 2$. Constructive.
- If MIN IDCODE has an α-approximation algorithm, then MIN DISCR. CODE has a 4α-approximation algorithm.

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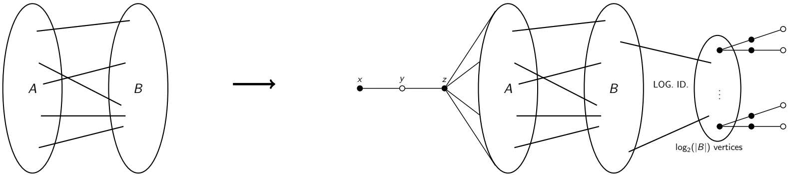

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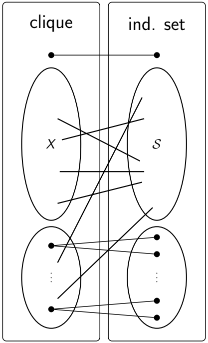
Corollary

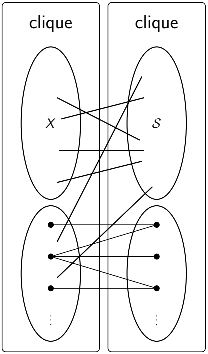
NP-hard to approximate MIN IDCODE within $o(\log(n))$
$\rightarrow$ even for **bipartite** graphs.

New non-approximability reductions

Similar reductions for split graphs and co-bipartite graphs.



split graphs



co-bipartite graphs

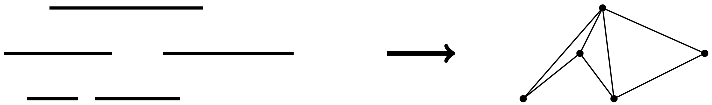
Theorem (F., 2012)

It is NP-hard to approximate MIN IDCODE within $o(\log(n))$
$\rightarrow$ even for **split** graphs and for **co-bipartite** graphs.

Interval graphs

Definition - Interval graph

Intersection graph of intervals of the real line.



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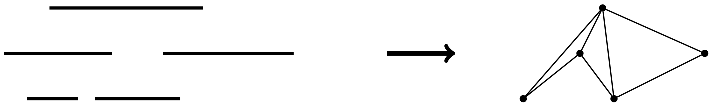


Remark

Many problems are efficiently solvable for interval graphs.
Example: DOMINATING SET

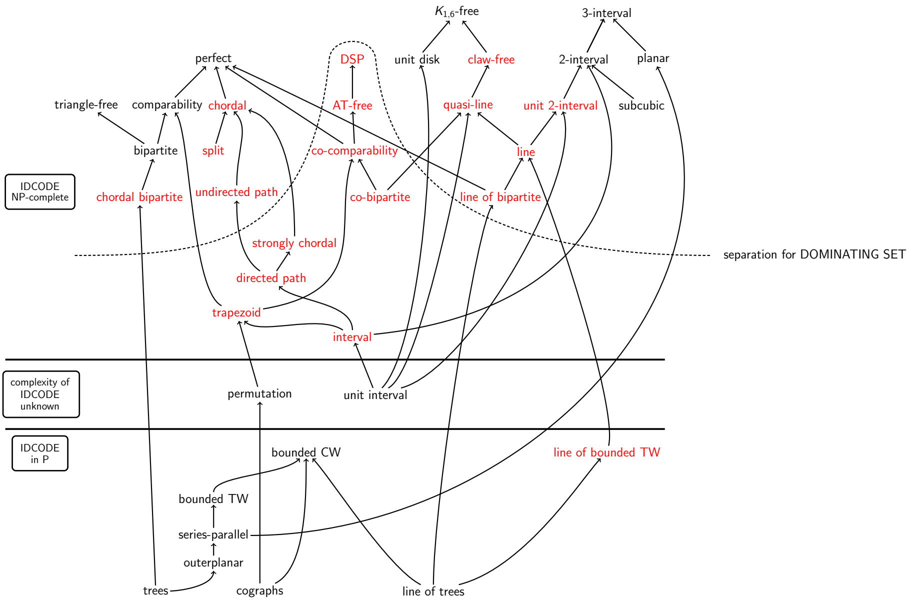
IDCODE for interval graphs

Theorem (F., Kosowski, Mertzios, Naserasr, Parreau, Valicov, 2012)

IDCODE is NP-complete for interval graphs. Reduction from 3-DIMENSIONAL MATCHING.

Main idea:
an interval can separate two pairs of intervals that are **far away** without affecting what lies in between.

Complexity of IDCODE for various graph classes



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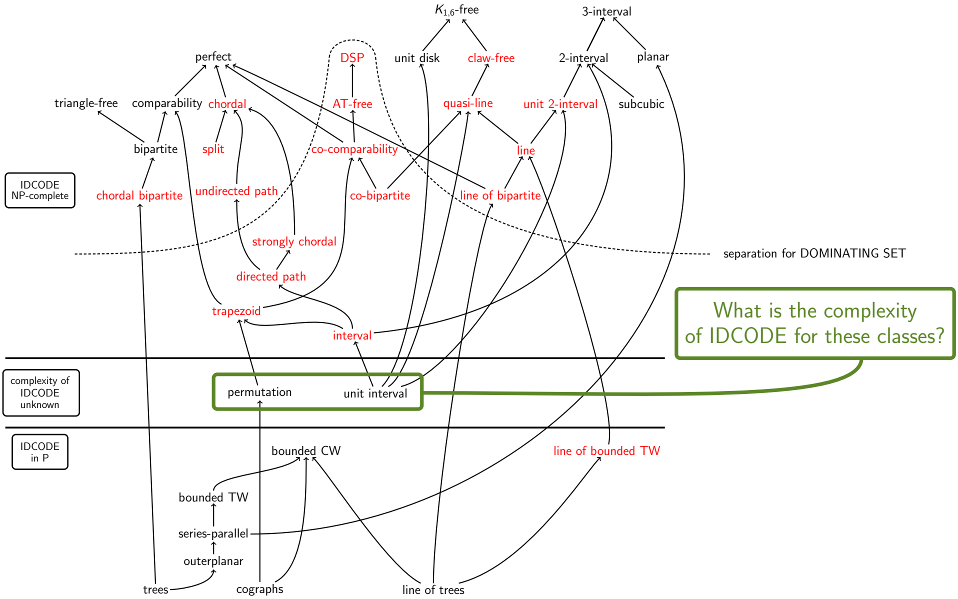

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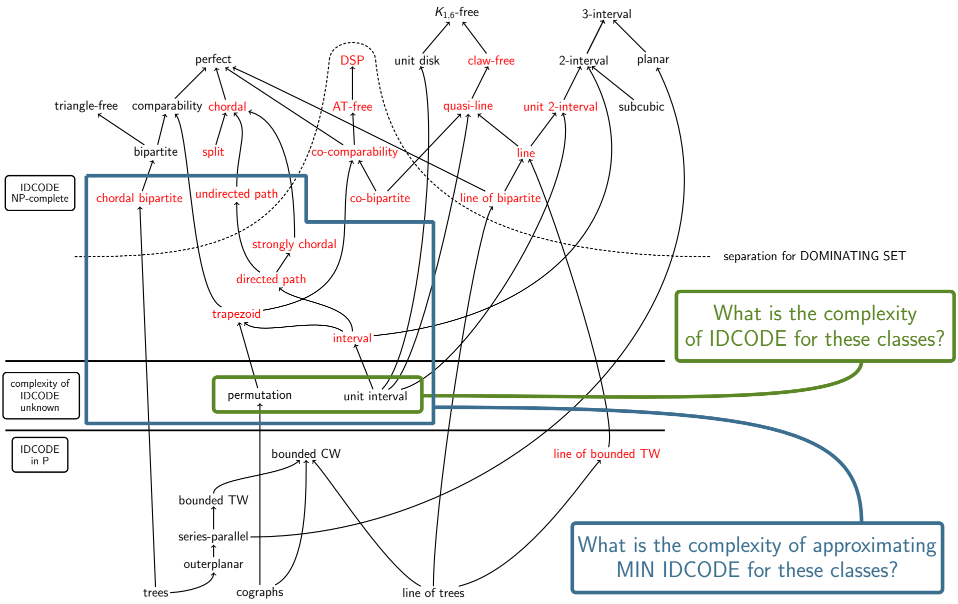

Perspectives

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  What are tight bounds on γ^{ID} for specific graph classes?
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Other perspectives:

- **Parameterized complexity** of IDCODE
- **Fractional** identifying codes