Identification problems in graphs

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joint work with:

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and
George Mertzios, Reza Naserasr, Aline Parreau, Petru Valicov

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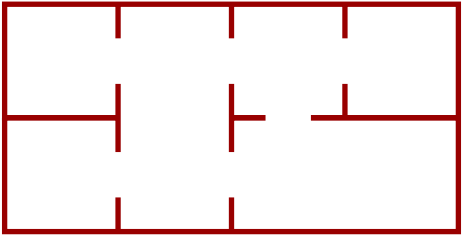
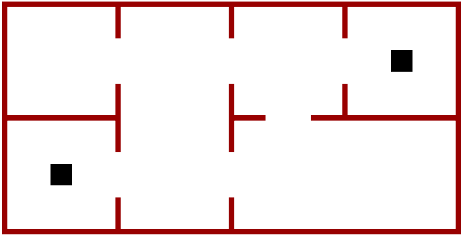
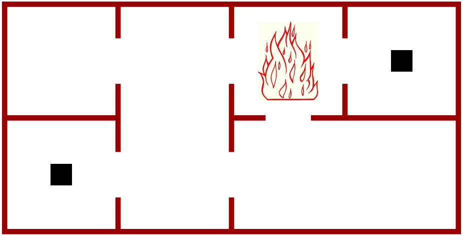
Part I: location-domination

Fire detection in a building





- Detector can detect fire in its room and its neighborhood (through a door).



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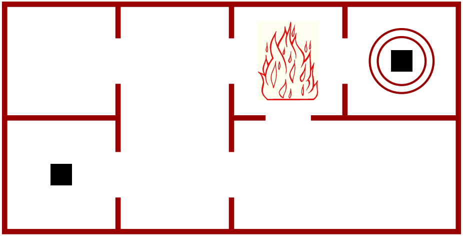

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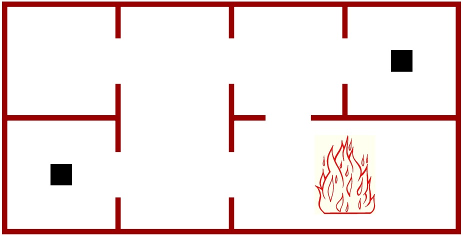

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- Each room must contain a detector or have one in an adjacent room.

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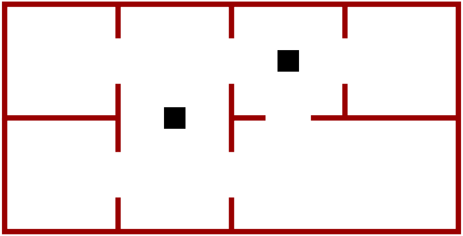

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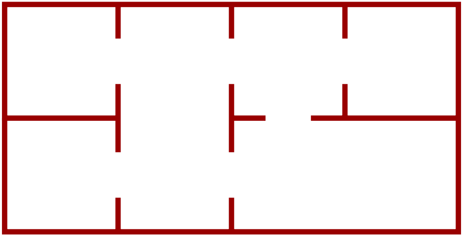
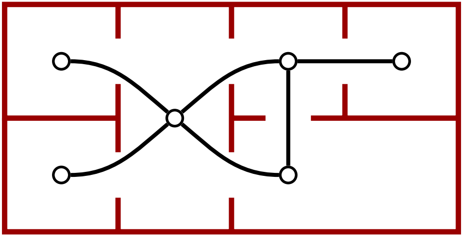


- Graph $G = (V, E)$. Vertices: rooms.
  Edges: between any two rooms connected by a door

Modelization with a graph



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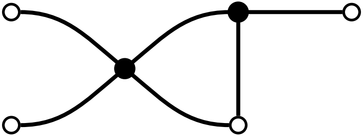

- Graph $G = (V, E)$. Vertices: rooms.
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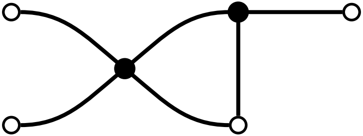
- Set of detectors = dominating set $D \subseteq V$: $\forall u \in V, N[u] \cap D \neq \emptyset$



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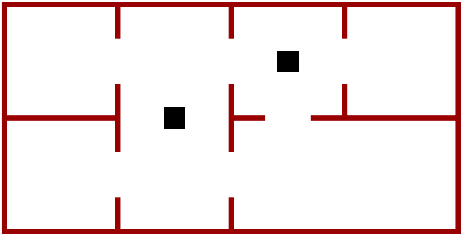
- Domination number $\gamma(G)$: smallest size of a dominating set of G

Back to the building



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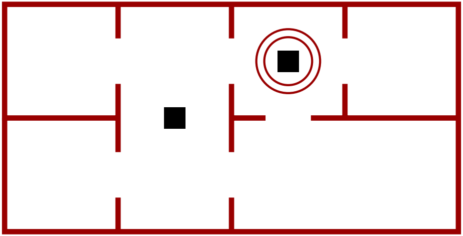
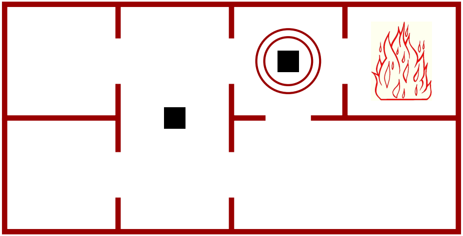


Where is the fire ?



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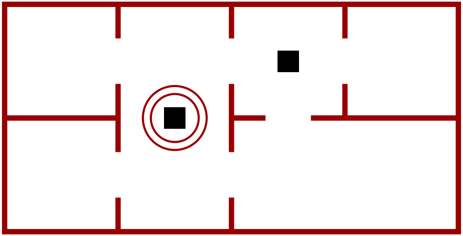

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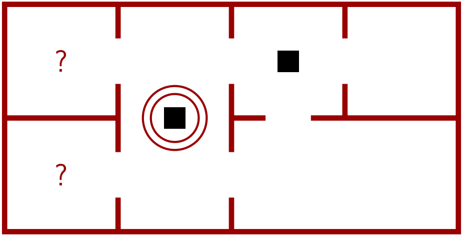
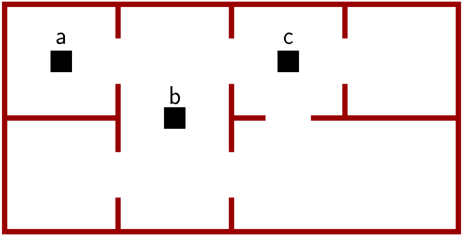


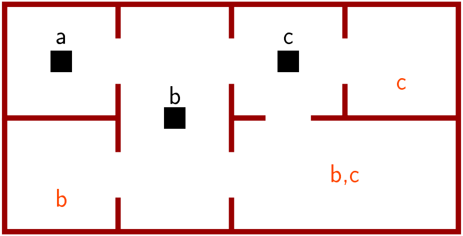
Where is the fire ?

To locate the fire, we need more detectors.

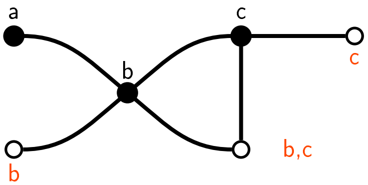
Locating the fire



Locating the fire



In each room with no detector, set of dominating detectors is distinct.



Slater, 1980's: Locating-dominating set D = subset of vertices of $G = (V, E)$ which is:

- dominating : $\forall u \in V, N[u] \cap D \neq \emptyset$,
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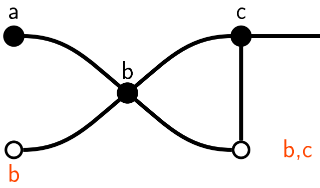

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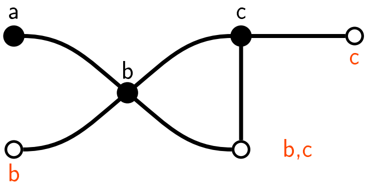

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Remark: $\gamma(G) \leq \gamma_L(G)$

Domination number: $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Examples: paths

Domination number: $\gamma(P_n) = \lceil \frac{n}{3} \rceil$

Location-domination number: $\gamma_L(P_n) = \lceil \frac{2n}{5} \rceil$

Upper bounds on the location-domination number

Upper bounds

Theorem (Domination bound — Ore, 1960's)

G graph of order n, no isolated vertices. Then $\gamma(G) \leq \frac{n}{2}$.

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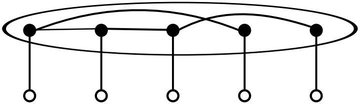
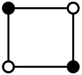
Tight examples:  

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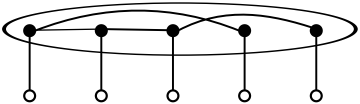
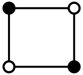
Tight examples:  

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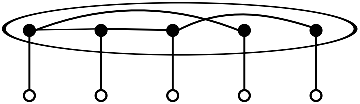
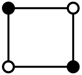
Tight examples:  

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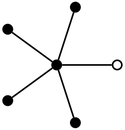
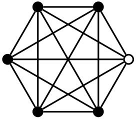
Tight examples:  

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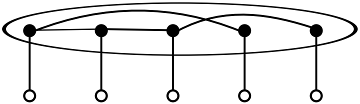

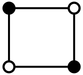

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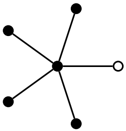

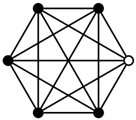

Remark: tight examples contain many twin-vertices!!

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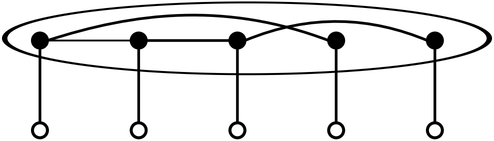
If true, tight: 1. domination-extremal graphs



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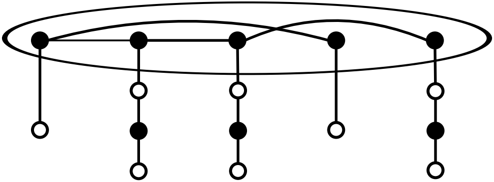

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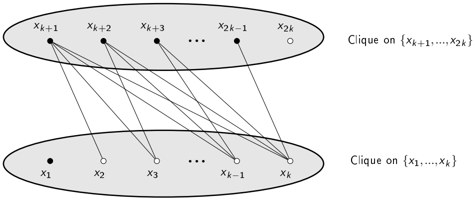
If true, tight: 3. a family with domination number 2



Upper bound - a conjecture (2)

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Theorem (Garijo, González & Márquez, 2014+)

Conjecture true if G has no 4-cycles, or if G is bipartite.

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Theorem (F., Henning, Löwenstein, Sasse, 2014+)

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Remark: Nontrivial proofs using very different techniques!
$\rightarrow$ Conjecture seems difficult.

Lower bounds on the location-domination number

Theorem (Slater, 1980's)

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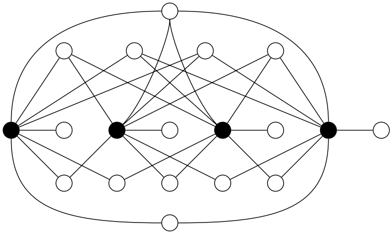
Tight example ($k = 4$):



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G tree of order n, $\gamma_L(G) = k$. Then $n \leq 3k - 1$, i.e. $\gamma_L(G) \geq \frac{n+1}{3}$.

Theorem (Rall & Slater, 1980's)

G planar graph, order n, $\gamma_L(G) = k$. Then $n \leq 7k - 10$, i.e. $\gamma_L(G) \geq \frac{n+10}{7}$.

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Tight examples:

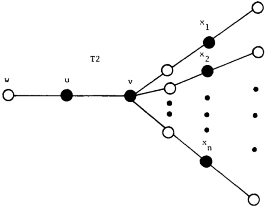


FIG. 2. Tree T2

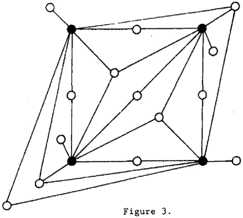
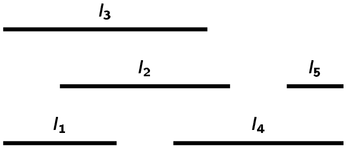
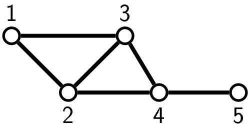


Figure 3.

Definition - Interval graph

Intersection graph of intervals of the real line.





Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2014+)

G interval graph of order n, $\gamma_L(G) = k$.

Then $n \leq \frac{k(k+3)}{2}$, i.e. $\gamma_L(G) = \Omega(\sqrt{n})$.

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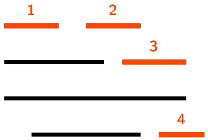

- Locating-dominating D of size k.
- Define zones using the right points of intervals in D.

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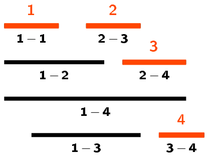

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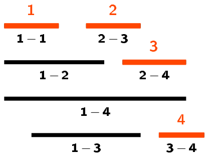

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$\rightarrow n \leq \sum_{i=1}^{k}(k-i)+k=\frac{k(k+3)}{2}$.

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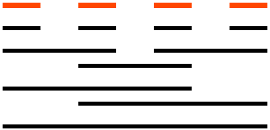
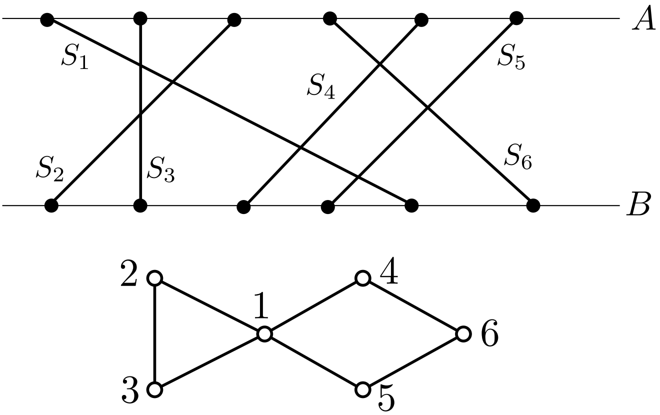
Tight:



Definition - Permutation graph

Given two parallel lines A and B:
intersection graph of segments joining A and B.



Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2014+)

G permutation graph of order n, $\gamma_L(G) = k$.
Then $n \leq k^2 + k - 2$, i.e. $\gamma_L(G) = \Omega(\sqrt{n})$.

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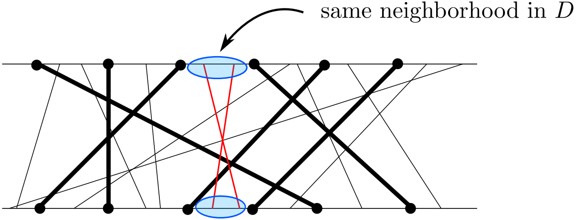

- Locating-sominating set D of size k: $k+1$ "top zones" and $k+1$ "bottom zones"
- Only one segment in $V \setminus D$ for one pair of zones

  $\rightarrow n \leq (k+1)^2 + k$
- Careful counting for the precise bound

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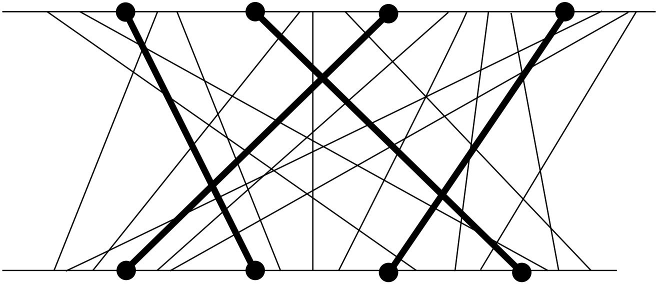
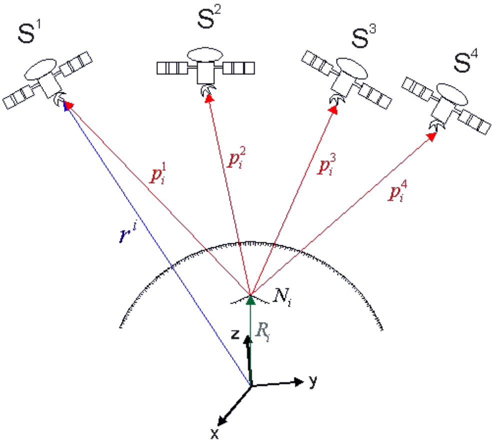
Tight:



Part II: metric dimension

GPS: need to know the exact position of 4 satellites + distance to them



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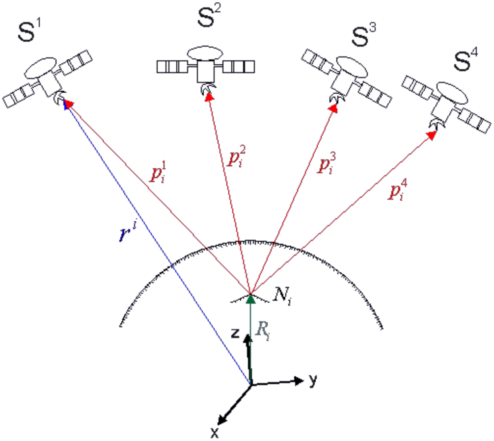

Question

Does the "GPS" approach also work in undirected unweighted graphs?

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)

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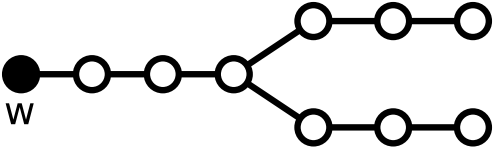

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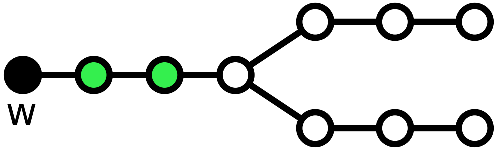

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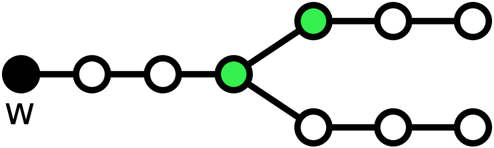

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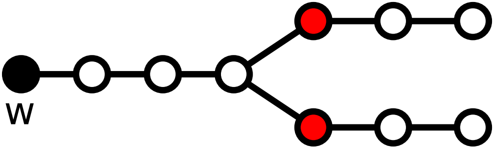

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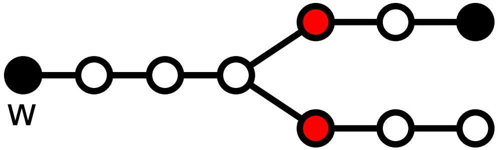

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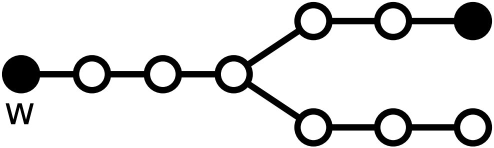

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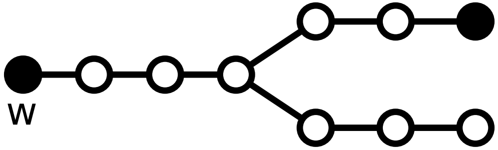

$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Remark

- Any locating-dominating set is a resolving set, hence $MD(G) \leq \gamma_L(G)$.
- A locating-dominating set can be seen as a "distance-1-resolving set".

Bounds with diameter

Example of path: no bound $n \leq f(MD(G))$ possible.

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→ Proofs are similar as for locating-dominating sets.

→ Bounds are tight (up to constant factors).

Perspectives

Future work:

- Solve the conjecture: $\gamma_L(G) \leq \frac{n}{2}$ if G twin-free?
- Investigate bounds for other "geometric" graphs, for MD and γ_L

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THANKS FOR YOUR ATTENTION