Tight algorithmic double-exponential bounds for treewidth

metric-based and identification-based graph problems

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joint works with:

Esther Galby, Liana Khazaliya, Shaohua Li, Fionn Mc Inerney, Roohani Sharma, Prafullkumar Tale (ICALP 2024)

Dipayan Chakraborty, Diptapriyo Majumdar, Prafullkumar Tale (ISAAC 2024)

LIMOS

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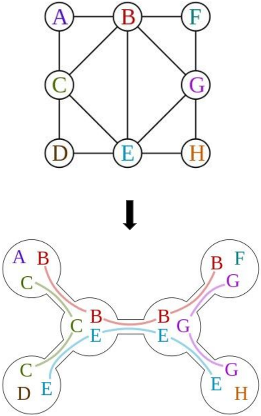
Treewidth

A tree decomposition of a graph $G = (V, E)$ is a tree T with nodes (bags) $X_1, \ldots, X_n$, where each X_i is a subset of V, satisfying

1. $X_1 \cup X_2 \cup \cdots \cup X_n = V$;
2. for all $v \in V$, the bags containing v form a **connected** subtree of T;
3. for all $uv \in E$, there exists a bag containing both u and v.



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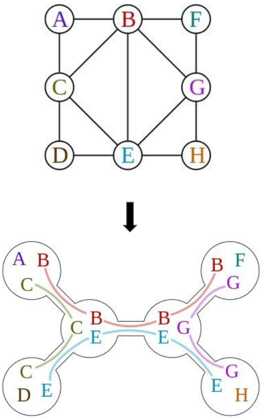
The width of a tree decomposition is the size of the **largest** bag minus one.



Treewidth

The treewidth $\mathtt{tw}(G)$ of G is the **minimum** width over all tree decompositions of G.

Treewidth: the **King** of Structural Parameters

Fixed parameter tractable (FPT)

Given a problem Π with input $\mathcal{I}$ and a parameter k, Π is FPT parameterized by k if it can be solved in time $f(k) \cdot |\mathcal{I}|^{O(1)}$, where f is a computable function.

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Many NP-hard problems are FPT parameterized by treewidth via dynamic programming on the tree decomposition.

In particular, graph problems expressible in Monadic Second-Order (MSO) logic are FPT parameterized by the treewidth plus the length of the MSO formula [Courcelle, 1990].

For a given signature (e.g graphs) τ, MSO has:

- element-variables $(x, y, z, \dots)$ and set-variables $(X, Y, Z, \dots)$
- relations $=$ (equation), $x \in X$ (membership), relations from τ
- quantifiers $\exists$, $\forall$ and operators $\wedge$, $\vee$, $\neg$

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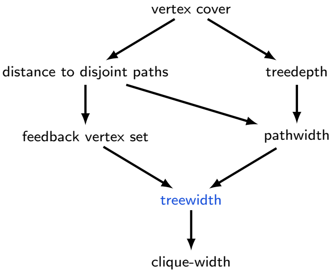

ETH-based conditional lower bounds on $f(\texttt{tw})$ for FPT algorithms

Exponential Time Hypothesis (ETH) [Impagliazzo, Paturi, 1990]

Roughly, n-variable 3-SAT cannot be solved in time $2^{o(n)}$.

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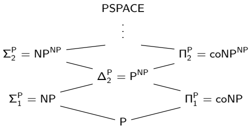
Rarer results: Unless the ETH fails,

- QSAT (PSPACE-complete) with k alternations admits a lower bound of a tower of exponents of height k in the tw of the primal graph [Fichte, Hecher, Pfandler, 2020];
- k-Choosability (Π_2^p-complete) and k-Choosability Deletion (Σ_3^p-complete) admit double- and triple-exponential lower bounds in tw, resp. [Marx, Mitsou, 2016];
- ∃∀-CSP (Σ_2^p-complete) admits a double-exponential lower bound in the vertex cover number [Lampis, Mitsou, 2017].



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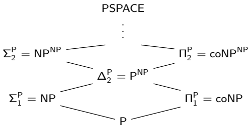


Common theme: problems are hard for complexity classes higher than NP.

Our results (ICALP 2024)

We prove the **first** (conditional) double-exponential lower bounds in the treewidth and vertex cover number for NP-complete problems.

We develop a **technique** and use it to prove such lower bounds for 3 NP-complete problems:

Theorem [F., Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2024]

METRIC DIMENSION and GEODETIC SET:

- can be solved in $2^{\mathtt{diam}^{O(\mathtt{tw})}} \cdot n^{O(1)}$ time
- no $2^{f(\mathtt{diam})^{o(\mathtt{tw})}} \cdot n^{O(1)}$ time algorithm assuming the ETH

Theorem [F., Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2024]

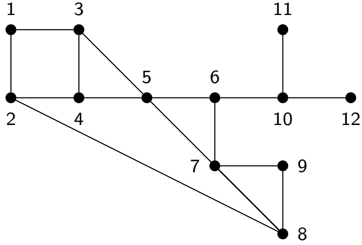
STRONG METRIC DIMENSION:

- can be solved in $2^{2^{O(\mathtt{vc})}} \cdot n^{O(1)}$ time
- no $2^{2^{o(\mathtt{vc})}} \cdot n^{O(1)}$ time algorithm assuming the ETH

Metric dimension

Metric dimension of a graph $G = (V, E)$ [Slater '75 + Harary, Melter '76]

$S \subseteq V$ is a resolving set of G if $\forall u, v \in V$, $\exists z \in S$ with $d(z, u) \neq d(z, v)$. The **minimum size** of a resolving set of G is the metric dimension of G.



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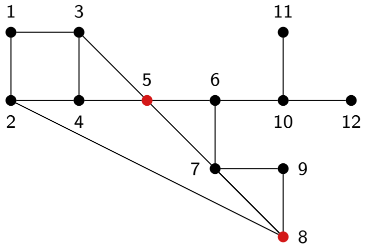

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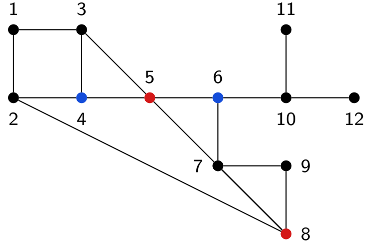

Vertices 4 and 6 are **not** resolved by 5 nor 8.

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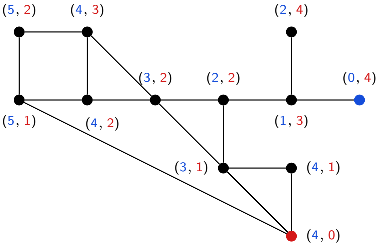

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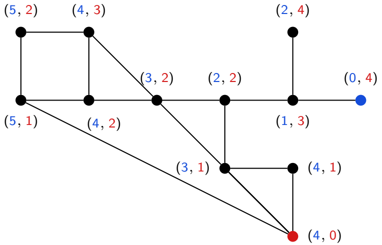

METRIC DIMENSION

Input: an undirected graph $G = (V, E)$ and an integer $k \geq 1$
Question: Is $MD(G) \leq k$?

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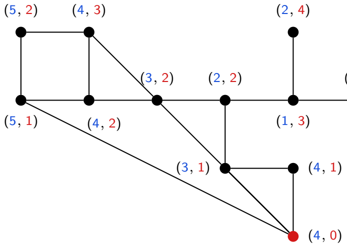


Polynomial-time

Trees	[Slater '75]
Cographs	[Epstein et al '15]
Chain	[Fernau et al '15]
Block	[Hoffmann et al '16]
Outerplanar	[Diaz et al '17]
Bipartite distance-hered.	[Moscarini '22]

NP-complete

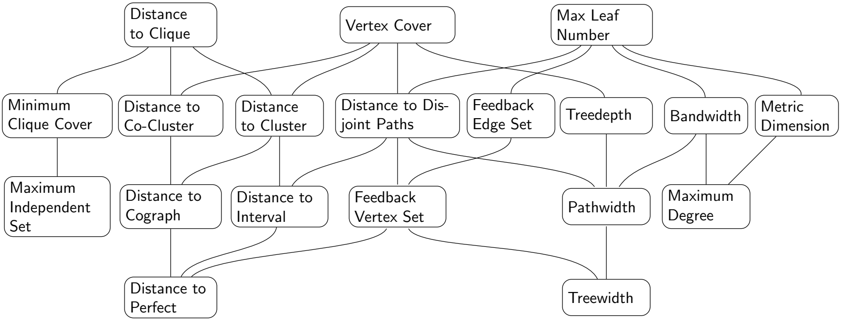
Arbitrary	[Garey, Johnson '79]
Split	[Epstein et al '15]
Bipartite	[Epstein et al '15]
Co-bipartite	[Epstein et al '15]
Planar	[Diaz et al '17]
Permutation	[Foucaud et al '17]
Interval	[Foucaud et al '17]

Parameterized complexity of METRIC DIMENSION

- FPT ($f(k) \cdot n^{O(1)}$-time algorithm)
- XP ($n^{f(k)}$-time algorithm)
- W[1]-hard (not FPT unless FPT=W[1])
- para-NP-hard (not XP unless P=NP)

n: size of input

k: size of parameter



A lower parameter is connected to a higher one if it is upper bounded by a function of the higher one

Parameterized complexity of Metric Dimension

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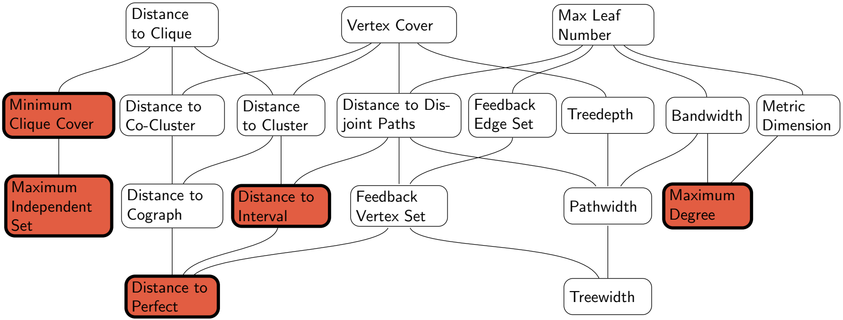

From NP-hardness results on previous slide

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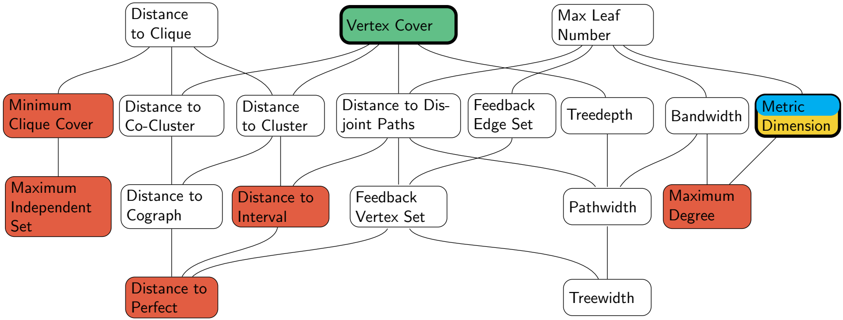

W[2]-hard parameterised by solution size [Hartung, Nichterlein '13]

FPT parameterised by Vertex Cover

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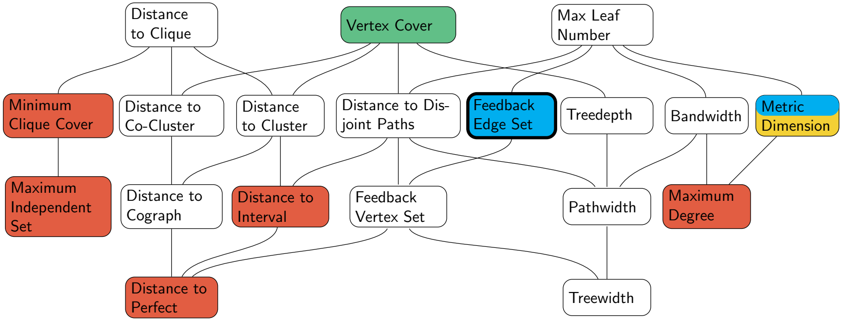

[Epstein, Levin, Woeginger '12]

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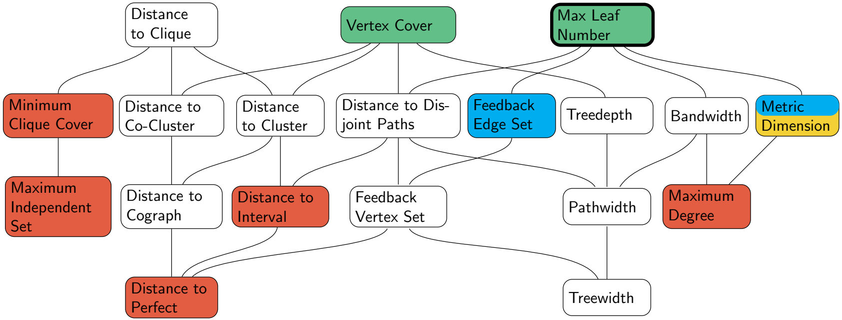

[Eppstein '15]

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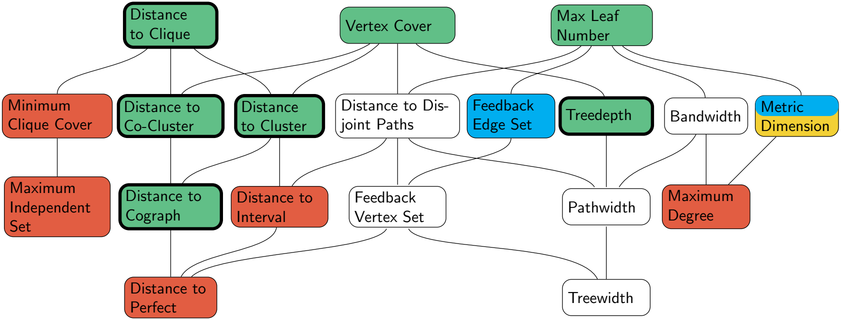

FPT parameterised by clique-width + diameter [Gima, Hanaka, Giyomi, Kobayashi, Otachi '21]

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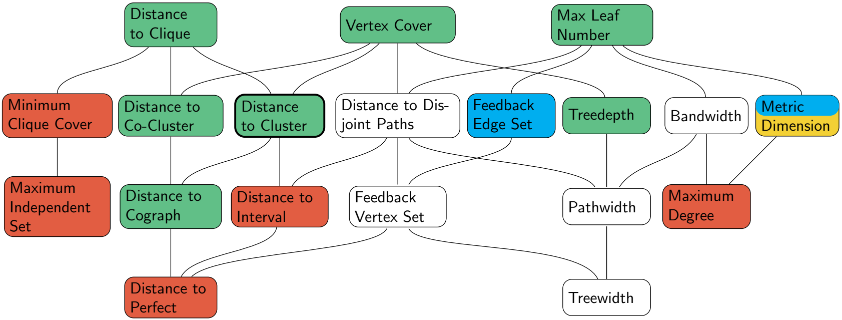

FPT parameterised by treelength + max degree [Belmonte, Fomin, Golovach, Ramanujan '17]

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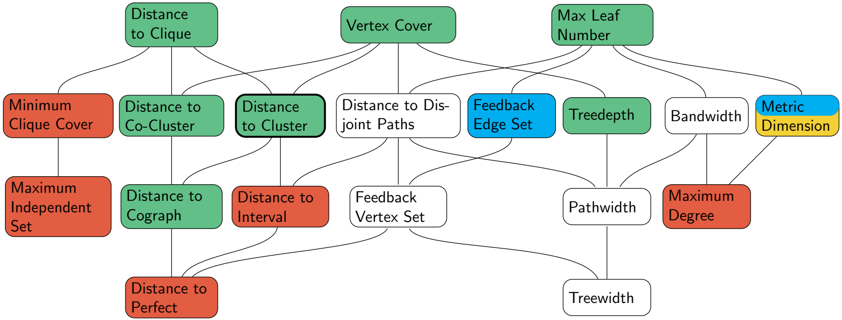

Q1: Complexity parameterised by **treewidth**? [Eppstein '15], [Belmonte et al '17], [Díaz, Potonen, Serna, van Leeuwen '17]

Q2: Complexity parameterised by **Feedback Vertex Set**? [Hartung, Nichterlein '13]

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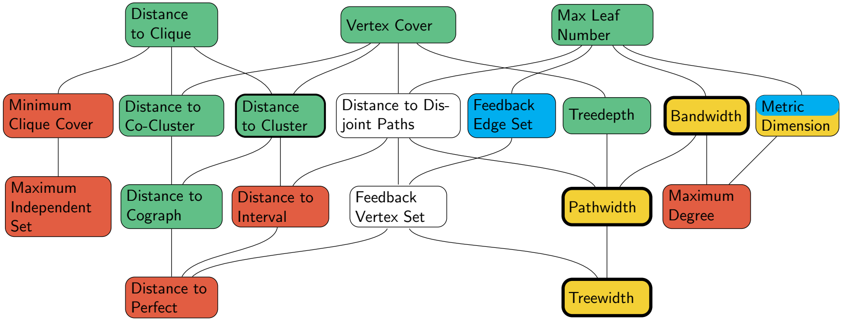

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Q1 answered first by [Bonnet, Purohit '21].

Parameterized complexity of Metric Dimension

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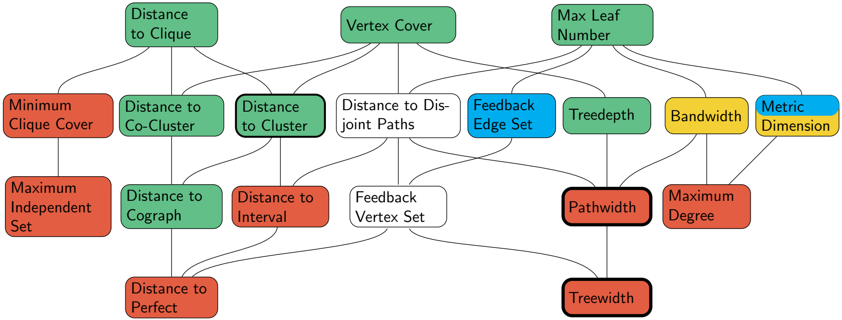

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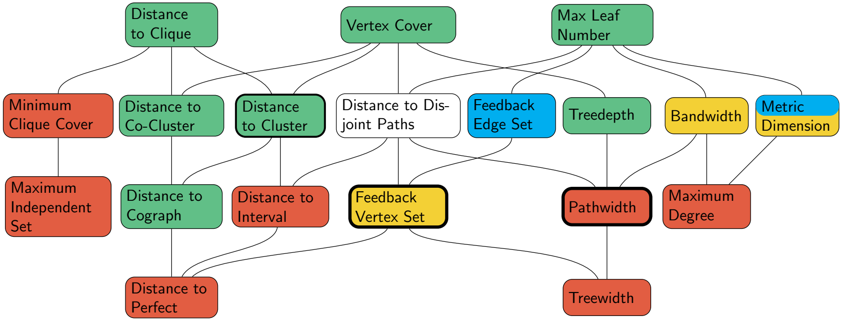

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Q2: Complexity parameterised by **Feedback Vertex Set**? [Hartung, Nichterlein '13]

Q2 answered for the combined parameter Feedback Vertex Set + Pathwidth

[Galby, Khazaliya, Mc Inerney, Sharma, Tale '23]

Hardness for Metric Dimension

Theorem [F., Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2024]

Unless the ETH fails, METRIC DIMENSION does not admit algorithms running in time $2^{f(\mathtt{diam})^{o(\mathtt{tw})}} \cdot n^{O(1)}$, for any computable function f.

Reduction.

3-PARTITIONED 3-SAT: φ $\rightarrow$ METRIC DIMENSION: (G, k)

$\mathtt{tw}(G) = \log(n)$

$\mathtt{diam}(G) = \text{const}$

3-Partitioned 3-SAT

3-PARTITIONED 3-SAT [Lampis, Melissinos, Vasilakis, 2023]

Input: 3-CNF formula φ with a partition of its variables into 3 disjoint sets X^α, X^β, and X^γ such that $|X^\alpha| = |X^\beta| = |X^\gamma| = n$ and each clause contains at most one variable from each of X^α, X^β, and X^γ

Question: Is ϕ satisfiable?

Theorem [Lampis, Melissinos, Vasilakis, 2023]

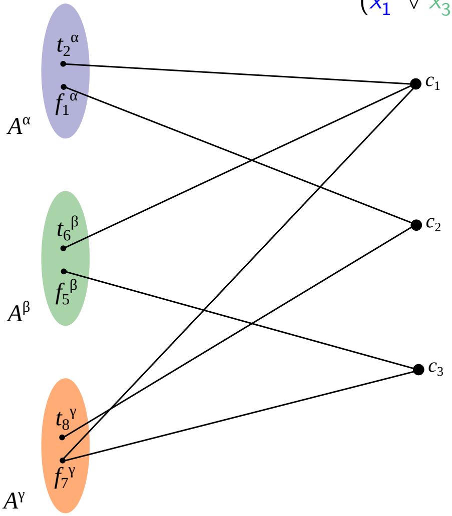
3-PARTITIONED 3-SAT has no $2^{o(n)}$ time algorithm assuming the ETH

Encode SAT via small separators

$$(x_1^{\alpha} \vee x_3^{\beta} \vee \overline{x_4}^{\gamma}) \wedge (\overline{x_1}^{\alpha} \vee x_4^{\gamma}) \wedge (\overline{x_3}^{\beta} \vee \overline{x_4}^{\gamma})$$

t_{2i}^{α} represents x_i^{α}

f_{2i-1}^{α} represents $\overline{x}_i^{\alpha}$



$(x_1^{\alpha} \vee x_3^{\beta} \vee \overline{x_4}^{\gamma})$

$(\overline{x_1}^{\alpha} \vee x_4^{\gamma})$

$(\overline{x_3}^{\beta} \vee \overline{x_4}^{\gamma})$

Set-Representation Gadget

$$(x_1^\alpha \vee x_3^\beta \vee \overline{x_4}^\gamma) \wedge (\overline{x_1}^\alpha \vee x_4^\gamma) \wedge (\overline{x_3}^\beta \vee \overline{x_4}^\gamma)$$

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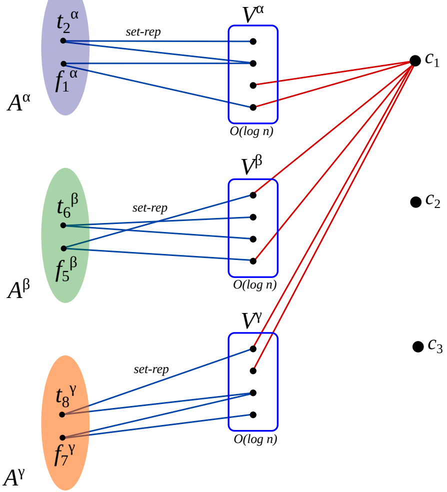

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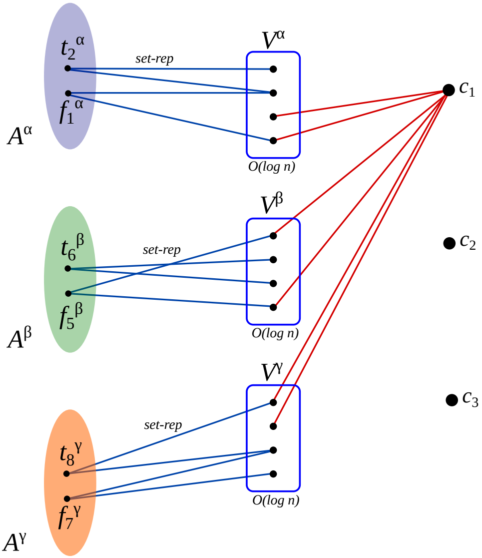


$(x_1^\alpha \vee x_3^\beta \vee \overline{x_4}^\gamma)$

F_p: collection of subsets of $\{1, \ldots, 2p\}$ of size p.

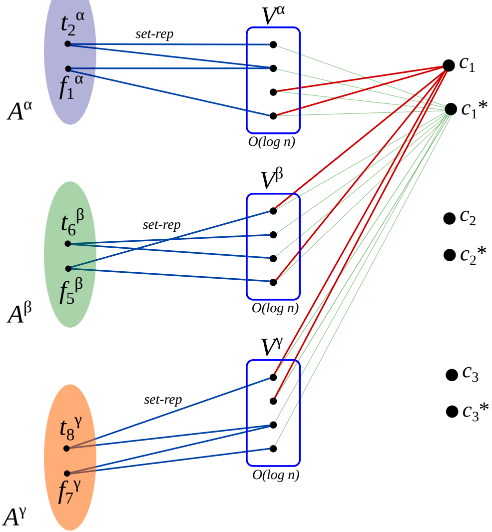
No set in F_p is contained in another set in F_p (Sperner family).

There exists $p = O(\log n)$ s.t. $\binom{2p}{p} \geq 2n$. We define a 1-to-1 function

set-rep : $\{1, \ldots, 2n\} \rightarrow F_p$.

t_2^α is the **only** vertex in A^α that **does not** share a **common neighbour** with c_1

Set-Representation Gadget



$(x_1^\alpha \vee x_3^\beta \vee \overline{x_4}^\gamma)$

F_p: collection of subsets of $\{1, \ldots, 2p\}$ of size p.

No set in F_p is contained in another set in F_p (Sperner family).

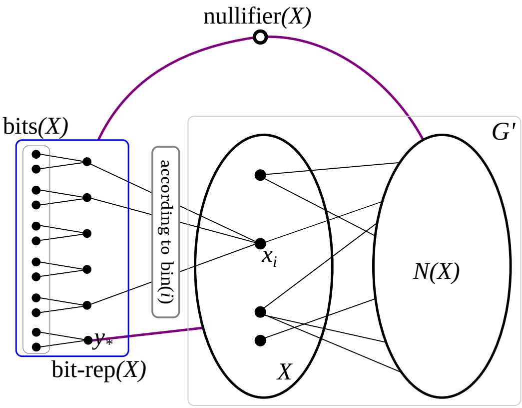
There exists $p = O(\log n)$ s.t. $\binom{2p}{p} \geq 2n$. We define a 1-to-1 function

set-rep : $\{1, \ldots, 2n\} \to F_p$.

t_2^α is the **only** vertex in A^α that **does not** share a **common neighbour** with c_1

Bit-representation Gadget

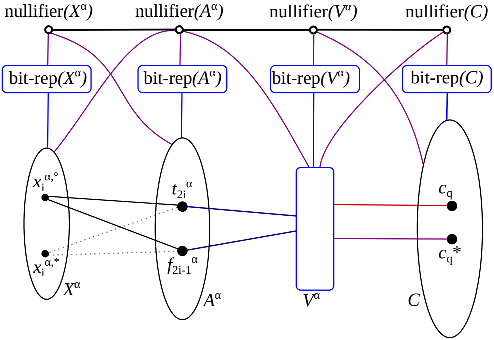
Observation. For any twins $u, v \in V(G)$ and any resolving set S of G, $S \cap \{u, v\} \neq \emptyset$.



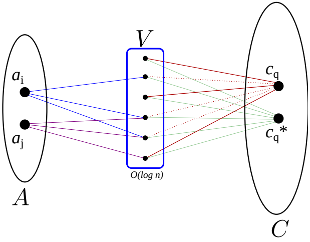
Purple edges represent all possible edges

- For any resolving set S, $|S \cap \text{bits}(X)| \geq \log(|X|) + 1$
- $|S \cap \text{bits}(X)|$ distinguishes each vertex in $X \cup \text{bit-rep}(X)$ from every other vertex in G
- nullifier(X) guarantees that the rest part of $V(G)$ is not affected by the gadget

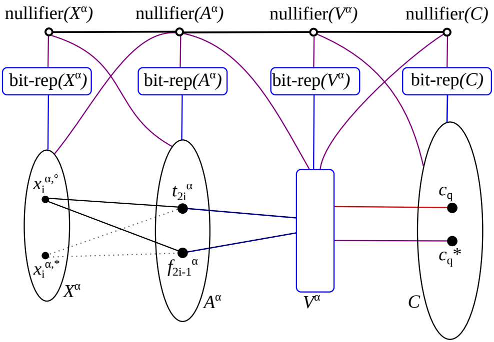
Lower bound for Metric Dimension parameterized by tw



Purple — all possible edges
Blue — set-rep
Red — complementary to blue



Lower bound for Metric Dimension parameterized by tw



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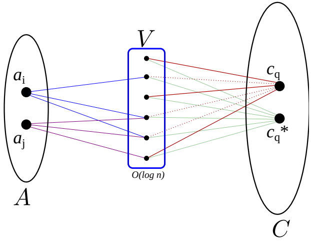
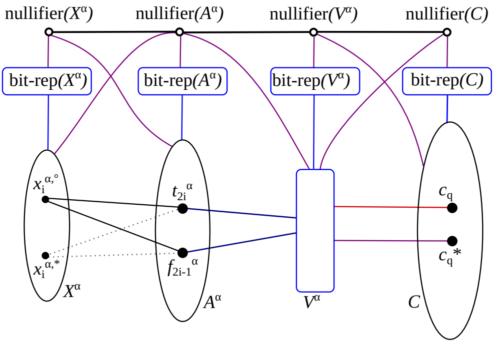


Note: $\texttt{tw}(G) = \log(n)$
$\texttt{diam}(G) = \text{const}$

Lower bound for Metric Dimension parameterized by tw



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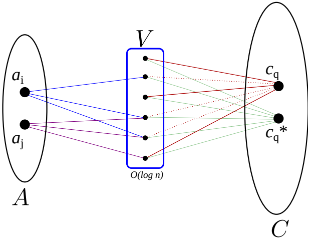

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$\texttt{diam}(G) = \text{const}$

Theorem [F., Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2024]

Metric Dimension has no $2^{f(\texttt{diam})^{o(\texttt{tw})}} \cdot n^{O(1)}$ time algorithm assuming the ETH

GEODETIC SET
Input: An undirected simple graph G
Question: Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any vertex $u \in V(G)$, there are two vertices $s_1, s_2 \in S$ such that a shortest path from s_1 to s_2 contains u?

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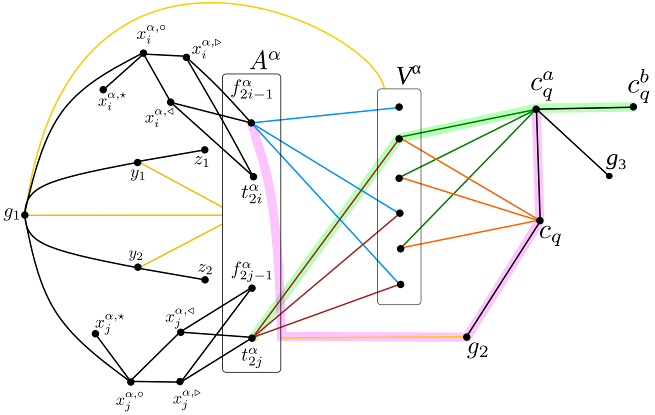


Strong Metric Dimension

Strong Metric Dimension
Input: An undirected simple graph G
Question: Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that either u lies on some shortest path between v and w, or v lies on some shortest path between u and w?

Theorem [F., Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2024]

Strong Metric Dimension has no $2^{2^{o(\mathrm{vc})}} \cdot n^{O(1)}$ time algorithm, assuming the ETH

Match with the Algorithms

Theorem [F., Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2024]

METRIC DIMENSION and GEODETIC SET:

- can be solved in $2^{\mathtt{diam}^{O(\mathtt{tw})}} \cdot n^{O(1)}$ time
- no $2^{f(\mathtt{diam})^{o(\mathtt{tw})}} \cdot n^{O(1)}$ time algorithm assuming the ETH

Theorem [F., Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2024]

STRONG METRIC DIMENSION:

- can be solved in $2^{2^{O(\mathtt{vc})}} \cdot n^{O(1)}$ time
- no $2^{2^{o(\mathtt{vc})}} \cdot n^{O(1)}$ time algorithm assuming the ETH

Other applications of the technique

Theorem [Chalopin, Chepoi, Mc Inerney, Ratel, COLT 2024]

Positive Non-Clashing Teaching Dimension for Balls in Graphs

- no $2^{2^{o(\mathtt{vc})}} \cdot n^{O(1)}$ time algorithm assuming the ETH

Theorem [Chakraborty, F., Majumdar, Tale, ISAAC 2024]

Locating-Dominating Set and Test Cover have

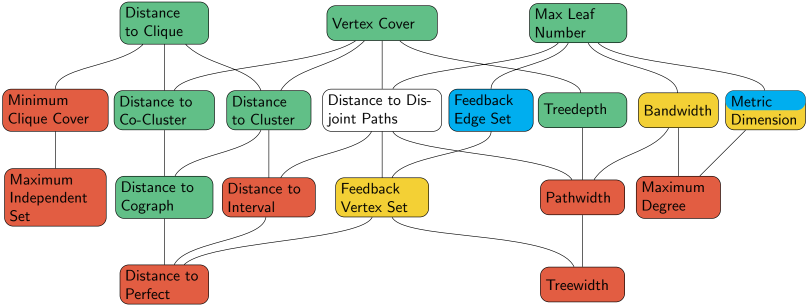
- no $2^{2^{o(\mathtt{tw})}} \cdot n^{O(1)}$ time algorithm assuming the ETH

Open questions for Metric Dimension

- FPT ($f(k) \cdot n^{O(1)}$-time algorithm)
- XP ($n^{f(k)}$-time algorithm)
- W[1]-hard (not FPT unless FPT=W[1])
- para-NP-hard (not XP unless P=NP)

n: size of input

k: size of parameter



- Poly-time for unit interval graphs / bipartite permutation graphs?
- XP or para-NP-hard parameterised by Feedback Vertex Set?
- W[1]-hard or FPT parameterised by Feedback Edge Set?
- W[1]-hard or FPT for Distance to Disjoint Paths?
- W[1]-hard or FPT for Feedback Vertex Set + solution size?

Open questions for Geodetic Set

- FPT ($f(k) \cdot n^{O(1)}$-time algorithm)
- XP ($n^{f(k)}$-time algorithm)
- W[1]-hard (not FPT unless FPT=W[1])
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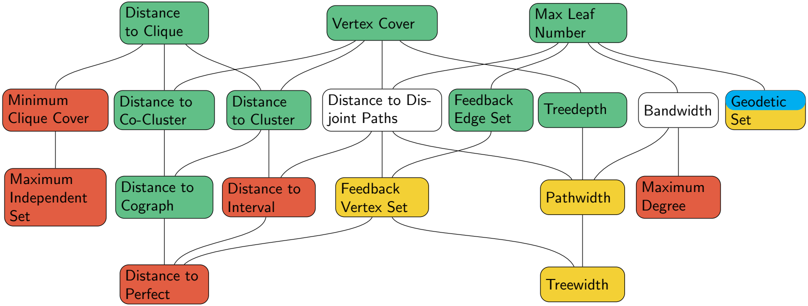

- XP or para-NP-hard parameterised by Treewidth / Pathwidth / FVS / Bandwidth?
- W[1]-hard or FPT parameterised by Bandwidth?
- W[1]-hard or FPT for Distance to Disjoint Paths?