# The complexity of homomorphisms of signed graphs

Florent Foucaud
(U. of Johannesburg + U. Paris-Dauphine)

joint work (in progress) with:

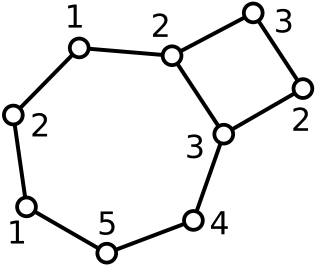
Richard Brewster (Thompson Rivers U., Kamloops)
Pavol Hell (Simon Fraser U., Vancouver)
Reza Naserasr (U. Paris-Sud, Orsay)

June 15th 2014

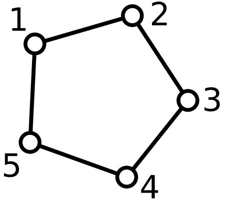
# Graph homomorphisms

**Definition** - Graph homomorphism of $G$ to $H$

**Mapping** from $V(G)$ to $V(H)$ which **preserves adjacency**.
If it exists, we note $G \to H$.



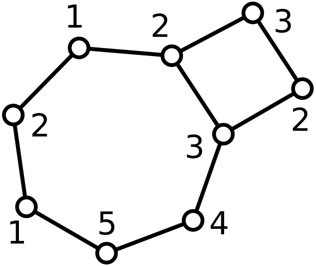

Target graph: $H = C_5$



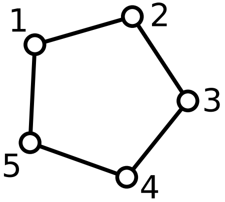
# Graph homomorphisms

**Definition** - Graph homomorphism of $G$ to $H$

**Mapping** from $V(G)$ to $V(H)$ which **preserves adjacency**.
If it exists, we note $G \to H$.




Target graph: $H = C_5$



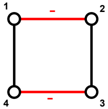

Homomorphisms generalize **proper vertex-colourings:**

$$G \to K_k \iff G \text{ is } k\text{-colourable}$$

# Signatures and signed graphs

**Signature Σ of graph G:** assignment of $+$ or $-$ sign to each edge of $G$.

$\Sigma$: set of $-$ edges.

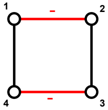



$\Sigma = \{12, 34\}$

# Signatures and signed graphs

**Signature $\mathbf{\Sigma}$ of graph G:** assignment of $+$ or $-$ sign to each edge of $G$.

$\Sigma$: set of $-$ edges.

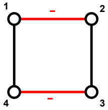



$\Sigma = \{12, 34\}$

**2-edge-coloured graph $(\mathbf{G}, \mathbf{\Sigma})$:** Graph $G$ with signature $\Sigma$.

**Signature $\Sigma$ of graph G:** assignment of $+$ or $-$ sign to each edge of $G$.

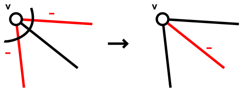
$\Sigma$: set of $-$ edges.




$\Sigma = \{12, 34\}$

**2-edge-coloured graph $(\mathbf{G}, \boldsymbol{\Sigma})$:** Graph $G$ with signature $\Sigma$.

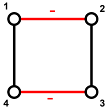
**Re-signing operation at v:** switch sign of each edge incident to $v$



# Signatures and signed graphs

**Signature $\Sigma$ of graph G:** assignment of $+$ or $-$ sign to each edge of $G$.

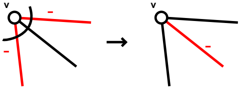
$\Sigma$: set of $-$ edges.



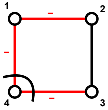

$\Sigma = \{12, 34\}$

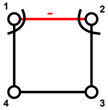
**2-edge-coloured graph $(\mathbf{G}, \boldsymbol{\Sigma})$:** Graph $G$ with signature $\Sigma$.

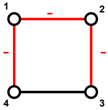
**Re-signing operation at v:** switch sign of each edge incident to $v$




$\Sigma$, $\Sigma'$ **equivalent** $(\Sigma \equiv \Sigma')$ if one can be obtained from the other via **re-signings**.

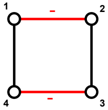


**Signature $\boldsymbol{\Sigma}$ of graph G:** assignment of $+$ or $-$ sign to each edge of $G$.

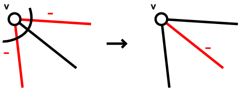
$\Sigma$: set of $-$ edges.




$\Sigma = \{12, 34\}$

**2-edge-coloured graph $(\mathbf{G}, \boldsymbol{\Sigma})$:** Graph $G$ with signature $\Sigma$.

**Re-signing operation at v:** switch sign of each edge incident to $v$

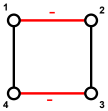



$\Sigma$, $\Sigma'$ **equivalent** ($\Sigma \equiv \Sigma'$) if one can be obtained from the other via **re-signings**.

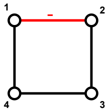
**Signed graph $[\mathbf{G}, \boldsymbol{\Sigma}]$:** $G$ with equivalence class $\mathcal{C}$ of signatures ($\Sigma \in \mathcal{C}$).

# Balance and unbalance

**Unbalanced cycle:** Cycle with an odd number of negative edges.



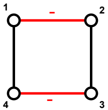

balanced $C_4$



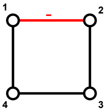

unbalanced $C_4$: $UC_4$

# Balance and unbalance

**Unbalanced cycle:** Cycle with an odd number of negative edges.




balanced $C_4$

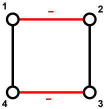



unbalanced $C_4$: $UC_4$

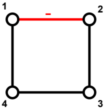
**Remark:** Re-signing **preserves the balance** of a cycle.

# Balance and unbalance

**Unbalanced cycle:** Cycle with an odd number of negative edges.




balanced $C_4$




unbalanced $C_4$: $UC_4$

**Remark:** Re-signing **preserves the balance** of a cycle.

**Theorem** (Zaslavsky, 1982)

Two signatures are **equivalent** if and only if they induce the **same set of unbalanced cycles**.

# Loops and digons

**Loops** are invariant under re-signing. Justification:

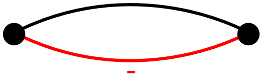
- sign changes twice;
- balance is preserved.

# Loops and digons

**Loops** are invariant under re-signing. Justification:

- sign changes twice;
- balance is preserved.


**Digon:** subgraph on 2 vertices with two parallel edges (+/-).

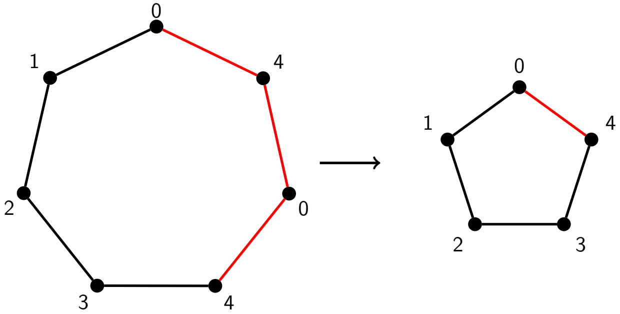


# 2-edge-coloured graph homomorphisms

Studied by Alon & Marshall, Brewster, Nesetřil & Raspaud...

**Definition** - Homomorphism of $(G, \Sigma)$ to $(H, \Pi)$

Homomorphism $f : G \to H$ that preserves edge-colors.

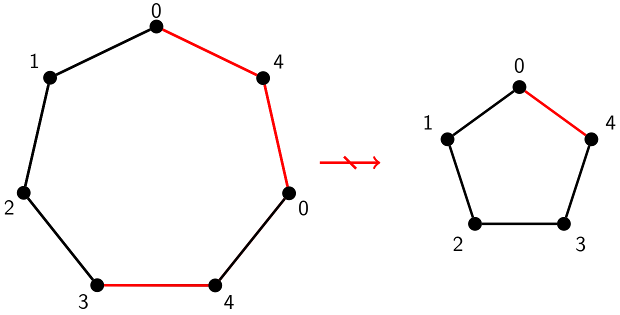


# 2-edge-coloured graph homomorphisms

Studied by Alon & Marshall, Brewster, Nesetřil & Raspaud...

**Definition** - Homomorphism of $(G, \Sigma)$ to $(H, \Pi)$

Homomorphism $f : G \to H$ that preserves edge-colors.



# Signed graph homomorphisms

Concept recently developed by Naserasr, Rollova, Sopena in 2012.

**Definition** - Signed graph homomorphism $[G, \Sigma] \to [H, \Pi]$

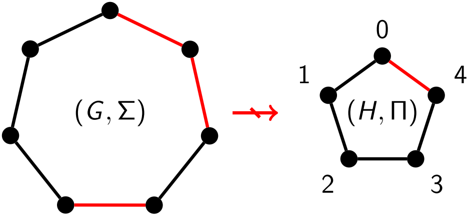
Homomorphism $f : G \to H$ such that there exists $\Sigma' \equiv \Sigma$ and $(G, \Sigma') \to (H, \Pi)$ [as 2-edge colored graphs].

# Signed graph homomorphisms

Concept recently developed by Naserasr, Rollova, Sopena in 2012.

**Definition** - Signed graph homomorphism $[G,\Sigma] \to [H,\Pi]$

Homomorphism $f : G \to H$ such that there exists $\Sigma' \equiv \Sigma$
and $(G,\Sigma') \to (H,\Pi)$ [as 2-edge colored graphs].

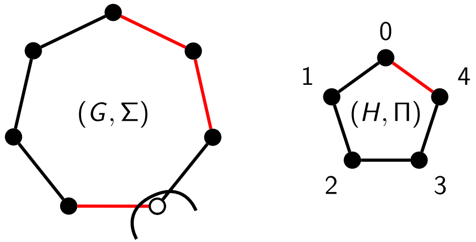


# Signed graph homomorphisms

Concept recently developed by Naserasr, Rollova, Sopena in 2012.

**Definition** - Signed graph homomorphism $[G, \Sigma] \to [H, \Pi]$

Homomorphism $f : G \to H$ such that there exists $\Sigma' \equiv \Sigma$ and $(G, \Sigma') \to (H, \Pi)$ [as 2-edge colored graphs].

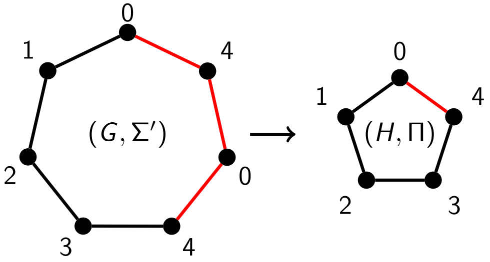


# Signed graph homomorphisms

Concept recently developed by Naserasr, Rollova, Sopena in 2012.

**Definition** - Signed graph homomorphism $[G, \Sigma] \to [H, \Pi]$

Homomorphism $f : G \to H$ such that there exists $\Sigma' \equiv \Sigma$
and $(G, \Sigma') \to (H, \Pi)$ [as 2-edge colored graphs].

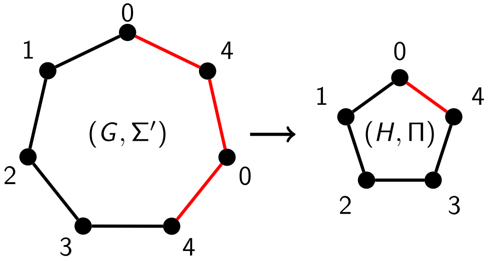


# Signed graph homomorphisms

Concept recently developed by Naserasr, Rollova, Sopena in 2012.

**Definition** - Signed graph homomorphism $[G, \Sigma] \to [H, \Pi]$

Homomorphism $f : G \to H$ such that there exists $\Sigma' \equiv \Sigma$ and $(G, \Sigma') \to (H, \Pi)$ [as 2-edge colored graphs].




**Note:** we can assume target signature is **fixed** $\to$ **no re-signing at target**.

# Complexity: classical homomorphisms

**$H$-Colouring**

INSTANCE: A graph $G$.
QUESTION: is it true that $G \to H$?

# Complexity: classical homomorphisms

**$H$-Colouring**

INSTANCE: A graph $G$.
QUESTION: is it true that $G \to H$?

**Theorem** (Karp, 1972)

$K_3$-Colouring is NP-complete.

# Complexity dichotomies

**Theorem** (Hell, Nešetřil, 1990)

$H$-COLOURING is polynomial (trivial) if $H$ is bipartite or has a loop.
Otherwise, NP-complete.

# Complexity dichotomies

**Theorem** (Hell, Nešetřil, 1990)

$H$-Colouring is polynomial (trivial) if $H$ is bipartite or has a loop. Otherwise, NP-complete.

**Conjecture** (Feder-Vardi, 1998: Dichotomy conjecture for CSP's)

For any **relational structure** $S$, $S$-Colouring (i.e. $S$-CSP) is either NP-complete or polynomial-time solvable.
(Equivalently: dichotomy for digraph-homomorphisms and for MMSNP)

Ladner, 1975: unless P=NP, no dichotomy for NP.

# Complexity dichotomies

**Theorem** (Hell, Nešetřil, 1990)

$H$-Colouring is polynomial (trivial) if $H$ is bipartite or has a loop.
Otherwise, NP-complete.

**Conjecture** (Feder-Vardi, 1998: Dichotomy conjecture for CSP's)

For any **relational structure** $S$, $S$-Colouring (i.e. $S$-CSP) is either NP-complete or polynomial-time solvable.
(Equivalently: dichotomy for digraph-homomorphisms and for MMSNP)

Ladner, 1975: unless P=NP, no dichotomy for NP.

**Proposition**

Dichotomy conjecture is equivalent to dichotomy for 2-edge-coloured graphs.

# Complexity: signed graph homomorphisms

$[H, \Pi]$-**Colouring**

INSTANCE: A signed graph $[G, \Sigma]$.
QUESTION: does $[G, \Sigma] \to [H, \Pi]$?

# Complexity: signed graph homomorphisms

**$[H, \Pi]$-Colouring**

INSTANCE: A signed graph $[G, \Sigma]$.
QUESTION: does $[G, \Sigma] \to [H, \Pi]$?

**Remark:** if $\Pi \equiv \emptyset$ (all positive) or $\Pi \equiv E(H)$ (all negative), $[H, \Pi]$-Colouring has same complexity as $H$-Colouring.

# Complexity: signed graph homomorphisms

**$[H,\Pi]$-Colouring**

INSTANCE: A signed graph $[G,\Sigma]$.
QUESTION: does $[G,\Sigma] \to [H,\Pi]$?

**Remark:** if $\Pi \equiv \emptyset$ (all positive) or $\Pi \equiv E(H)$ (all negative), $[H,\Pi]$-Colouring has same complexity as $H$-Colouring.

**Proposition** (signed $C_5$)

For every $\Pi \subseteq E(C_5)$, $[C_5,\Pi]$-Colouring is NP-complete.

**Proof:** Either $\Pi \equiv \emptyset$ or $\Pi \equiv E(C_5)$. Both $(C_5,\emptyset)$-Colouring and $(C_5,E(C_5))$-Colouring are NP-complete. □

**$[H, \Pi]$-Colouring**

INSTANCE: A signed graph $[G, \Sigma]$.
QUESTION: does $[G, \Sigma] \to [H, \Pi]$?

**Remark:** if $\Pi \equiv \emptyset$ (all positive) or $\Pi \equiv E(H)$ (all negative), $[H, \Pi]$-Colouring has same complexity as $H$-Colouring.

**Proposition** (signed $C_5$)

For every $\Pi \subseteq E(C_5)$, $[C_5, \Pi]$-Colouring is NP-complete.

**Proof:** Either $\Pi \equiv \emptyset$ or $\Pi \equiv E(C_5)$. Both $(C_5, \emptyset)$-Colouring and $(C_5, E(C_5))$-Colouring are NP-complete. □

**Proposition** (2-edge-coloured $C_5$)

$(C_5, \Pi)$-Colouring is NP-complete if $\Pi = \emptyset$ or $\Pi = E(C_5)$; poly-time otherwise.

# (Trivial) poly-time cases

**$[H,\Pi]$-Colouring**

INSTANCE: A signed graph $[G,\Sigma]$.
QUESTION: does $[G,\Sigma] \to [H,\Pi]$?

**Proposition**

$[H,\Pi]$-COLOURING is poly-time if:

(a) If $H$ is bipartite and $\Pi \equiv \emptyset \equiv E(H)$ (i.e. $[H,\Pi]$ retracts to an edge);

(b) If $[H,\Pi]$ retracts to a single vertex with loop(s);

(c) If $H$ is bipartite and $[H,\Pi]$ contains (retracts to) a digon.

# (Trivial) poly-time cases

**$[H, \Pi]$-Colouring**

INSTANCE: A signed graph $[G, \Sigma]$.
QUESTION: does $[G, \Sigma] \to [H, \Pi]$?

**Proposition**

$[H, \Pi]$-COLOURING is poly-time if:

(a) If $H$ is bipartite and $\Pi \equiv \emptyset \equiv E(H)$ (i.e. $[H, \Pi]$ retracts to an edge);

(b) If $[H, \Pi]$ retracts to a single vertex with loop(s);

(c) If $H$ is bipartite and $[H, \Pi]$ contains (retracts to) a digon.

**Proof.** Input $[G, \Sigma]$.

(a) YES $\Leftrightarrow$ $G$ bipartite and $\Sigma \equiv \emptyset$.

(b) YES $\Leftrightarrow$ $\Sigma \equiv \emptyset$ (+ loop), $\Sigma \equiv E(G)$ (- loop), or always true (both loops).

(c) YES $\Leftrightarrow$ $G$ is bipartite. □

# A tool to capture them all

Construction of targets that are **invariant under re-signing**

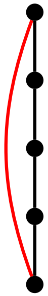
(Zaslavsky, Brewster and Graves, Klostermeyer and MacGillivray ...)

# A tool to capture them all

Construction of targets that are **invariant under re-signing**

(Zaslavsky, Brewster and Graves, Klostermeyer and MacGillivray ...)

$(H, \Pi)$

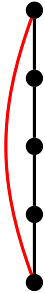


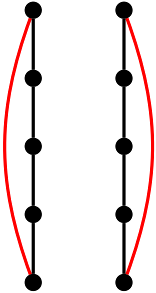
# A tool to capture them all

Construction of targets that are **invariant under re-signing**

(Zaslavsky, Brewster and Graves, Klostermeyer and MacGillivray ...)

$(H, \Pi)$





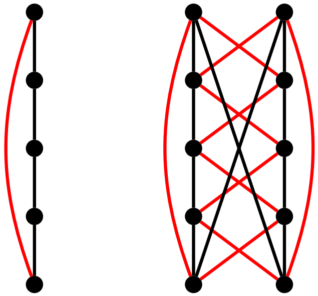
$P(H, \Pi)$

# A tool to capture them all

Construction of targets that are **invariant under re-signing**

(Zaslavsky, Brewster and Graves, Klostermeyer and MacGillivray ...)

$(H, \Pi)$



$P(H, \Pi)$

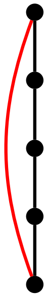
$\rightarrow$ For every $\Pi' \equiv \Pi$, $P(H, \Pi)$ contains $(H, \Pi')$ as a subgraph.

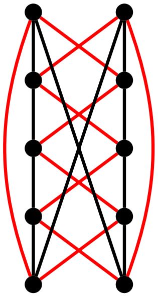
# A tool to capture them all

Construction of targets that are **invariant under re-signing**

(Zaslavsky, Brewster and Graves, Klostermeyer and MacGillivray ...)

$(H, \Pi)$





$P(H, \Pi)$

$\rightarrow$ For every $\Pi' \equiv \Pi$, $P(H, \Pi)$ contains $(H, \Pi')$ as a subgraph.

**Theorem**

$$[G, \Sigma] \rightarrow [H, \Pi] \text{ if and only if } (G, \Sigma) \rightarrow P(H, \Pi).$$

# Hard cases: bipartite graphs

$[BC_k]$: class of balanced $k$-cycles.
$[UC_k]$: class of unbalanced $k$-cycles.

- $[BC_{2k+1}]$-Colouring equivalent to $[C_{2k+1}, \emptyset]$-Colouring: NP-c.
- $[UC_{2k+1}]$-Colouring equivalent to $[C_{2k+1}, E(C_{2k+1})]$-Colouring: NP-c.
- $[BC_{2k}]$-Colouring equivalent to $[K_2, \emptyset]$-Colouring: poly-time.

**Theorem** (F., Naserasr, 2012)

$[UC_{2k}]$-Colouring is NP-complete for every $k \geq 2$.

# Hard cases: bipartite graphs

**Theorem** (Brewster, F., Hell, Naserasr)

If $[H,\Pi]$ is digon-free and has an unbalanced even cycle, then $[H,\Pi]$-Colouring is NP-complete. True even with the presence of (single) loops.

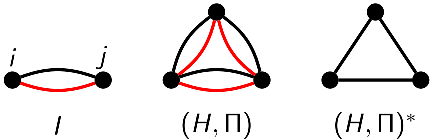
# Hard cases: bipartite graphs

**Theorem** (Brewster, F., Hell, Naserasr)

If $[H,\Pi]$ is digon-free and has an unbalanced even cycle, then $[H,\Pi]$-Colouring is NP-complete. True even with the presence of (single) loops.

*Indicator-construction:* classic tool for reductions (see e.g. Hell-Nešetřil).

- Indicator: subgraph $I$ with two distinguished vertices $i, j$.
- $H^*$: graph on $V(H)$ with an edge $uv$ iff $f : I \to H$ with $f(i) = u$, $f(j) = v$.
- ${}^*G$: replace each edge of $G$ by a copy of $I$.




**Proposition**

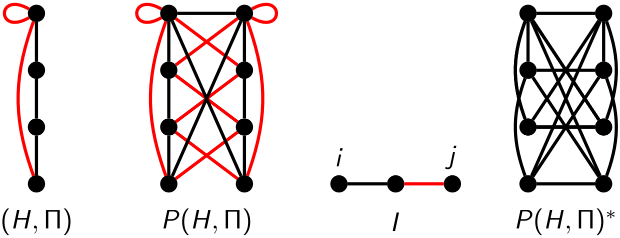
$$G \to H^* \iff {}^*G \to H$$

# Hard cases: bipartite graphs

**Theorem** (Brewster, F., Hell, Naserasr)

If $[H, \Pi]$ is digon-free and has an unbalanced even cycle, then $[H, \Pi]$-COLOURING is NP-complete. True even with the presence of (single) loops.

**Proof:** Use indicator-gadget on $P(H, \Pi)$:



# Hard cases: odd cycles

**Theorem** (Brewster, F., Hell, Naserasr)

$[H, \Pi]$-COLOURING is NP-complete if $[H, \Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H, \Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H, \Pi)^*$ has an odd cycle and no loop

# Hard cases: odd cycles

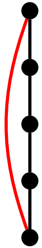
**Theorem** (Brewster, F., Hell, Naserasr)

$[H, \Pi]$-Colouring is NP-complete if $[H, \Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H, \Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H, \Pi)^*$ has an odd cycle and no loop

$(H, \Pi)$



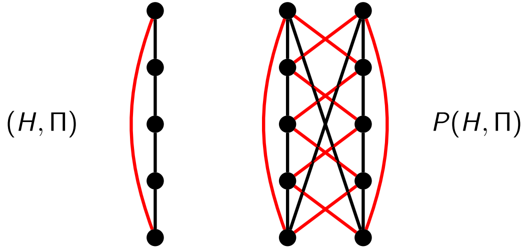
# Hard cases: odd cycles

**Theorem** (Brewster, F., Hell, Naserasr)

$[H,\Pi]$-Colouring is NP-complete if $[H,\Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H,\Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H,\Pi)^*$ has an odd cycle and no loop



# Hard cases: odd cycles

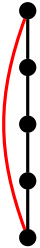
**Theorem** (Brewster, F., Hell, Naserasr)

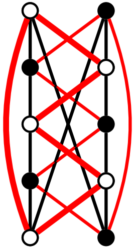
$[H,\Pi]$-Colouring is NP-complete if $[H,\Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H,\Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H,\Pi)^*$ has an odd cycle and no loop

$(H,\Pi)$





$P(H,\Pi)$

# Hard cases: odd cycles

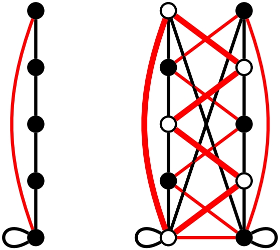
**Theorem** (Brewster, F., Hell, Naserasr)

$[H,\Pi]$-COLOURING is NP-complete if $[H,\Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H,\Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H,\Pi)^*$ has an odd cycle and no loop

$(H,\Pi)$



$P(H,\Pi)$

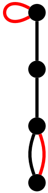
# Hard cases: odd cycles

**Theorem** (Brewster, F., Hell, Naserasr)

$[H, \Pi]$-COLOURING is NP-complete if $[H, \Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H, \Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H, \Pi)^*$ has an odd cycle and no loop



$(H, \Pi)$

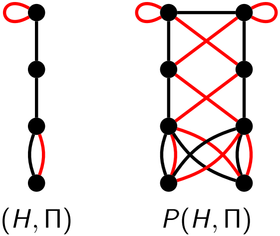
# Hard cases: odd cycles

**Theorem** (Brewster, F., Hell, Naserasr)

$[H,\Pi]$-COLOURING is NP-complete if $[H,\Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H,\Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H,\Pi)^*$ has an odd cycle and no loop



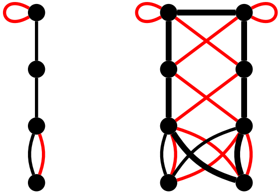
# Hard cases: odd cycles

**Theorem** (Brewster, F., Hell, Naserasr)

$[H, \Pi]$-COLOURING is NP-complete if $[H, \Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H, \Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H, \Pi)^*$ has an odd cycle and no loop



$(H, \Pi)$ $P(H, \Pi)$

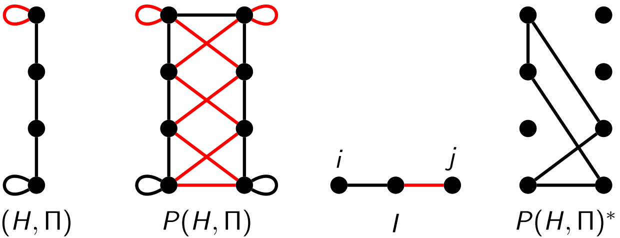
# Hard cases: odd cycles

**Theorem** (Brewster, F., Hell, Naserasr)

$[H,\Pi]$-Colouring is NP-complete if $[H,\Pi]$ is non-bipartite and:
(a) no loops, no digon;
(b) (single) loops but no digon;
(c) a digon and only one kind of (single) loops.

**Proof:** Either

- $P(H,\Pi)$ has a $+$ odd cycle and no $+$ loop (or vice-versa), or
- $P(H,\Pi)^*$ has an odd cycle and no loop



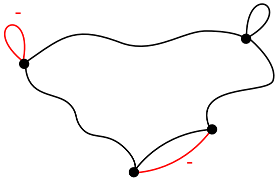
# Concluding remarks

- Besides trivially poly-time cases, everything (we know of) is NP-complete.

# Concluding remarks

- Besides trivially poly-time cases, everything (we know of) is NP-complete.

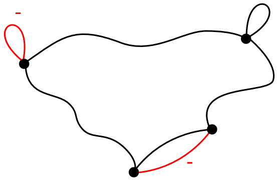
- missing cases: targets with digons and both kinds of loops.



# Concluding remarks

- Besides trivially poly-time cases, everything (we know of) is NP-complete.

- missing cases: targets with digons and both kinds of loops.




- Sharp contrast to 2-edge-coloured graphs, where dichotomy is very difficult.