Parameterized complexity of modification problems on edge-coloured and signed graph homomorphisms

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17 may 2019

Overview

1. Modification Problems
2. *H*-Colouring
3. Vertex Deletion and Edge Deletion
4. Switching *H*-colouring

Overview

Problem

Take $\mathcal{P}$ a graph property. Can we modify G so that $\mathcal{P}(G)$ is true?

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- Edge Deletion
- Switching

Examples

$\mathcal{P}$	Vertex Deletion	Edge Deletion

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$\mathcal{P}$	Vertex Deletion	Edge Deletion
Being Edgeless		

Examples

$\mathcal{P}$	Vertex Deletion	Edge Deletion
Being Edgeless	VERTEX COVER	

Examples

$\mathcal{P}$	Vertex Deletion	Edge Deletion
Being Edgeless	VERTEX COVER	Trivial

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$\mathcal{P}$	Vertex Deletion	Edge Deletion
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Being Bipartite		

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Being Edgeless	VERTEX COVER	Trivial
Being Bipartite	ODD CYCLE TRANSVERSAL	EDGE BIPARTIZATION
Being Planar	PLANAR VERTEX DELETION	PLANAR EDGE DELETION
$\chi(G) \leq 12$	VD K_{12}-COLOURING	ED K_{12}-COLOURING

Many more: EDGE DOMINATING SET, EULERIAN DELETION, FEEDBACK VERTEX SET...

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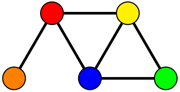

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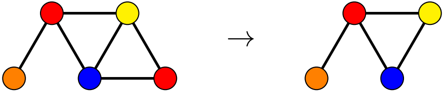

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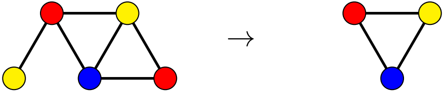

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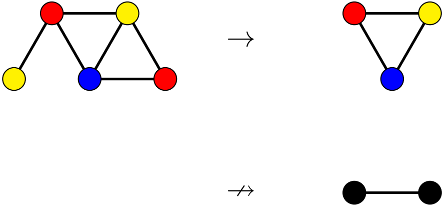

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Theorem (Hell and Nešetřil)

H-COLOURING is polynomial if H has a loop or is bipartite. It is NP-Complete otherwise.

Definition (core)

The core of H is the smallest subgraph C of H such that $H \to C$.
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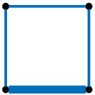

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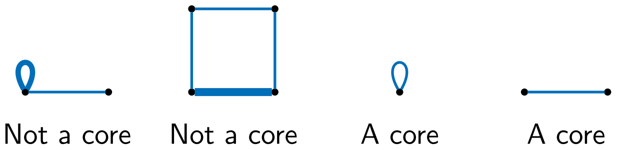

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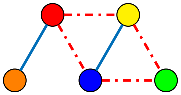

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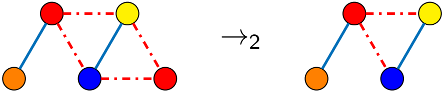

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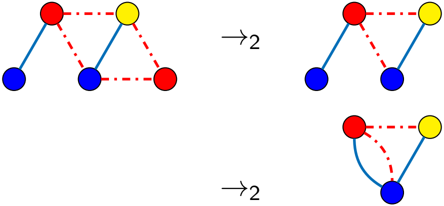

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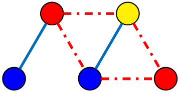

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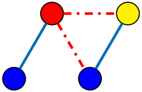

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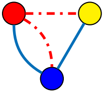

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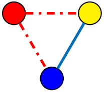

Problem (H-Colouring$_t$)

Input: A t-edge-coloured graph G.
Question: Does there exist a homomorphism from G to H?

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If H-COLOURING$_t$ is NP-complete then so is *VD* H-COLOURING$_t$.

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Theorem (R. C. Brewster, R. Dedić, F. Huard and J. Queen)

The H-Colouring$_t$ problem is polynomial in the following cases:

- *H has order 2,*
- *H has order 3, is loop-free and contains no monochromatic triangle.*

If H-$\textsc{Colouring}_t$ is NP-complete then so is *VD* H-$\textsc{Colouring}_t$.

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Theorem (R. C. Brewster and P. Hell)

The H-Colouring$_t$ problem is polynomial for some cycles.

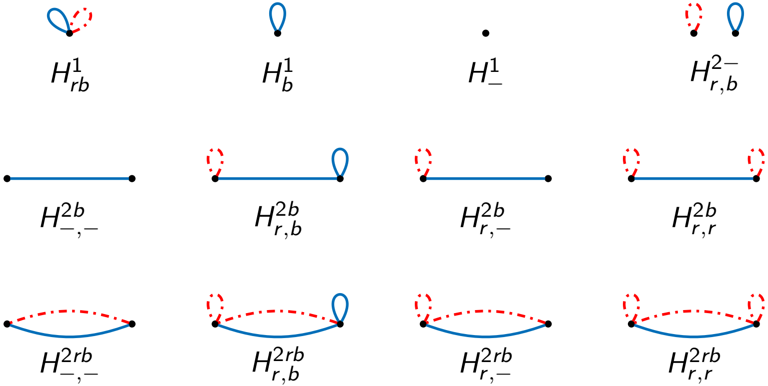

Figure: The twelve 2-edge-coloured graphs of order at most 2.

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Corollary (of the proof)

The $\mathcal{P}$ Vertex-Deletion problem for nontrivial properties $\mathcal{P}$, on loopless t-edge-coloured graphs, that are hereditary on induced subgraphs and true for all independent sets is NP-hard.

Theorem

For a core H, the problem is trivial iff H is composed of one vertex with t coloured loops of different colours. In this case, all t-edge-coloured graphs G verify $G \to_t H$.

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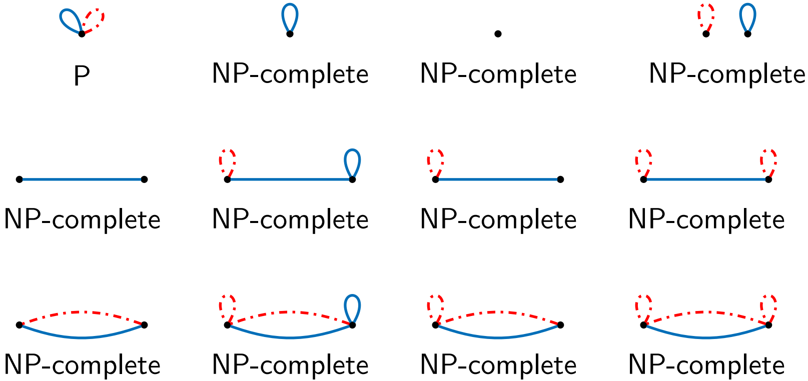
Theorem

The problem VERTEX DELETION H-COLOURING$_t$ *for a t-edge-coloured graph H is polynomial if H contains a vertex having all t coloured loops and NP-complete otherwise.*


P
NP-complete
NP-complete
NP-complete
NP-complete
NP-complete
NP-complete
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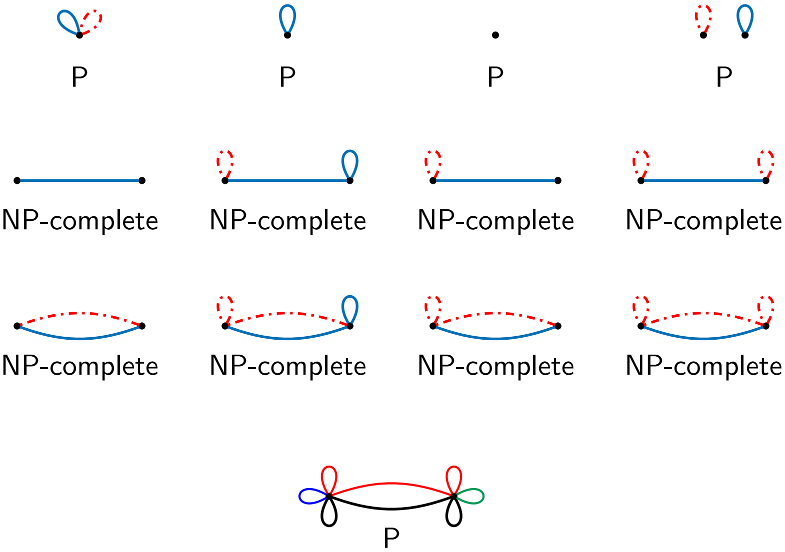
NP-complete

Theorem

Let H be an edge-coloured core of order at most 2. If each colour of H induces a set of loops or contains all three possible edges, then EDGE DELETION H-COLOURING$_t$ *lies in P, otherwise it is NP-complete.*

Alternate formulation, if H contains one of these two, then it is NP-complete:


P
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P

Theorem (R. C. Brewster, R. Dedić, F. Huard and J. Queen)

*The H-*Colouring$_t$ *problem is polynomial when H has order 2 by reduction to 2-*Sat.

$E_i(H)$	Clause
$\emptyset$	$\bot$
$\{00\}$	$(\overline{x_u})(\overline{x_v})$
$\{01\}$	$(x_u + x_v)(\overline{x_u} + \overline{x_v})$
$\{11\}$	$(x_u)(x_v)$
$\{00, 01\}$	$(\overline{x_u} + \overline{x_v})$
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$\{00, 01, 11\}$	$\top$

Problem (Variable Deletion Almost 2-Sat)

Input: A 2-CNF formula F, an integer k.
Parameter: k.
Question: Is there a set of k variables that can be deleted from F (together with the clauses containing them) so that the resulting formula is satisfiable?

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Clause Deletion Almost 2-Sat and Variable Deletion Almost 2-Sat were proved to be FPT (I. Razgon and B. O'Sullivan for the former and M. Cygan *et al.* for the later).

$E_i(H)$	Clause before	Clause after modification
$\varnothing$	$\perp$	$(x_u + x_v)(\overline{x_u} + x_v)(x_u + \overline{x_v})(\overline{x_u} + \overline{x_v})$
$\{00\}$	$(\overline{x_u})(\overline{x_v})$	$(\overline{x_u} + x_v)(\overline{x_u} + \overline{x_v})(\overline{x_v} + x_u)$
$\{01\}$	$(x_u + x_v)(\overline{x_u} + \overline{x_v})$	$(x_u + x_v)(\overline{x_u} + \overline{x_v})$
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Theorem

If H has order 2 then there exists an FPT algorithm for VERTEX DELETION *H*-COLOURING$_t$.

Problem (Group Deletion Almost 2-Sat)

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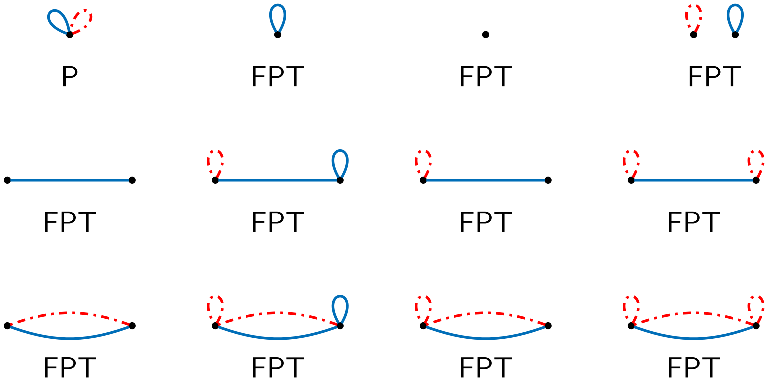
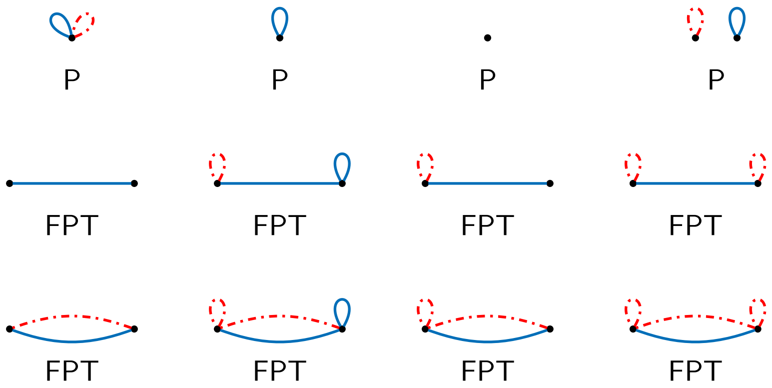
Corollary

If H has order 2 then there exists an FPT algorithm for Edge Deletion H-Colouring$_t$.

Vertex Deletion



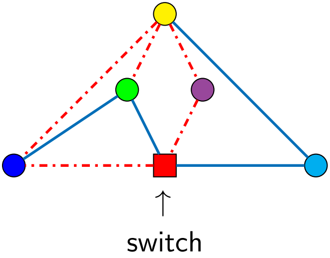
Edge Deletion



Overview

Definition

A signed graph G is a graph where each edge can be either positive or negative. Moreover we can switch at each vertex v. Switching at v consists in inverting the signs of all edges incident with v.



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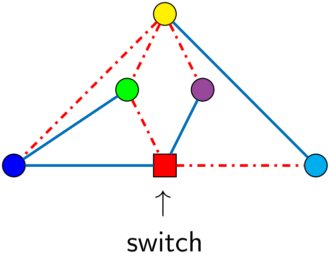

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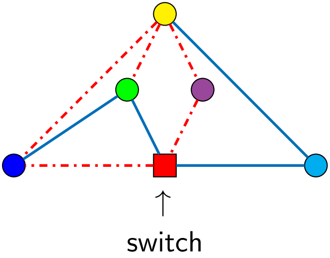

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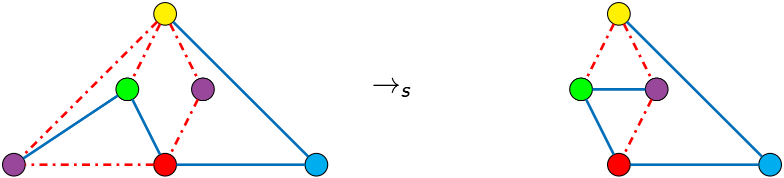

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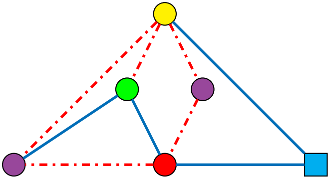

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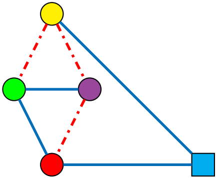

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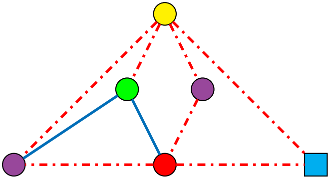

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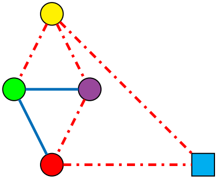
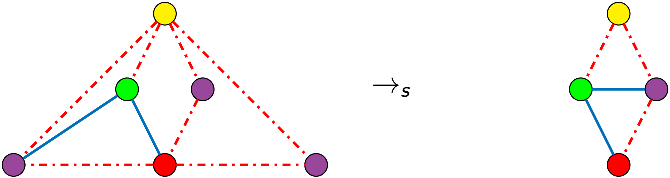


Definition

A signed graph G is a graph where each edge can be either positive or negative. Moreover we can switch at each vertex v. Switching at v consists in inverting the signs of all edges incident with v.



We consider two types of homomorphisms:

- $G \to_s H$: We can switch on G an arbitrary number of times to get G' and $G' \to_2 H$,
- $G \to_s^{\leq k} H$: We can switch on G at most k times to get G' and $G' \to_2 H$.

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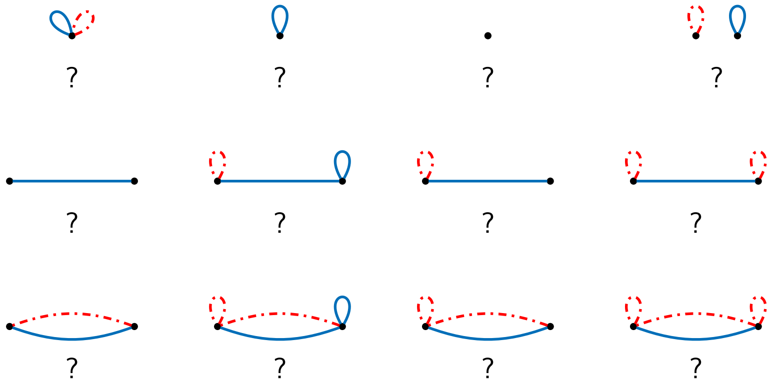
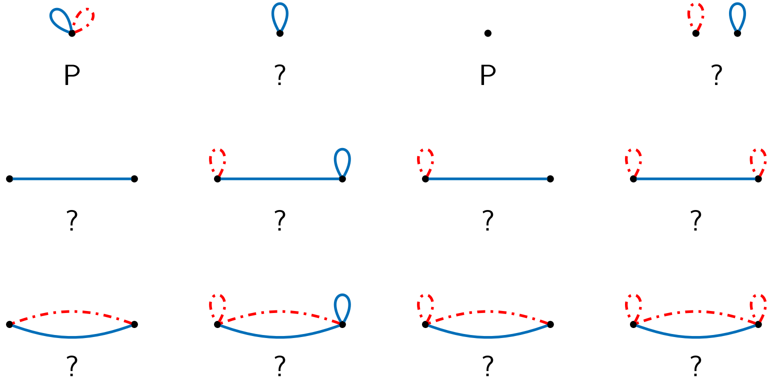
Problem (Signed H-colouring)

Input: G.
Question: Does $G \to_s H$?

Problem (Switching H-colouring)

Input : G and k.
Parameter: k.
Question: Does $G \to_s^{\leq k} H$?

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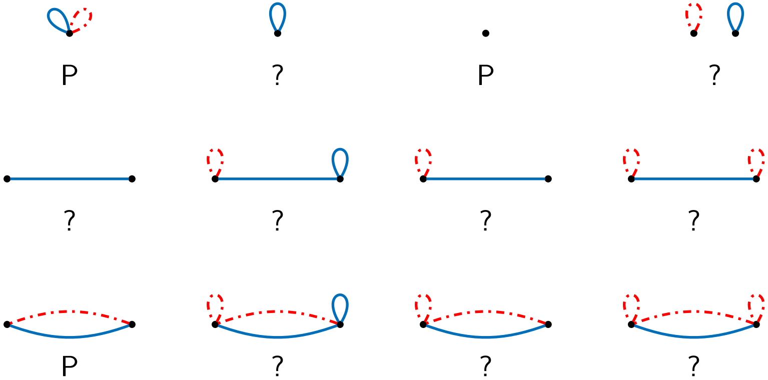
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An extension of a result from Zaslavsky gives:

Theorem

If G is a signed graph with a given signature (i.e. edge-colouring), then we can test in polynomial time if G can be made all positive. Moreover we can compute in polynomial time the number of switchings required to do so.

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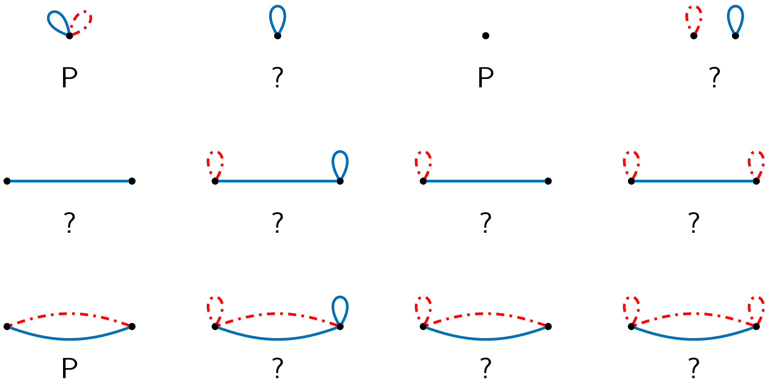
Theorem (Brewster, Foucaud, Hell, Naserasr and Brewster, Siggers)

Let H be a signed graph. Signed H-Colouring *is polynomial if the switching core of H has at most two edges, and NP-complete otherwise.*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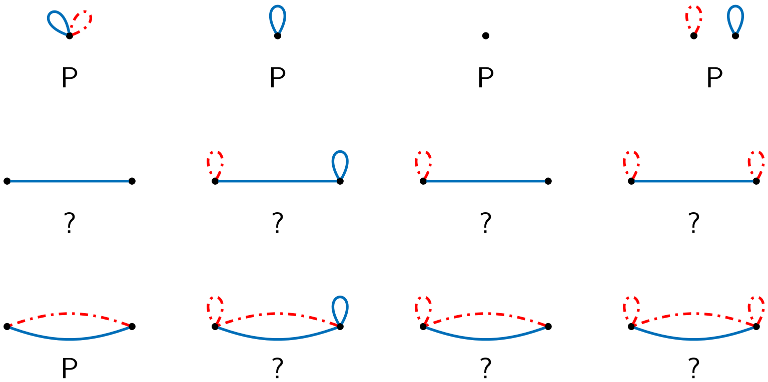
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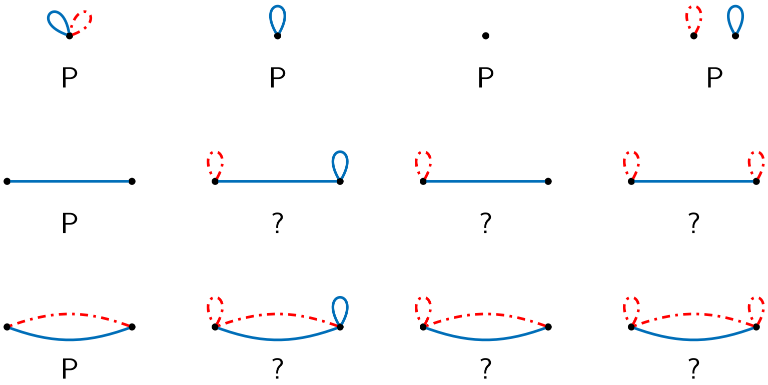
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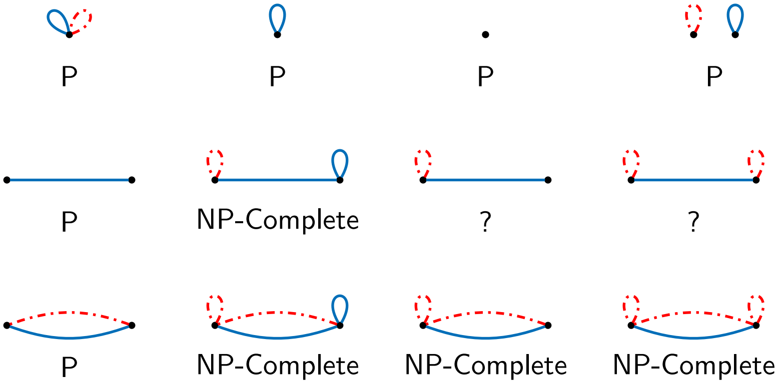
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For $H =$  , $G \rightarrow_2 H$ iff G has no cycles with an odd number of positive edges (Bawar, Brewster and Marcotte).

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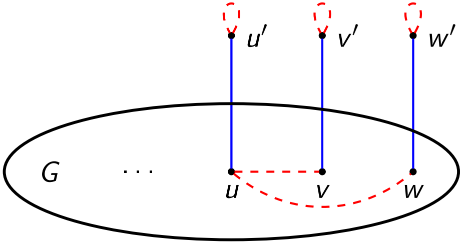
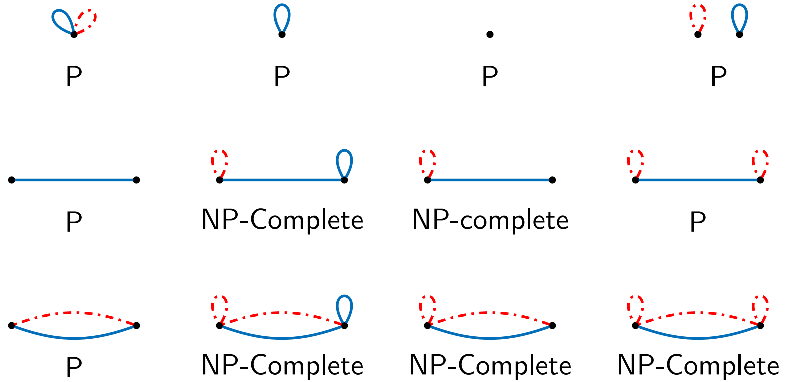
$\Rightarrow$ The parity of the number of positive edges of a cycle does not change by switching.

For H = , reduction from VERTEX COVER.

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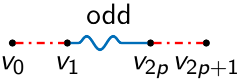
$\Rightarrow$ We have an FPT branching algorithm.

For $H =$ , $G \rightarrow_2 H$ iff 

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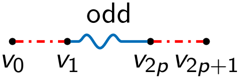
For $H =$ , $G \to_2 H$ iff 

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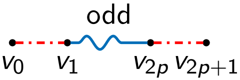
The only way to remove such a path is to switch at one of $v_0, v_1, v_{2p}, v_{2p+1}$.

For $H =$ , $G \to_2 H$ iff 

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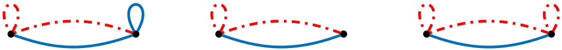
The only way to remove such a path is to switch at one of $v_0, v_1, v_{2p}, v_{2p+1}$. Thus we can have an FPT branching algorithm.

What about :



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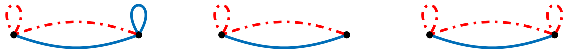
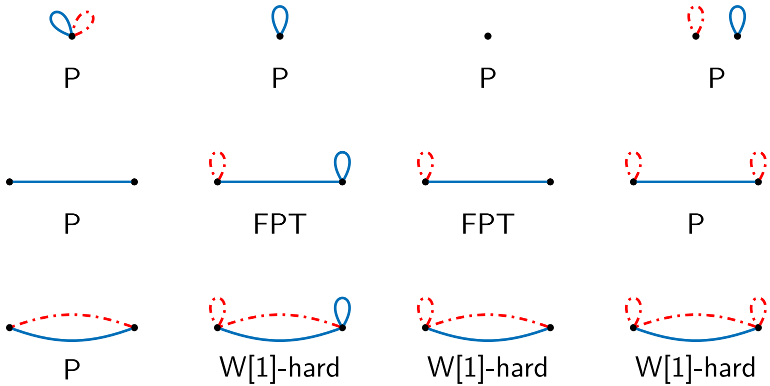


Theorem

For these three graphs SWITCHING H-COLOURING *is W[1]-hard.*

Switching



Open questions

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Thank you for your attention!