Broadcast domination and multipacking in graphs and digraphs

Florent Foucaud (Université Clermont Auvergne, France)

joint works with:

Laurent Beaudou (Université Clermont Auvergne, France)
Richard C. Brewster (Thompson Rivers University, Canada)

Benjamin Gras (Université d'Orléans, France)
Anthony Perez (Université d'Orléans, France)
Florian Sikora (Université Paris-Dauphine, France)

Sandip Das (Indian Statistical Institute, India)
Sk Samim Islam (Ramakrishna Mission Vivekananda Edu. and Res. Institute, India)
Joydeep Mukherjee (Ramakrishna Mission Vivekananda Edu. and Res. Institute, India)

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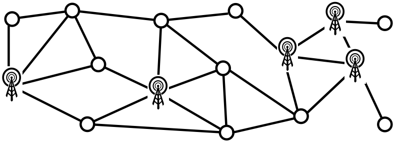
Covering: cover the vertices of a graph using as few structures as possible

Example: dominating set: covering using 1-balls
→ domination number $\gamma(G)$



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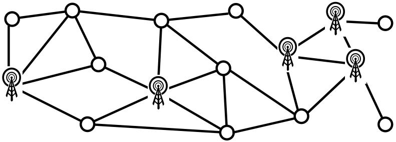
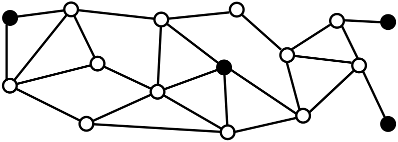


Packing: pack as many structures as possible without interference

Example: dist. 3-independent set / 2-packing: packing 1-balls without overlap
→ 2-packing number $\rho_2(G)$



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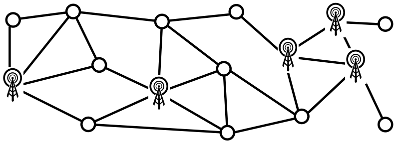

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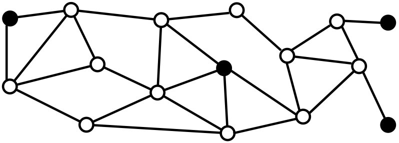


These problems are dual (in the sense of LP) and $\rho_2(G) \leq \gamma(G)$.

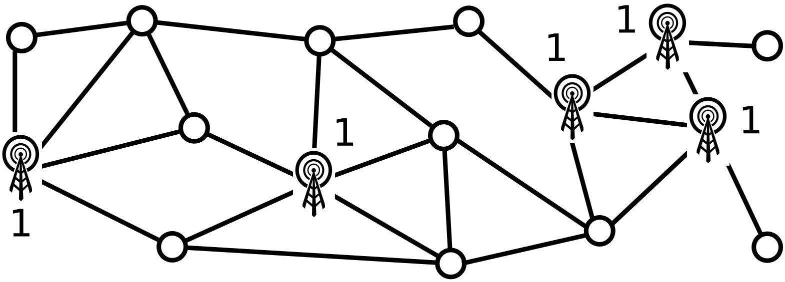
Definition - Dominating broadcast of graph G (Erwin, 2001)

A function $f : V(G) \to \mathbb{N}$ s.t. for every $v \in V(G)$, there exists $x \in V(G)$ with

- $f(x) > 0$ and
- $f(x) \geq d_G(x, v)$.

The cost of f is $\sum_{v \in V(G)} f(v)$.

Broadcast number $\gamma_b(G)$: smallest cost of a dominating broadcast of G.



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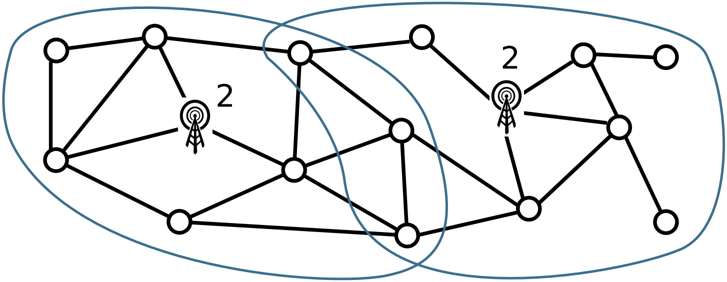

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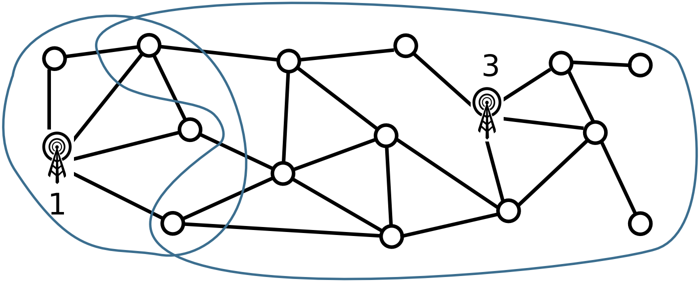

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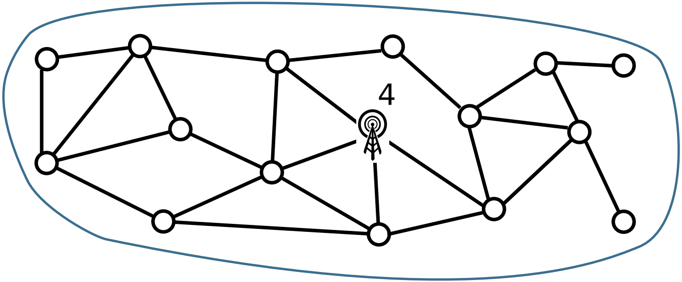

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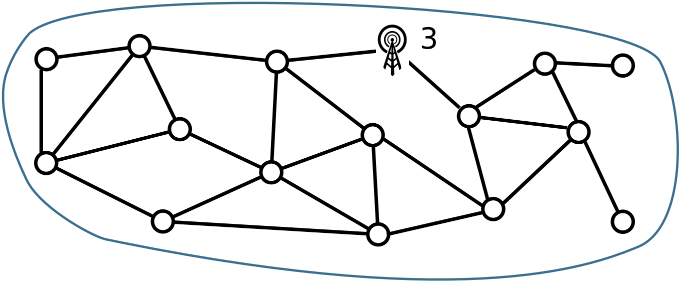

Theorem (Heggernes-Lokshtanov, 2006)

We can find a minimum-cost dominating broadcast in polynomial time $O(n^6)$.

Proof idea:

- sufficient to find an efficient dominating broadcast (Erwin, 2001)
- The structure of covering balls is a path or a cycle
- Dynamic programming on this structure

Vertices: $v_1, \ldots, v_n$.

$x_{i,k} \in \{0,1\}$: whether vertex v_i broadcasts with radius k

We want to minimize:

$$\sum_{k=1}^{n} \sum_{i=1}^{n} k \cdot x_{i,k}$$

subject to:

$$\sum_{d(v_i,v_j) \leq k} x_{i,k} \geq 1 \text{ for each vertex } v_j.$$

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Dual ILP:

We want to maximize:

$$\sum_{i=1}^{n} y_i$$

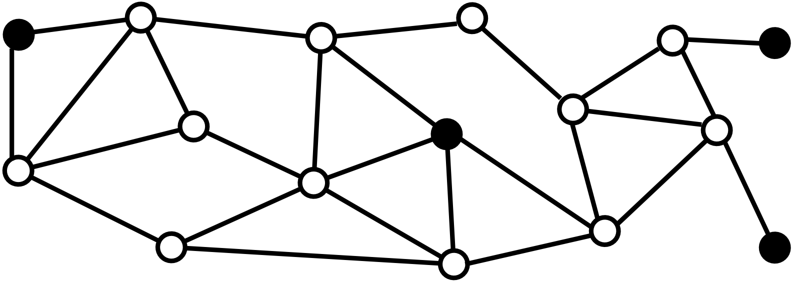
subject to:

$$\sum_{d(v_i,v_j) \leq k, y_j \geq 0} y_i \leq k \text{ for each vertex } v_j \text{ and integer } k \leq n.$$

Definition - Multipacking of graph G (Brewster-Mynhardt-Teshima, 2014)

A set S of vertices s.t. for every $v \in V(G)$ and every $d \in \mathbb{N}$, the d-ball $B_d(v)$ contains at most d vertices of S.

Multipacking number $mp(G)$: largest size of a multipacking of G.



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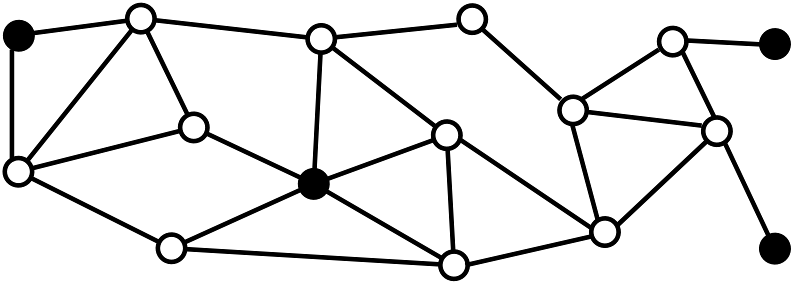

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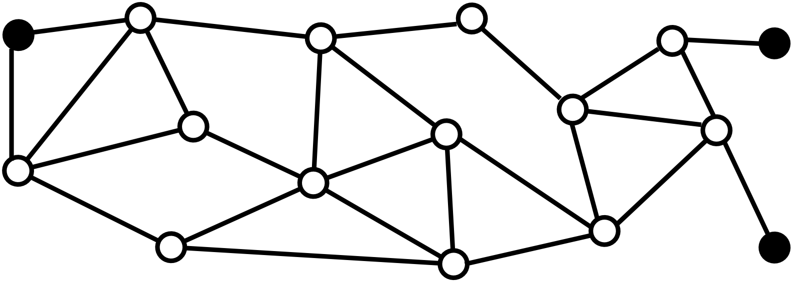

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Open question

Can one compute a maximum-size multipacking in polynomial time?

Bounds for undirected graphs : general graphs

The two problems are dual (in the sense of LP).

Proposition

For every graph G, we have $mp(G) \leq \gamma_b(G)$.

Equality holds for:

- trees (Mynhardt-Teshima, 2017)
- more generally, strongly chordal graphs (Brewster-MacGillivray-Yang, 2019)
- rectangular grids (Beaudou-Brewster, 2019)

A chain of inequalities

diameter $diam(G)$: largest distance between two vertices in G

eccentricity of a vertex v: largest possible distance from v to another vertex

radius $rad(G)$: smallest eccentricity among all vertices

Proposition (Erwin, 2001 + Hartnell-Mynhardt, 2014)

For any graph G, $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq rad(G) \leq diam(G)$.

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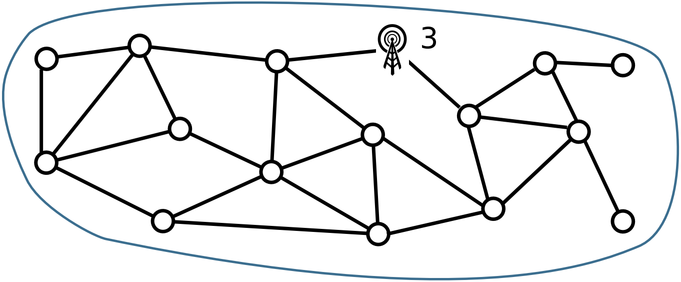
$\gamma_b(G) \leq rad(G)$: consider a radial vertex v. Set $f(v) = rad(G)$.



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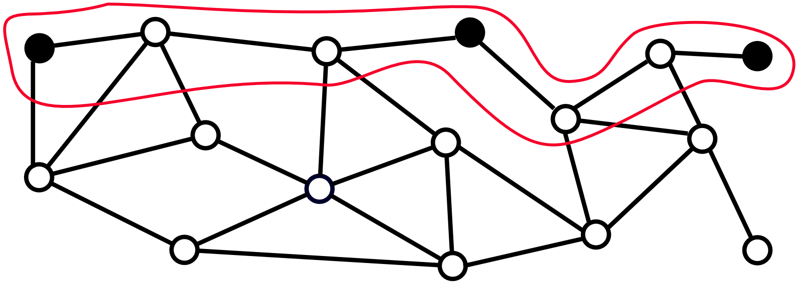
$\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G)$: consider a diametral path P, select every third vertex.



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For any graph G, $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq rad(G) \leq diam(G)$.

Corollary

For any graph G, we have $\gamma_b(G) < 3mp(G)$, hence $\frac{\gamma_b(G)}{mp(G)} < 3$.

Question (Hartnell-Mynhardt, 2014)

What is the largest possible ratio $\frac{\gamma_b(G)}{mp(G)}$?

Theorem (Beaudou, Brewster, F., 2019)

For any graph G, we have $\gamma_b(G) \leq 2mp(G) + 3$, hence $\frac{\gamma_b(G)}{mp(G)} \leq 2 + \epsilon$.

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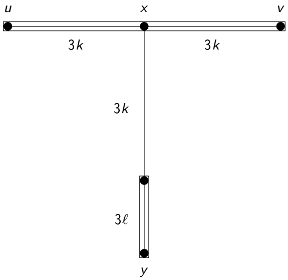
Lemma

Proof sketch

Let u, v, x, y be 4 vertices with:

- $d(u, v) = 6k$
- $d(x, u) = d(x, v) = 3k$
- $d(x, y) = 3k + 3\ell$.

Then, $mp(G) \geq 2k + \ell$.



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Let $diam(G) = 6k + i$ and $rad(G) = 3k + 3\ell + j$ $(0 \leq i < 6$ and $0 \leq j < 3)$

Apply the lemma with x, a vertex of eccentricity $rad(G)$.

$$\begin{aligned} mp(G) &\geq 2k + \ell \\ &\geq \frac{diam(G)}{3} + \frac{rad(G)}{3} - \frac{diam(G)}{6} - c \\ &\geq \frac{rad(G)}{2} - c \\ &\geq \frac{\gamma_b(G)}{2} - c \end{aligned}$$

$\square$

Theorem (Beaudou, Brewster, F., 2019)

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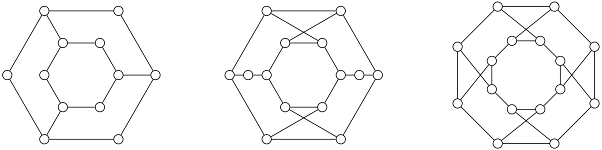
Conjecture would be tight — infinitely many graphs G s.t. $\gamma_b(G) = 2mp(G)$:



$mp(G) = 2$ and $\gamma_b(G) = 4$

Connected graphs, general case

Conjecture

For any graph G, we have $\gamma_b(G) \leq 2mp(G)$.

Question

What happens for connected graphs?

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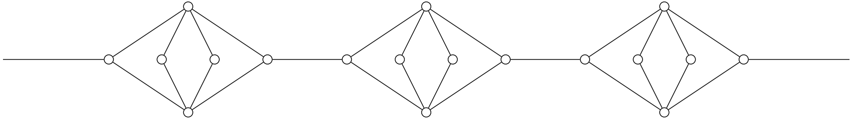
What happens for connected graphs?

Closest known connected family: $\gamma_b(G) = \frac{4}{3}mp(G)$ (Hartnell-Mynhardt, 2014)



Here: $mp(G) = 3$ and $\gamma_b(G) = 4$.

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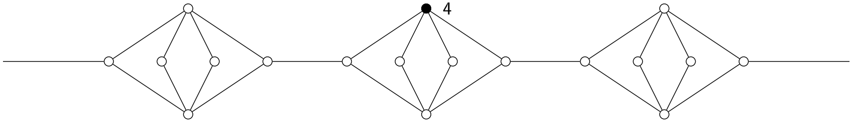

Here: $mp(G) = 3$ and $\gamma_b(G) = 4$.

Corollary

For connected graphs G, $\frac{4}{3} \leq \lim_{mp(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{mp(G)} \right\} \leq 2 + \epsilon$.

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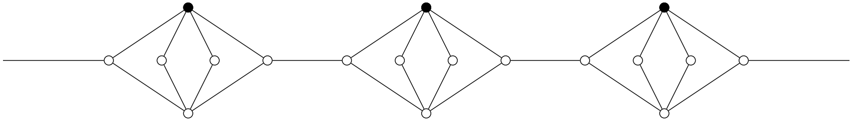

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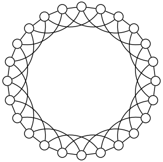
What happens for connected graphs?

Proposition

There is a connected graph G of order 24 with $mp(G) = 3$ and $\gamma_b(G) = 5$.



Connected graphs, small case

Conjecture

For any graph G, we have $\gamma_b(G) \leq 2mp(G)$.

Question

What happens for connected graphs?

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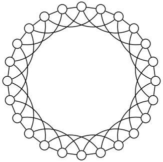


Question

Is there a connected graph G with $mp(G) = 3$ and $\gamma_b(G) = 6$?

Bounds for undirected graphs : chordal graphs

Chordal graph: graph where every cycle of length 4 or more has a *chord*

Proposition

If G is a chordal graph, then $\gamma_b(G) \leq \left\lceil \frac{3}{2} mp(G) \right\rceil$.

Chordal graph: graph where every cycle of length 4 or more has a *chord*

Proposition

If G is a chordal graph, then $\gamma_b(G) \leq \left\lceil \frac{3}{2} mp(G) \right\rceil$.

This can be proved using the following two theorems:

Theorem (Laskar, Shier, 1983)

If G is a chordal graph with radius r and diameter d, then $2r \leq d + 2$.

Theorem (Erwin 2001 & Hartnell-Mynhardt 2014)

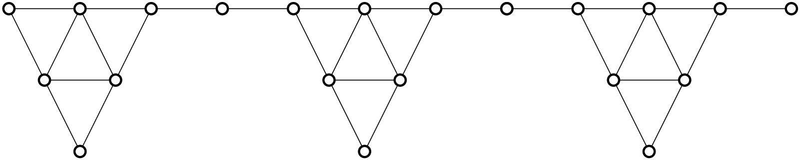
If G is a connected graph of order at least 2 having radius r, diameter d, multipacking number $mp(G)$, broadcast domination number $\gamma_b(G)$ and domination number $\gamma(G)$, then $\left\lceil \frac{d+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq min\{\gamma(G), r\}$.

Our results

Theorem (Das, F., Islam, Mukherjee, 2023)

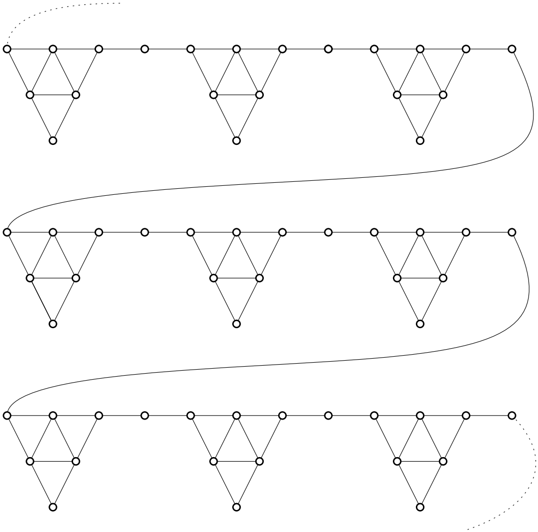
For connected chordal graphs G, $\frac{10}{9} \leq \lim_{mp(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{mp(G)} \right\} \leq \frac{3}{2}$.

Consider the graph G_1.



Proof sketch

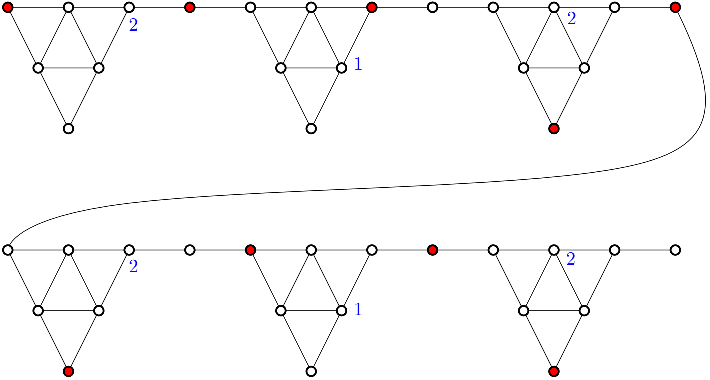
Join k copies of G_1 to form G_k.



Proof sketch

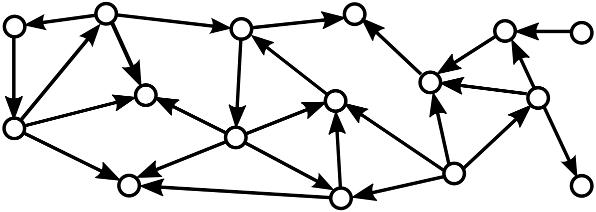
Lemma

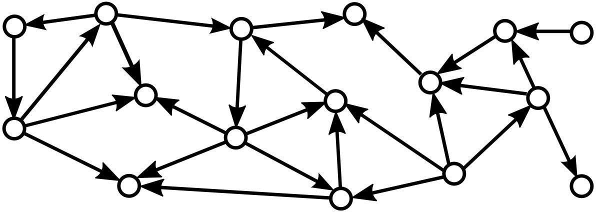
$mp(G_{2k}) \leq 9k$ and $\gamma_b(G_{2k}) = 10k$.



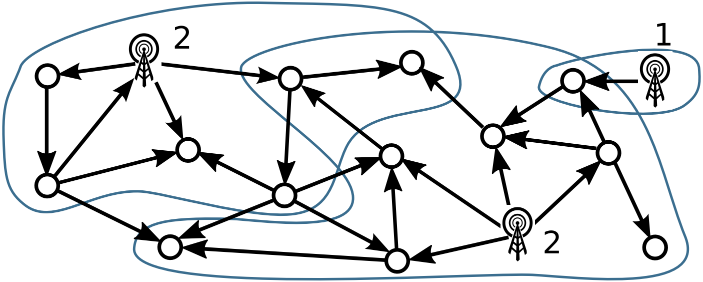
Complexity & algorithms for directed graphs

Broadcast domination in directed graphs





Note: an undirected graph can be seen as a symmetric directed graph!



Broadcast domination for directed graphs:

A vertex v with $f(v) = r$ broadcasts to all vertices at directed distance up to r.

BROADCAST DOM
Input: A (directed) graph G, an integer k.
Question: Does G have a dominating broadcast of cost at most k?

Theorem (Heggernes-Lokshtanov, 2006)

BROADCAST DOM can be solved in polynomial time $O(n^6)$ for **undirected graphs**.

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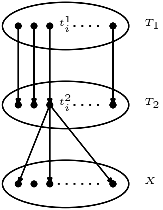
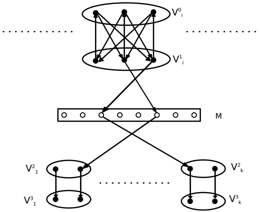
Theorem (F., Gras, Perez, Sikora, 2020)

BROADCAST DOM is NP-hard and $W[2]$-hard: likely no algorithm of the form $f(k)poly(n)$, for any computable function f.

Proof: Reductions from SET COVER.





BROADCAST DOM
Input: A (directed) graph G, an integer k.
Question: Does G have a dominating broadcast of cost at most k?

Theorem (F., Gras, Perez, Sikora, 2020)

BROADCAST DOM is NP-hard and $W[2]$-hard: likely no algorithm of the form $f(k)poly(n)$, for any computable function f.

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Theorem (F., Gras, Perez, Sikora, 2020)

There is an $O(c^k n)$-time algorithm for BROADCAST DOM for directed acyclic graphs.

Proof:

Lemma: There always exists an optimal broadcast where each broadcasting vertex is covered only by itself.

$\rightarrow$ iterative branching, starting from the sources.

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Theorem (F., Gras, Perez, Sikora, 2020)

There is a linear-time algorithm for BROADCAST DOM on single-source layered directed graphs.

Proof:

Lemma: there always exists an optimal broadcast where the broadcasting vertices are all in layers of size 1, and no vertex is covered twice.

$\rightarrow$ Easy top-down procedure.

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Complexity for BROADCAST DOM (4)

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Theorem (F., Gras, Perez, Sikora, 2020)

BROADCAST DOM is FPT parameterizd by solution cost k and maximum degree d.

Proof:

A YES-instance has at most $k(k+1)d^k$ vertices.

MULTIPACKING
Input: A (directed) graph G, an integer k.
Question: Does G have a multipacking of size at least k?

(Note: OPEN for undirected graphs.)

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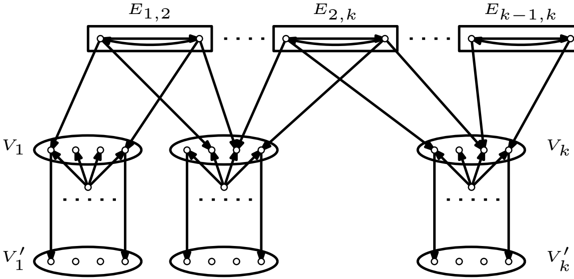
Proof: Reduction from INDEPENDENT SET.



Complexity of Multipacking (2)

MULTIPACKING
Input: A (directed) graph G, an integer k.
Question: Does G have a multipacking of size at least k?

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Theorem (F., Gras, Perez, Sikora, 2020)

There is a linear-time algorithm for MULTIPACKING on single-source layered directed graphs.

Proof:

Lemma: There always exists an optimal multipacking that intersects each layer at most once.

$\rightarrow$ Bottom-up dynamic programming.

MULTIPACKING
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Theorem (F., Gras, Perez, Sikora, 2020)

MULTIPACKING is FPT parameterizd by solution cost k and maximum degree d.

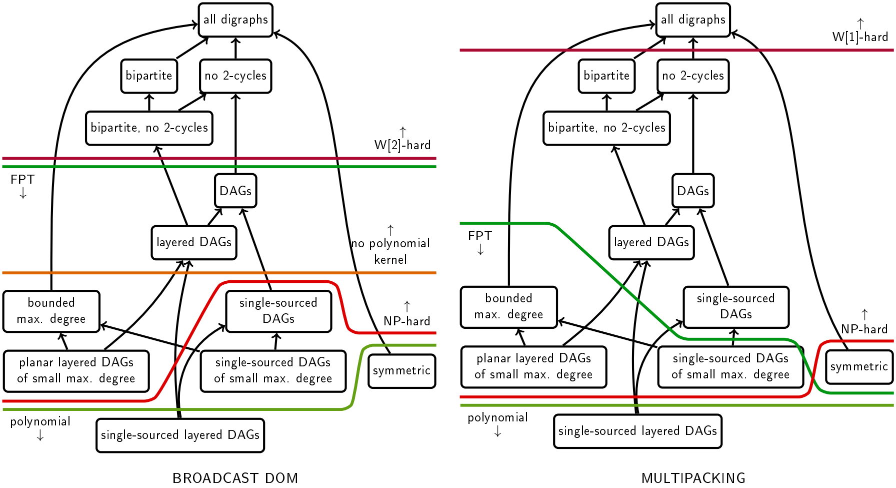
Proof:

If G has a path of length $3k-3$: return YES.

If there is a *minimum absorbing set* of size k (computable by reduction to HITTING SET): return YES.

Otherwise: the instance has at most $d^{O(k)}$ vertices.

Complexity landscapes



Open questions

Bounds:

- Is the conjecture true that for any undirected graph G, $\gamma_b(G) \leq 2mp(G)$?
- What is a tight bound for connected undirected graphs? $\gamma_b(G) \leq \frac{4}{3}mp(G)$?
- What is a tight bound for connected undirected chordal graphs? $\gamma_b(G) \leq \frac{10}{9}mp(G)$?
- What about directed graphs?

Complexity:

- Is MULTIPACKING NP-hard on undirected graphs?
- Is MULTIPACKING FPT for DAGs?

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- Is MULTIPACKING NP-hard on undirected graphs?
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Thanks!