Broadcast domination and multipacking in graphs and digraphs

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joint works with:

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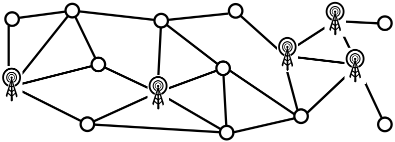
Covering: cover the vertices of a graph using as few structures as possible

Example: dominating set: covering using 1-balls
$\rightarrow$ domination number $\gamma(G)$



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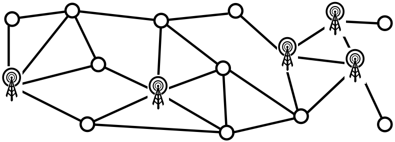
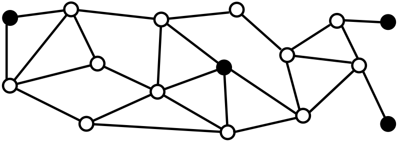


Packing: pack as many structures as possible without interference

Example: dist. 3-independent set / 2-packing: packing 1-balls without overlap
$\rightarrow$ 2-packing number $\rho_2(G)$



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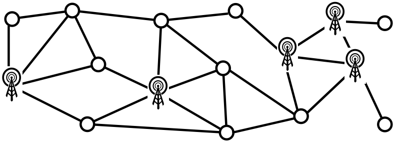

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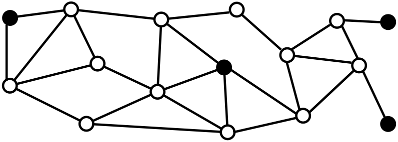


These problems are dual (in the sense of LP) and $\rho_2(G) \leq \gamma(G)$.

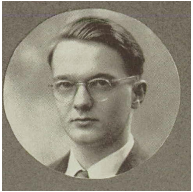
Classic packing and covering problems are typically **NP-hard** (Karp, 1971):

$\Rightarrow$ Unless $P = NP$, there exists no efficient algorithm to solve them.

"Efficient algorithm": polynomial-time in terms of the size of the input graph (Cobham, Edmonds, 1965)

Richard C. Karp (1935-)

Jack Edmonds (1934-)



Alan B. Cobham (1927-2011)

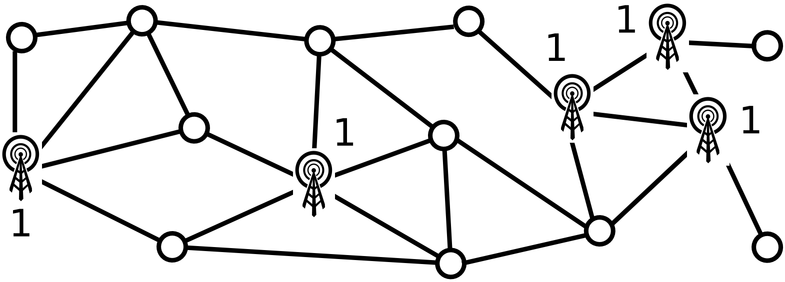
Definition - Dominating broadcast of graph G (Erwin, 2001)

A function $f : V(G) \to \mathbb{N}$ s.t. for every $v \in V(G)$, there exists $x \in V(G)$ with

- $f(x) > 0$ and
- $f(x) \geq d_G(x, v)$.

The cost of f is $\sum_{v \in V(G)} f(v)$.

Broadcast number $\gamma_b(G)$: smallest cost of a dominating broadcast of G.



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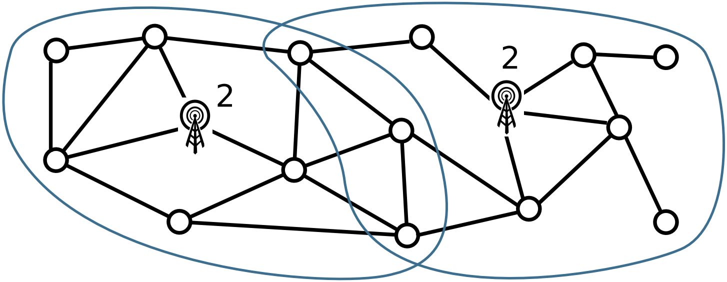

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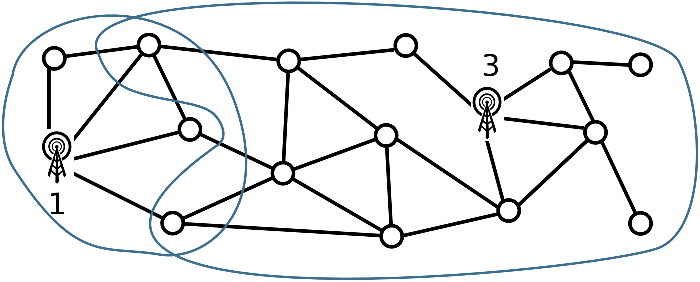

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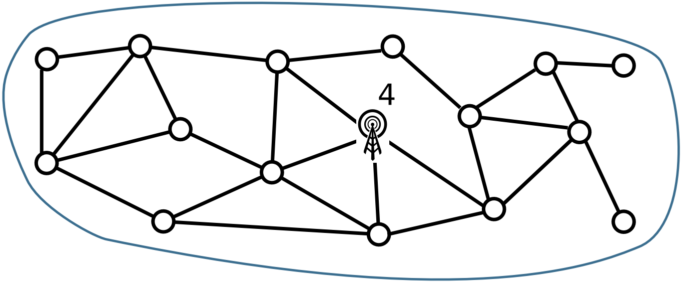

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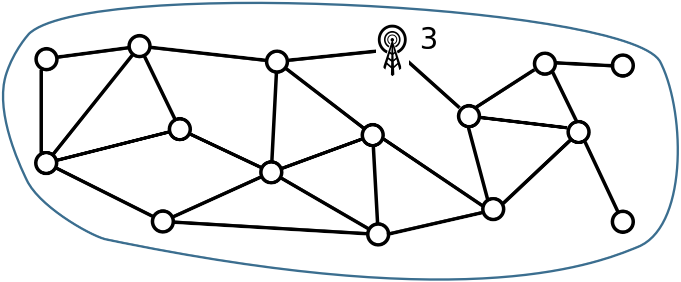


n : number of vertices of the input graph

Theorem (Heggernes-Lokshtanov, 2006)

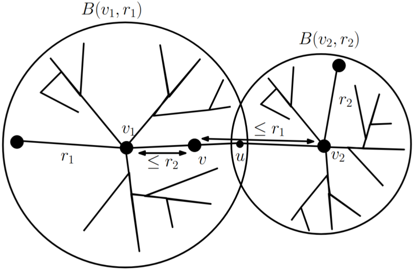
We can find a minimum-cost dominating broadcast in polynomial time $O(n^6)$.

Proof idea:

- sufficient to find an efficient dominating broadcast (Erwin, 2001)
- The structure of broadcasting balls is a path or a cycle
- Dynamic programming on this structure

Lemma (Erwin, 2001)

In an optimum broadcast which minimizes the number of broadcasting balls, no two balls intersect.



Assume $r_1 \geq r_2$ and let u be in $B(v_1, r_1) \cap B(v_2, r_2)$.

Let u be the vertex on the shortest $v_1 - v$ path at distance r_2 from v_1.

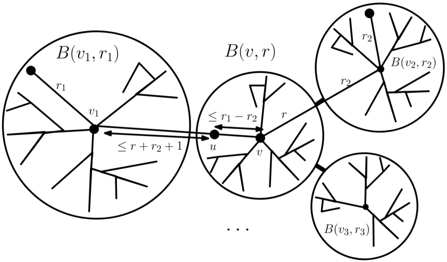
$\rightarrow$ Replace $B(v_1, r_1)$ and $B(v_2, r_2)$ by $B(u, r_1 + r_2)$.

Image credit: Behsaz-Salavatipour, 2015

Domination graph has broadcasting balls as vertex set, and two balls are adjacent IFF there is an edge joining some vertices of the balls in G.

Lemma (Heggernes-Lokshtanov, 2006)

In an optimum efficient broadcast which minimizes the number of broadcasting balls, every ball has maximum degree 2 in the domination graph.



Assume $r_1 \geq r_2 \geq r_3$.

u : vertex on shortest $v - v_1$ path at distance $\min\{r_1 + r + 1, r_1 - r_2\}$ from v.

$\rightarrow$ Replace the four balls by $B(u, r + r_1 + r_2 + 1)$.

Image credit: Behsaz-Salavatipour, 2015

Vertices: $v_1, \ldots, v_n$.

$x_{i,k} \in \{0,1\}$: whether vertex v_i broadcasts with radius k

We want to minimize:

$$\sum_{k=1}^{n} \sum_{i=1}^{n} k \cdot x_{i,k}$$

subject to:

$$\sum_{d(v_i,v_j) \leq k} x_{i,k} \geq 1 \text{ for each vertex } v_j.$$

Broadcast domination: ILP formulation

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Dual ILP:

We want to maximize:

$$\sum_{i=1}^{n} y_i$$

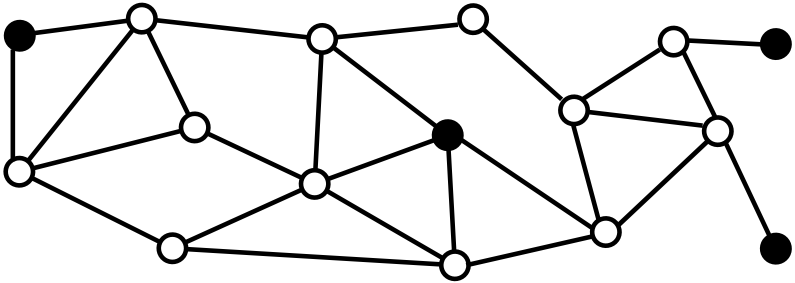
subject to:

$$\sum_{d(v_i,v_j) \leq k, y_j \geq 0} y_i \leq k \text{ for each vertex } v_j \text{ and integer } k \leq n.$$

Definition - Multipacking of graph G (Brewster-Mynhardt-Teshima, 2014)

A set S of vertices s.t. for every $v \in V(G)$ and every $d \in \mathbb{N}$, the d-ball $B_d(v)$ contains at most d vertices of S.

Multipacking number $mp(G)$: largest size of a multipacking of G.



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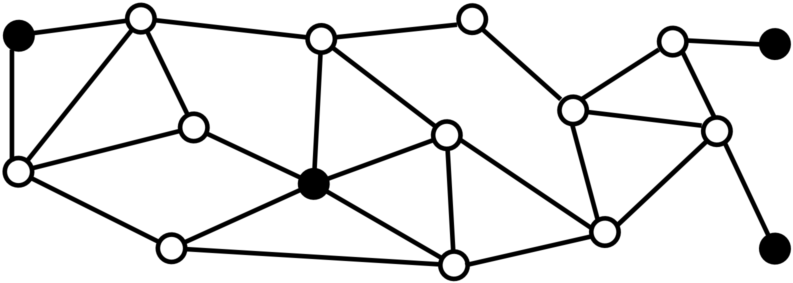

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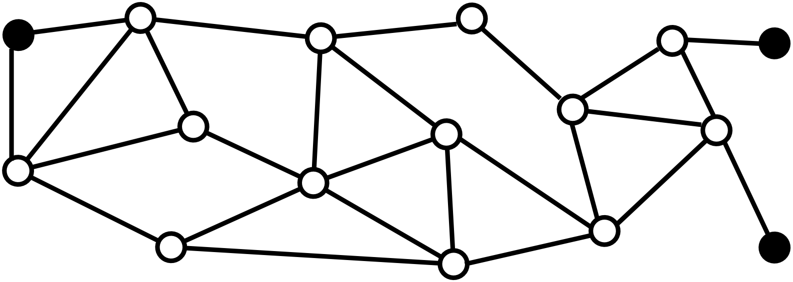

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Open question

Can one compute a maximum-size multipacking in polynomial time?

Bounds for undirected graphs : general graphs

The two problems are dual (in the sense of LP).

Proposition

For every graph G, we have $mp(G) \leq \gamma_b(G)$.

Equality holds for:

- trees (Mynhardt-Teshima, 2017)
- more generally, strongly chordal graphs (Brewster-MacGillivray-Yang, 2019)
- rectangular grids (Beaudou-Brewster, 2019)

A chain of inequalities

diameter $diam(G)$: largest distance between two vertices in G

eccentricity of a vertex v: largest possible distance from v to another vertex

radius $rad(G)$: smallest eccentricity among all vertices

Proposition (Erwin, 2001 + Hartnell-Mynhardt, 2014)

For any graph G, $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq rad(G) \leq diam(G)$.

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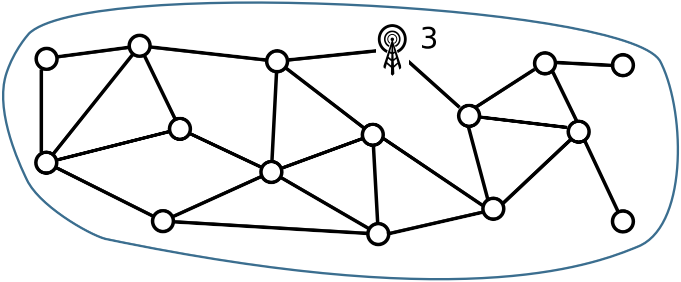
$\gamma_b(G) \leq rad(G)$: consider a radial vertex v. Set $f(v) = rad(G)$.



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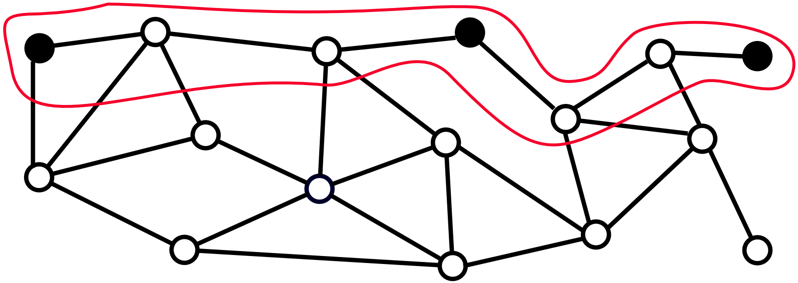
$\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G)$: consider a diametral path P, select every third vertex.



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For any graph G, $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq rad(G) \leq diam(G)$.

Corollary

For any graph G, we have $\gamma_b(G) < 3mp(G)$, hence $\frac{\gamma_b(G)}{mp(G)} < 3$.

Question (Hartnell-Mynhardt, 2014)

What is the largest possible ratio $\frac{\gamma_b(G)}{mp(G)}$?

Our theorem

Theorem (Beaudou, Brewster, F., 2019)

For any graph G, we have $\gamma_b(G) \leq 2mp(G) + 3$, hence $\frac{\gamma_b(G)}{mp(G)} \leq 2 + \epsilon$.

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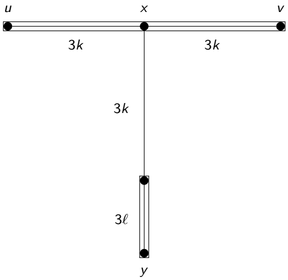
Lemma

Proof sketch

Let u, v, x, y be 4 vertices with:

- $d(u, v) = 6k$
- $d(x, u) = d(x, v) = 3k$
- $d(x, y) = 3k + 3\ell$.

Then, $mp(G) \geq 2k + \ell$.



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Then, $mp(G) \geq 2k + \ell$.

Let $diam(G) = 6k + i$ and $rad(G) = 3k + 3\ell + j$ ($0 \leq i < 6$ and $0 \leq j < 3$)
Apply the lemma with x, a vertex of eccentricity $rad(G)$.

$$
\begin{aligned}
mp(G) &\geq 2k + \ell \\
&\geq \frac{diam(G)}{3} + \frac{rad(G)}{3} - \frac{diam(G)}{6} - c \\
&\geq \frac{rad(G)}{2} - c \\
&\geq \frac{\gamma_b(G)}{2} - c
\end{aligned}
$$

□

Conjecture

Theorem (Beaudou, Brewster, F., 2019)

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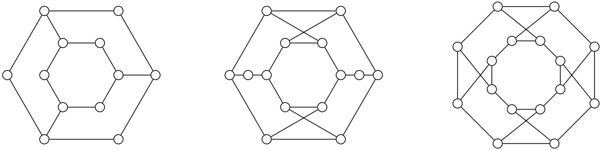
Conjecture would be tight — infinitely many graphs G s.t. $\gamma_b(G) = 2mp(G)$:



$mp(G) = 2$ and $\gamma_b(G) = 4$

Connected graphs

Question

What happens for connected graphs?

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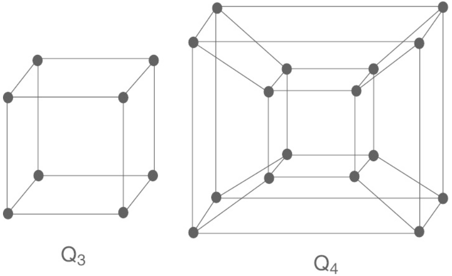
Theorem (Brešar, Špacapan, 2019)

For the hypercube H_d: $\gamma_b(H_d) = d - 1$.

Theorem (Rajendraprasad, Sani, Sasidharan, Sen, 2025+)

For the hypercube H_d: $\lfloor \frac{d}{2} \rfloor \leq mp(H_d) \leq \frac{d}{2} + 6\sqrt{2d}$.



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Corollary

For connected graphs G, $\lim_{mp(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{mp(G)} \right\} = 2$.

Bounds for undirected graphs : chordal graphs

Chordal graph: graph where every cycle of length 4 or more has a *chord*

Proposition

If G is a chordal graph, $\gamma_b(G) \leq \left\lceil \frac{3}{2} mp(G) \right\rceil$.

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Proposition

If G is a chordal graph, $\gamma_b(G) \leq \left\lceil \frac{3}{2} mp(G) \right\rceil$.

This can be proved using the following two theorems:

Theorem (Laskar, Shier, 1983)

If G is a chordal graph with radius r and diameter d, then $2r \leq d + 2$.

Theorem (Erwin 2001 & Hartnell-Mynhardt 2014)

If G is a connected graph of order at least 2 having radius r, diameter d, multipacking number $mp(G)$, broadcast domination number $\gamma_b(G)$ and domination number $\gamma(G)$, then $\left\lceil \frac{d+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq min\{\gamma(G), r\}$.

Bound for chordal graphs

Theorem (Das, F., Islam, Mukherjee, 2023)

For connected chordal graphs G, $\frac{10}{9} \leq \lim_{mp(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{mp(G)} \right\} \leq \frac{3}{2}$.

Theorem (Das, F., Islam, Mukherjee, 2023)

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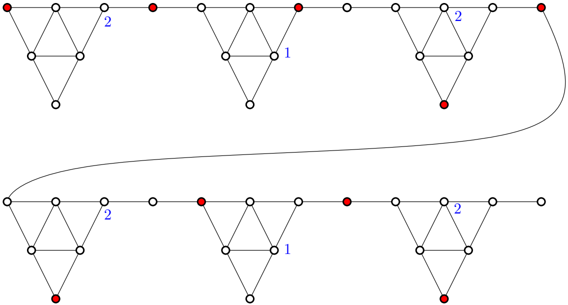
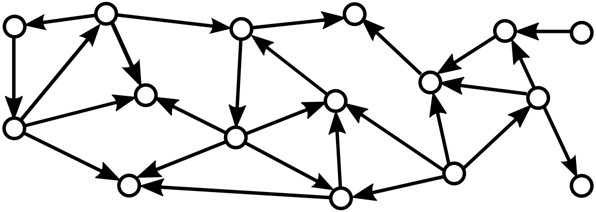
Lemma

$mp(G_{2k}) \leq 9k$ and $\gamma_b(G_{2k}) = 10k$.



Complexity & algorithms for directed graphs

Broadcast domination in directed graphs



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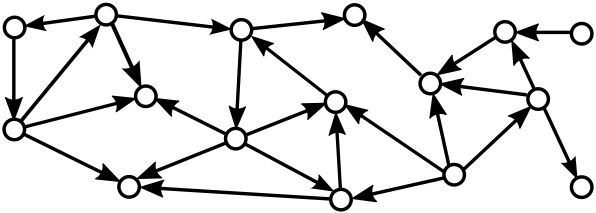
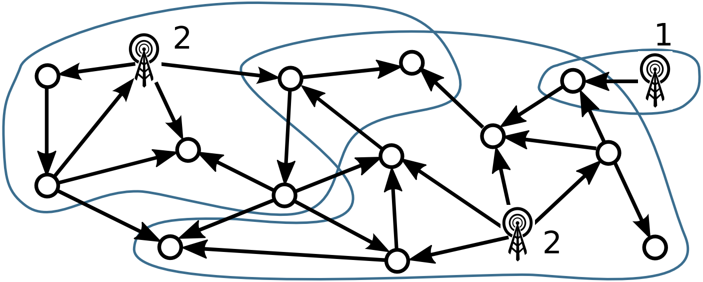
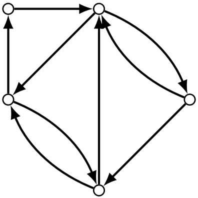


Note: an undirected graph can be seen as a symmetric directed graph!

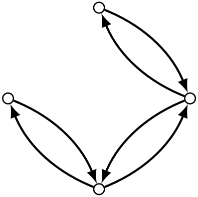


Broadcast domination for directed graphs:

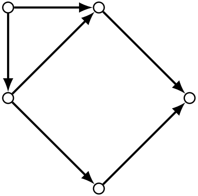
A vertex v with $f(v) = r$ broadcasts to all vertices at directed distance up to r.



General directed graphs (digraphs)



Symmetric digraphs = undirected graphs



Directed acyclic graphs (DAGs)

BROADCAST DOM
Input: A (directed) graph G, an integer k.
Question: Does G have a dominating broadcast of cost at most k?

Theorem (Heggernes-Lokshtanov, 2006)

BROADCAST DOM can be solved in polynomial time $O(n^6)$ for **undirected graphs**.

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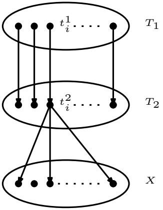
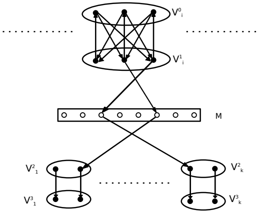
Theorem (F., Gras, Perez, Sikora, 2020)

BROADCAST DOM is NP-hard and $W[2]$-hard: likely no algorithm of the form $f(k)poly(n)$, for any computable function f.

Proof: Reductions from SET COVER.





Algorithmic complexity for BROADCAST DOM (2)

BROADCAST DOM
Input: A (directed) graph G, an integer k.
Question: Does G have a dominating broadcast of cost at most k?

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FPT $k^{O(k)}n$-time algorithm for BROADCAST DOM for directed acyclic graphs.

Proof: *Lemma: There always exists an optimal broadcast where each broadcasting vertex is covered only by itself.*
$\rightarrow$ iterative branching, starting from the sources: a previously uncovered vertex is either covered by itself, or by one of its broadcasting predecessors. □

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(Note: OPEN for undirected graphs.)

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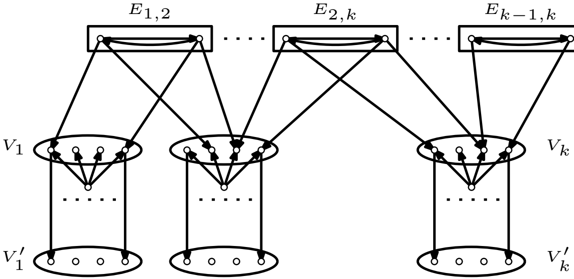
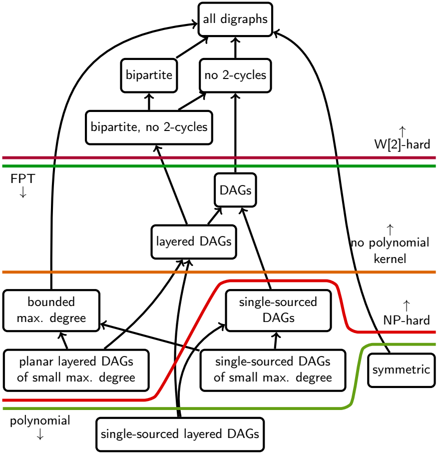
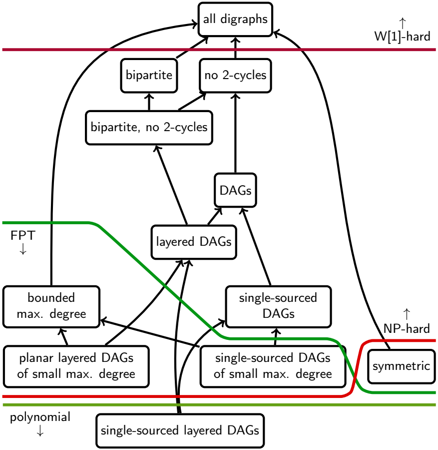
Proof: Reduction from INDEPENDENT SET.





BROADCAST DOM



MULTIPACKING

Concluding remarks

Takeaway: Problems tend to become more difficult on digraphs (even DAGs)

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Open problems:

Bounds:

- Conjecture: $\gamma_b(G) \leq 2mp(G)$ for any undirected graph G
- Better bounds for the multipacking number of the hypercube H_d?
- What is a tight bound for connected undirected chordal graphs? $\gamma_b(G) \leq \frac{10}{9}mp(G)$?

Algorithmic complexity:

- Is MULTIPACKING NP-hard on undirected graphs?
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- Complexity of both problems on oriented trees?

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Thanks!