Bounds on the order of a graph of given metric dimension and diameter

studies for standard graph classes

Florent Foucaud

CanaDAM 2021

Miniymposium: The Metric Dimension of a Graph and its Variants - Part I (CM15)

based on joint works with:

George Mertzios, Reza Naserasr, Aline Parreau, Petru Valicov

Laurent Beaudou, Peter Dankelmann, Mike Henning, Arnaud Mary, Aline Parreau




MICHELIN

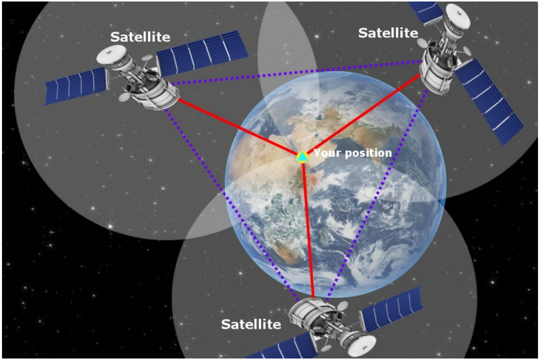


Pascal


Saint Nectaire

Determination of Position in 3D euclidean space

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



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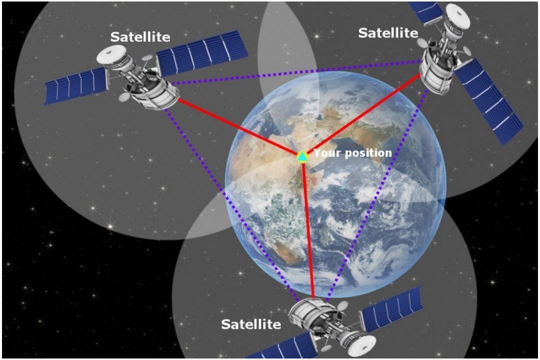

Question

How can we transpose the "GPS" approach for graphs?

Metric dimension

$w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

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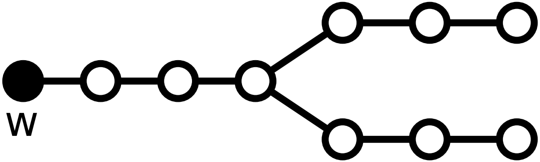

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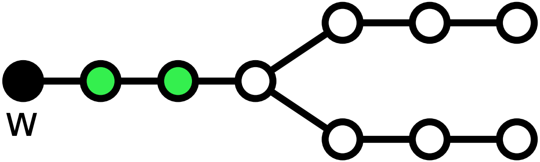

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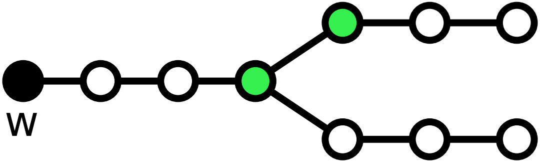

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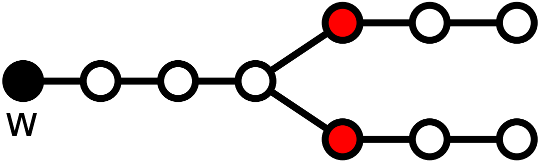

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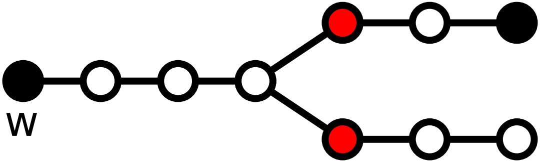

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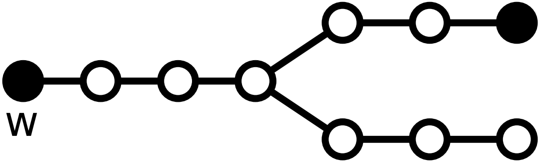

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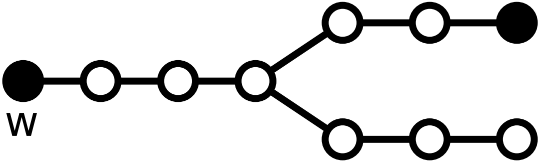
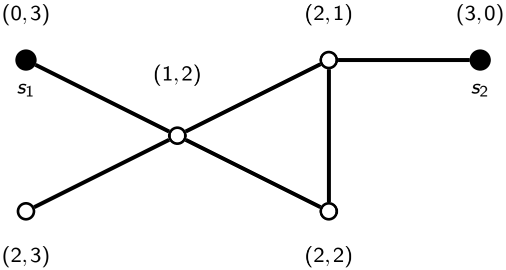


$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Example

The resolving set $\{s_1, s_2\}$ assigns unique coordinates to each vertex:



$S = \{s_1, s_2\}$

$MD(G) = 2$

Proposition

$MD(G) = 1 \Leftrightarrow G$ is a path

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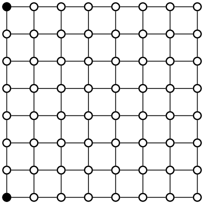

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For any square grid G, $MD(G) = 2$.

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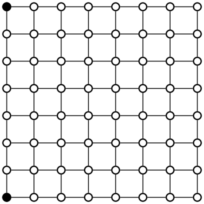

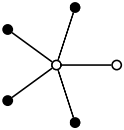

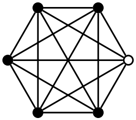

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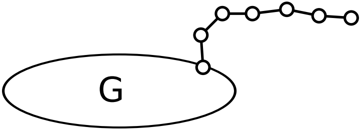
Proposition

$$MD(K_n) = n - 1 \text{ and } MD(K_{1,n-1}) = n - 2.$$

Trees

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.



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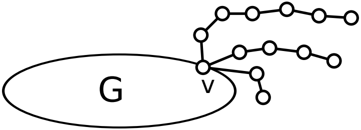

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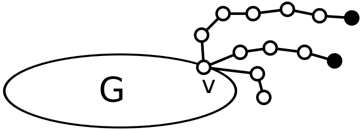

Observation

R resolving set. If v has k legs, at least $k-1$ legs contain a vertex of R.

Simple leg rule: if v has $k \geq 2$ legs, select $k-1$ leg endpoints.

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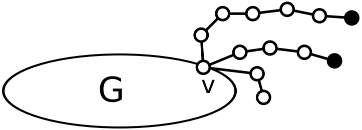

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Theorem (Slater, 1975 )

For any tree, the simple leg rule produces an optimal resolving set.

See also:

- Harary-Melter, 1976  
- Chartrand, Eroh, Johnson, Oellermann, 2000    
- Khuller, Raghavachari & Rosenfeld, 2002   

Theorem (Khuller, Raghavachari & Rosenfeld, 2002   )

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter: maximum distance between two vertices)

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Theorem (Hernando, Mora, Pelayo, Seara, Wood 2010     )

G of order n, diameter D, $MD(G) = k$.

Then $n \leq (\lfloor \frac{2D}{3} \rfloor + 1)^k + k \sum_{i=1}^{\lceil D/3 \rceil} (2i-1)^{k-1}$. (Tight.)

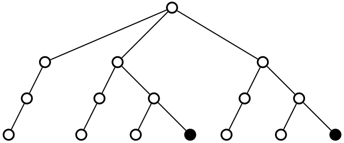
Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

T a tree with diameter D and $MD(T) = k$, then

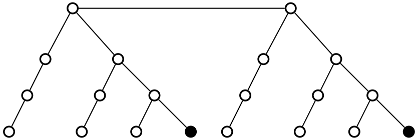
$$n \leq \begin{cases} \frac{1}{8}(kD+4)(D+2) & \text{if } D \text{ even,} \\ \frac{1}{8}(kD-k+8)(D+1) & \text{if } D \text{ odd.} \end{cases} = \Theta(kD^2)$$

Bounds are tight.

$k = 2,\ D = 6$



$k = 2,\ D = 7$



Bounds using diameter: using tree decompositions

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

G of order n, diameter D, $MD(G) = k$ with tree decomposition of width w and length ℓ. Then

$$n = O(kD^2(2\ell+1)^{3w+1})$$

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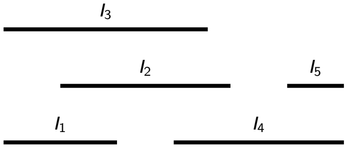
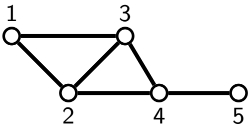
Applications :

- chordal graphs : $\ell = 1 \longrightarrow n = O(kD^2 2^{O(w)}) = O(D^2 2^{2^{O(k)}})$
- tree-width $w \longrightarrow n = O(kD^{3w+3})$

Definition - Interval graph

Intersection graph of intervals of the real line.





Bounds using diameter: interval graphs

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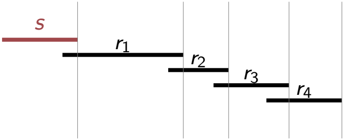

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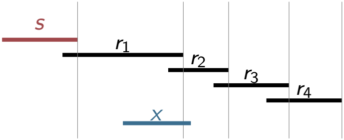

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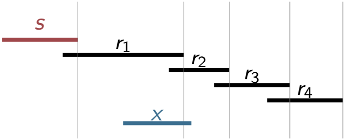


$r_1, r_2, ..$: rightmost path

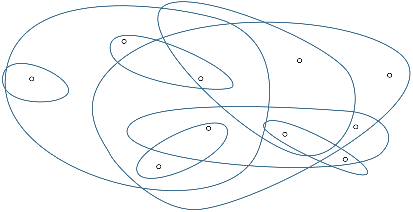
$d(x, s) = 2$

- Partition of $\mathbb{R}$ into $kD + 1$ parts
- Any interval is uniquely determined by beginning + end zone: $n - k \leq (kD + 1)^2$

Vapnik-Červonenkis dimension

Measure of intersection complexity of sets in a hypergraph $(X, \mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered: for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.



V-C dimension of H: maximum size of a shattered set in H

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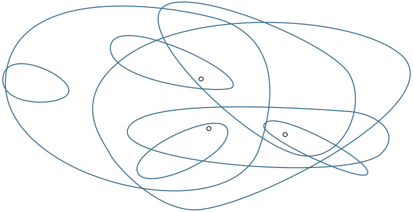

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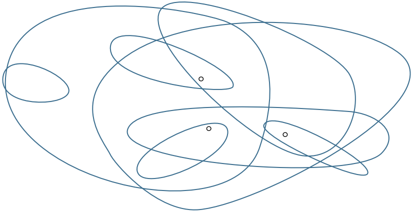

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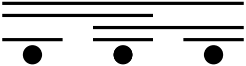
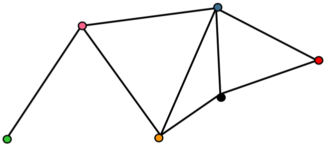
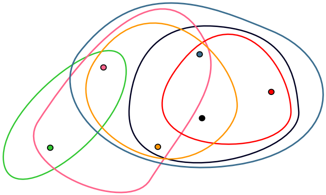
Typically bounded for geometric hypergraphs: 

V-C dimension of a graph: V-C dimension of its closed neighbourhood hypergraph





Vapnik-Červonenkis dimension - graphs

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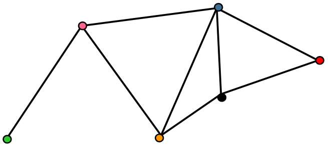

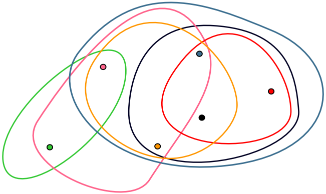

Typically bounded for geometric intersection graphs:

→ interval graphs ($d = 2$)
→ C_4-free graphs ($d = 2$)
→ line graphs ($d = 4$)
→ permutation graphs ($d = 3$)
→ unit disk graphs ($d = 3$)
→ planar graphs ($d = 4$)

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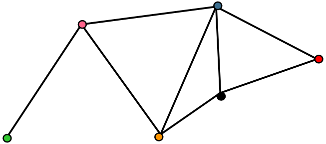
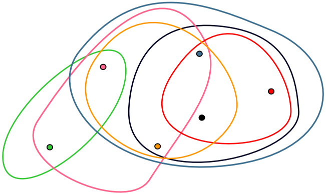
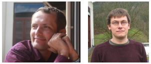
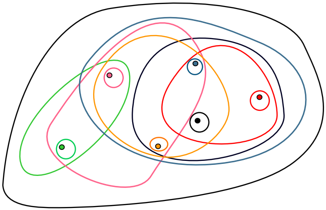






Distance V-C dimension of a graph (Bousquet-Thomassé, 2015): V-C dimension of its ball hypergraph



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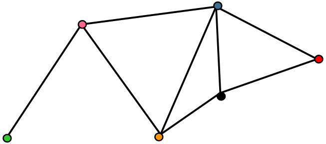

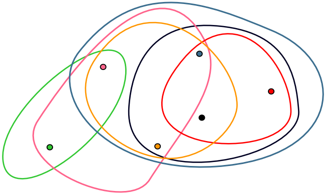

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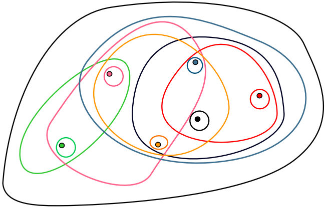

Bounded for:

- graphs of bounded rank-width
- interval graphs
- graphs with no K_t-minor

(Bousquet-Thomassé, 2015)

Bounds using diameter: the V-C dimension

dual distance V-C dimension of G: V-C dimension of the dual of its ball hypergraph.

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

G of order n, diameter D, $MD(G) = k$ with dual distance V-C dimension d^*:

$$n \leq (Dk+1)^{d^*} + 1.$$

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If G is K_t-minor-free, then $d^* \leq t-1$.

(Proof based on the ideas of Bousquet and Thomassé, 2015.)

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Corollary (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

G planar with diameter D and $MD(G) = k$, then $n = O(k^4 D^4)$.

Bounds using diameter: planar graphs

Using distance-V-C dimension:

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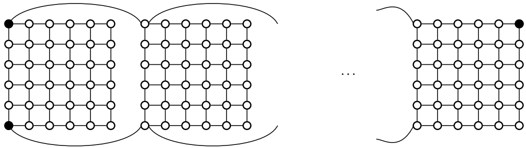
Tight? Example with $k = 3$ and $n = \Theta(D^3)$:



Conclusion

Graph class	Upper bound for n	Worst example
All	$O(D^k)$	$\Theta(D^k)$
Trees	$O(D^2 k)$	$\Theta(D^2) k$
Chordal	$2^{2^{O(k)}} D^2$	
Tree-width w	kD^{3w+3}	
Interval	$O(Dk^2)$	$\Theta(Dk^2)$
Unit interval	$O(Dk)$	$\Theta(Dk)$
Permutation	$O(Dk^2)$	$\Theta(Dk^2)$
Bipartite permutation	$O(Dk)$	$\Theta(Dk)$
Cographs	$O(k)$	$\Theta(k)$
Planar	$D^4 k^4$	$\Theta(D^3 k)$
Outerplanar	$O(D^2 k)$	$\Theta(D^2 k)$
K_t-minor-free	$D^{t-1} k^{t-1}$	
Dual dist. V-C dim. d^*	$k^{d^*} D^{d^*}$	

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Some questions:

- Chordal: $n \leq 2^{2^{O(k)}} D^2$. Better bound?
- Tree-width w: $n = O(kD^{3w+3})$. Better bound?
- Planar: $n = O(k^4 D^4)$. Does $n \leq f(k) D^3$ hold?
- K_4-minor-free (aka series-parallel/t-width 2): $n = O(k^3 D^3)$ and $n = O(kD^9)$. Does $n = O(kD^2)$ hold, like for outerplanar graphs?

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THANKS FOR YOUR ATTENTION!

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