Extremal graphs for domination-based identification

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LIMOS

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Location-domination in graphs

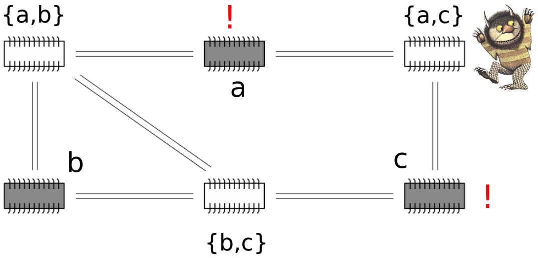
Location-domination

Definition - Locating-dominating set (Slater, 1980's) 

$D \subseteq V(G)$ locating-dominating set of G:

- for every $u \in V$, $N[v] \cap D \neq \emptyset$ (domination).
- $\forall u \neq v$ of $V(G) \setminus D$, $N(u) \cap D \neq N(v) \cap D$ (location).

Motivation: fault-detection in networks.
$\rightarrow$ The set D of grey processors is a set of fault-detectors.



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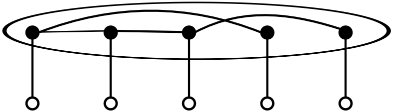
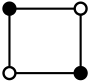
Notation. location-domination number $LD(G)$,
smallest size of a locating-dominating set of G

Domination number: $DOM(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Location-domination number: $LD(P_n) = \left\lceil \frac{2n}{5} \right\rceil$

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Tight examples:  

Upper bounds

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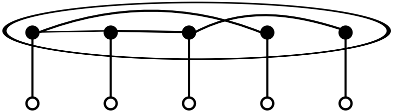
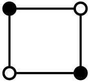
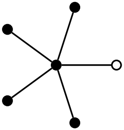
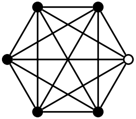
Tight examples:  

Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Tight examples:  

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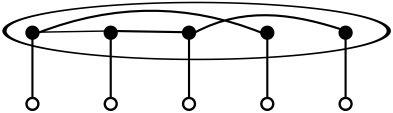
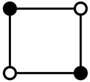
Tight examples:  

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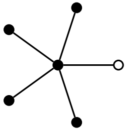
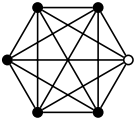
Tight examples:  

Remark: tight examples contain many twin-vertices!!

Upper bound: a conjecture

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Remark:

- twins are easy to detect
- twins have a trivial behaviour w.r.t. location-domination

Upper bound: a conjecture

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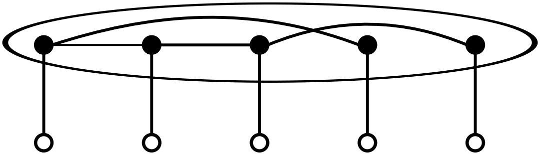
If true, tight: 1. domination-extremal graphs



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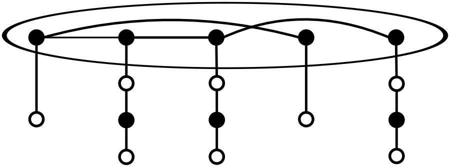
If true, tight: 2. a similar construction



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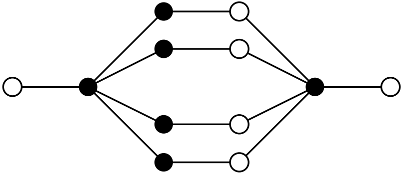
If true, tight: 3. a family with domination number 2



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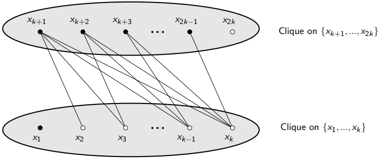
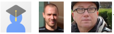
If true, tight: 4. family with dom. number 2: complements of half-graphs



Conjecture (Garijo, González & Márquez, 2014 )

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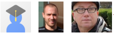
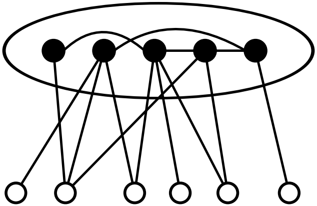
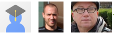
Theorem (Garijo, González & Márquez, 2014 )

Conjecture true if G has independence number $\geq n/2$.
(in particular, if bipartite)

Proof: every vertex cover is a locating-dominating set

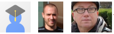


Upper bound: a conjecture - special graph classes

Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

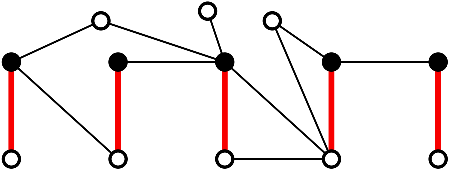
$\alpha'(G)$: matching number of G

Theorem (Garijo, González & Márquez, 2014 )

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Proof:

- Consider special maximum matching M
- Select one vertex in each edge of M



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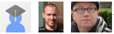
Conjecture (Garijo, González & Márquez, 2014 )

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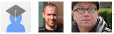
Theorem (Garijo, González & Márquez, 2014 )

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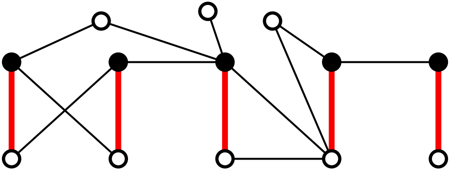
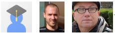


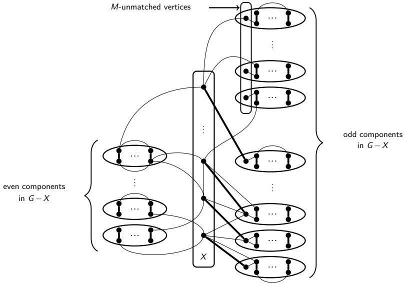
Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

Conjecture true if G is cubic.

Proof: Involved argument using maximum matching and Tutte-Berge theorem.



Upper bound: a conjecture - special graph classes

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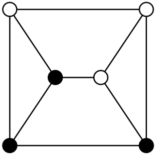
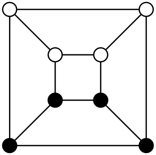
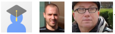
Bound is tight:





Question

Do we have $LD(G) = \frac{n}{2}$ for other cubic graphs?

Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

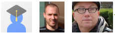
Conjecture true if G is cubic.

Question

Are there twin-free (cubic) graphs with $LD(G) > \alpha'(G)$?

(if not, conjecture is true)

Upper bound: a conjecture - special graph classes

Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2016 )

Conjecture true if G is split graph or complement of bipartite graph.

Line graph of G: intersection graph of the edges of G.

Theorem (F., Henning, 2017 )

Conjecture true if G is a line graph.

Proof: By induction on the order, using edge-locating-dominating sets

Upper bound: a conjecture - general bound

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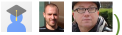
Conjecture (Garijo, González & Márquez, 2014 )

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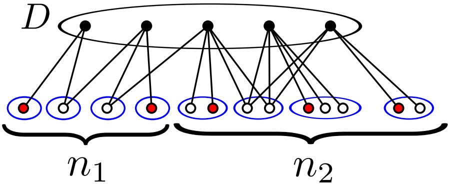
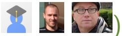


Conjecture (Garijo, González & Márquez, 2014 )

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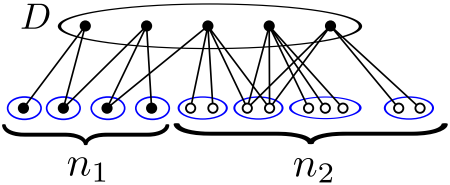

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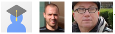
Conjecture (Garijo, González & Márquez, 2014 )

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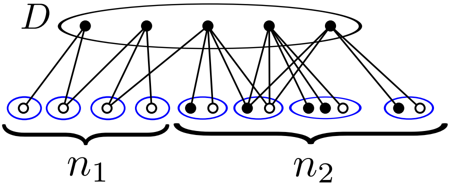

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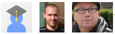
Conjecture (Garijo, González & Márquez, 2014 )

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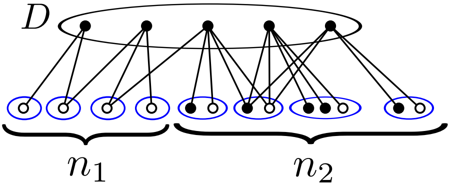
• there is a LD-set of size $|D| + n_1$; there is a LD-set of size $n - n_1 - n_2$

• $\min\{|D| + n_1, n - n_1 - n_2\} \leq \frac{2}{3}n$



Open neighbourhood location-domination

Open neighbourhood locating-dominating sets

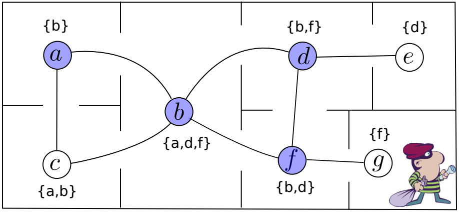
G: undirected graph $N(u)$: set of neighbours of v

Definition - OLD set (Seo, Slater, 2010  )

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating set**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Notation. $OLD(G)$: OLD number of G,
minimum size of an OLD-set in G



Examples: paths

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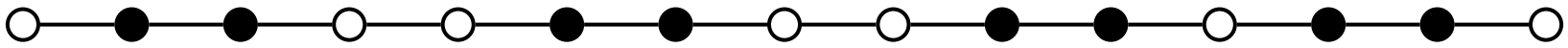
Total domination number: $\gamma_t(P_n) \approx \lceil \frac{n}{2} \rceil$



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Total domination number: $\gamma_t(P_n) \approx \lceil \frac{n}{2} \rceil$

OLD-number: $OLD(P_n) \approx \lceil \frac{2n}{3} \rceil$

Remark

Not all graphs have an OLD set!

An **isolated vertex** cannot be totally dominated.

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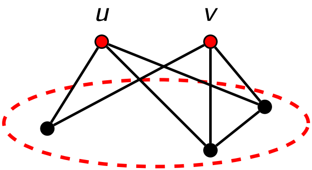
Not all graphs have an OLD set!

An **isolated vertex** cannot be totally dominated.

Open twins = pair u, v such that $N(u) = N(v)$.



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An **isolated vertex** cannot be totally dominated.

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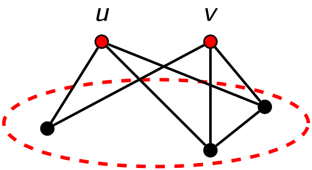
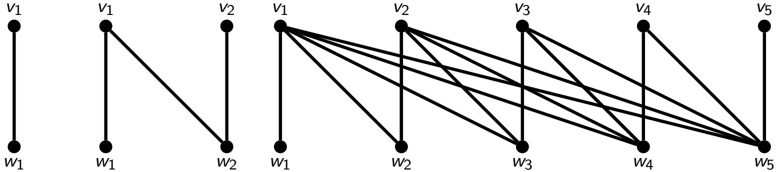


Proposition

A graph is **locatable** if and only if it has no **isolated vertices** and **open twins**.

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.



$H_1 = P_2$ $H_2 = P_4$ H_5

Upper bound on $OLD(G)$?

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Some vertices are **forced** to be in any OLD-set because of **domination**

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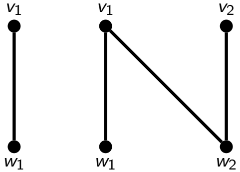

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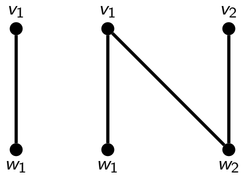

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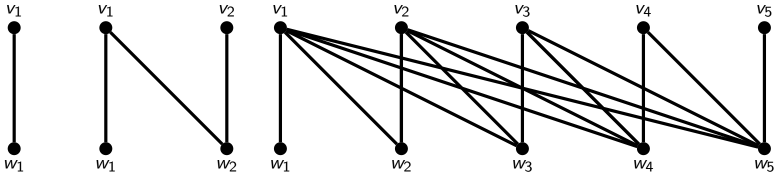

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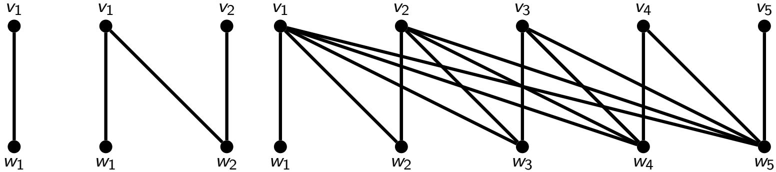

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Some vertices are **forced** to be in any OLD-set because of **domination** or **location**

Proposition

For every half-graph H_k of order $n = 2k$, $OLD(H_k) = n$.

Characterizing “bad graphs” for OLD-sets

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.

Then, $OLD(G) = n$ if and only if G is a half-graph.

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$G' = G - \{x, y\}$ is locatable, connected and has $OLD(G') = n - 2$.

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By induction, G' is a half-graph. We can conclude that G is a half-graph too.

$\square$

Identifying codes in graphs

Identifying codes

G: undirected graph

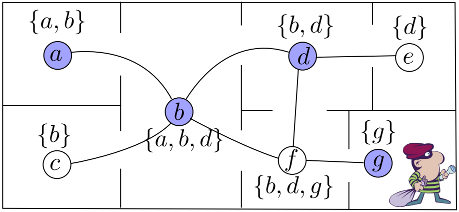
$N[u]$: set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that:

- C is a **dominating set**: $\forall u \in V(G)$, $N[u] \cap C \neq \emptyset$, and
- C is a **separating set**: $\forall u \neq v$ of $V(G)$, $N[u] \cap C \neq N[v] \cap C$

$ID(G)$: identifying code number of G,
minimum size of an identifying code in G



Examples: paths

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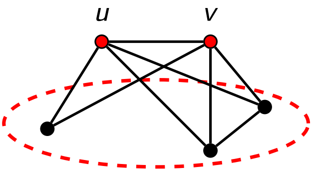
Domination number: $DOM(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Identifying code number: $ID(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$

Remark

Not all graphs have an identifying code!

Closed twins = pair u, v such that $N[u] = N[v]$.



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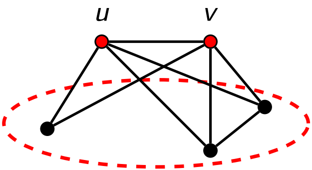

Proposition

A graph is **identifiable** if and only if it has **no closed twins**.

n: number of vertices

Theorem (Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

G identifiable graph on n vertices with at least one edge:

$$ID(G) \leq n-1$$

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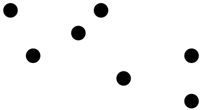
$ID(G) = n \Leftrightarrow G$ has no edges



Bounds on $ID(G)$

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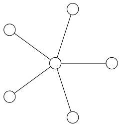

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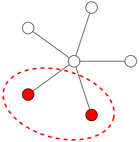

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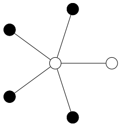

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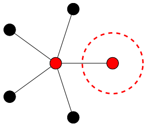

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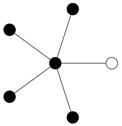

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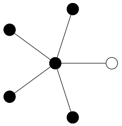

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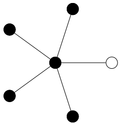

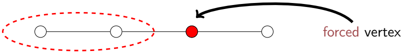

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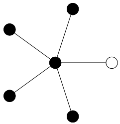

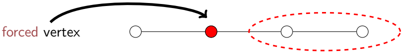

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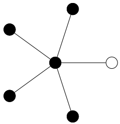

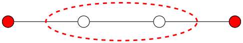

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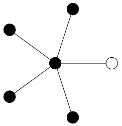
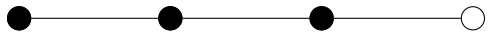




Question

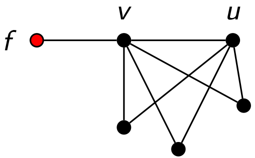
What are the graphs G with n vertices and $ID(G) = n - 1$?

Forced vertices

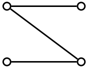
u, v such that $N[v] \ominus N[u] = \{f\}$:

f belongs to **any identifying code**

$\rightarrow$ f forced by u, v.



Special path powers: $A_k = P_{2k}^{k-1}$



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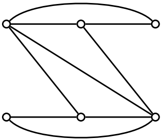

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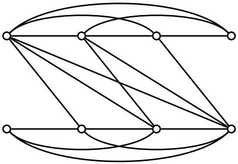

$A_4 = P_8^3$

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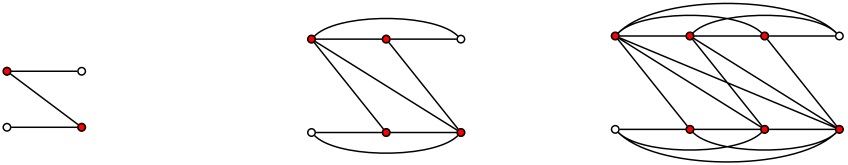

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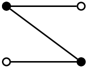

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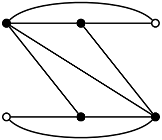

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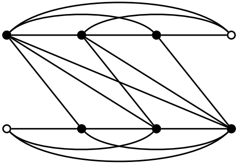

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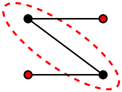

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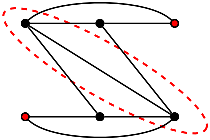

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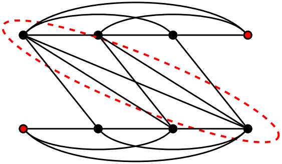

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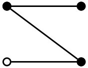

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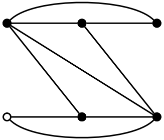

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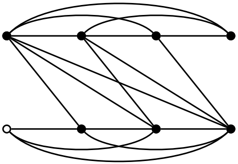

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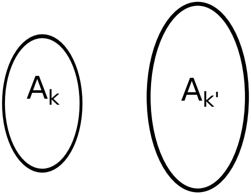
Proposition

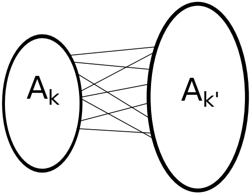
$$ID(A_k) = n - 1$$

Constructions using joins

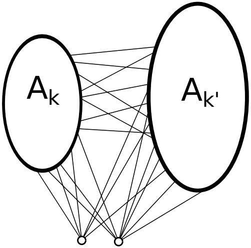


Two graphs A_k and $A_{k'}$

Constructions using joins

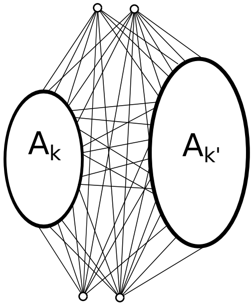


Join: add all edges between them



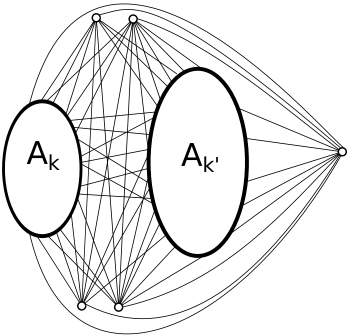
Join the new graph to two non-adjacent vertices ($\overline{K_2}$)

Constructions using joins



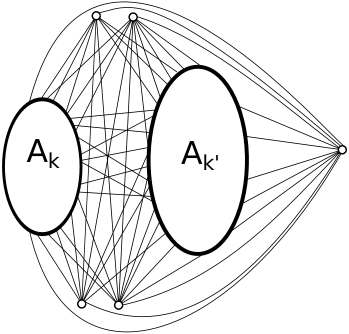
Join the new graph to two non-adjacent vertices, again

Constructions using joins



Finally, add a **universal vertex**

Constructions using joins



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Proposition

At each step, the constructed graph has $ID = n - 1$

A characterization

(1) stars

(2) $A_k = P_{2k}^{k-1}$

(3) joins between 0 or more members of (2) and 0 or more copies of $\overline{K_2}$

(4) (2) or (3) with a universal vertex

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

G connected identifiable graph, n vertices:

$$ID(G) = n - 1 \Leftrightarrow G \in (1), (2), (3) \text{ or } (4)$$

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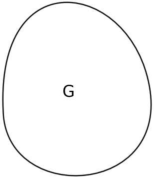

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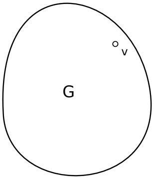

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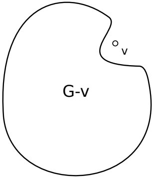

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$\Rightarrow$ By minimality of G:

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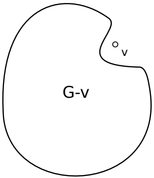

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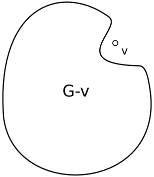

- Put v back $\Rightarrow$ **contradiction:** no counterexample exists!

Lower bounds

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Proposition

G graph on n vertices: $n \leq 2^{LD(G)} + LD(G) - 1 \implies LD(G) \geq \log_2(n) - 1$.

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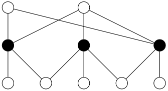
Tight examples:

$LD(G) = \lceil \log_2(n) \rceil - 1$



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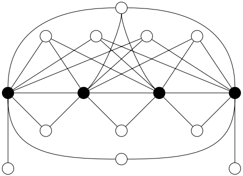

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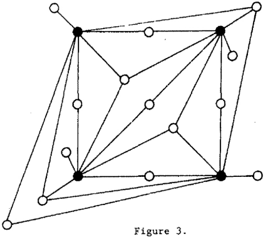
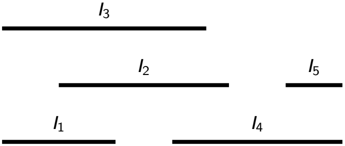
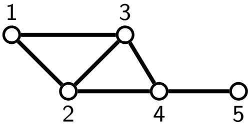


Figure 3.

Definition - Interval graph

Intersection graph of intervals of the real line.





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Then $n \leq \frac{k(k+1)}{2}$, i.e. $LD(G) = \Omega(\sqrt{n})$.

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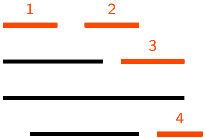

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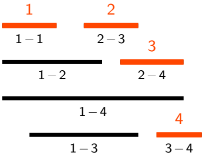

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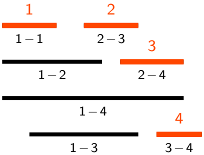

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$\rightarrow n \leq \sum_{i=1}^{k}(k-i) = \frac{k(k+1)}{2}$.

Lower bound for interval graphs

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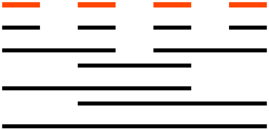
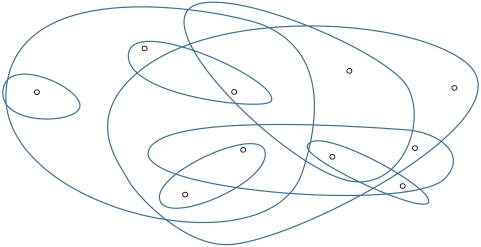


Vapnik-Červonenkis dimension

Measure of intersection complexity of sets in a hypergraph $(X, \mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered:

for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.



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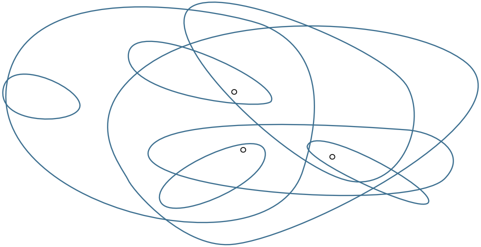

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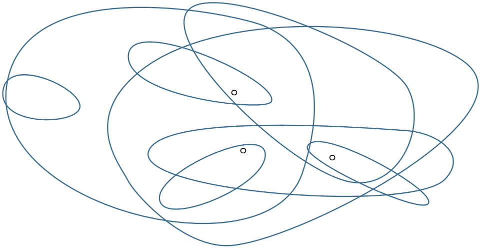

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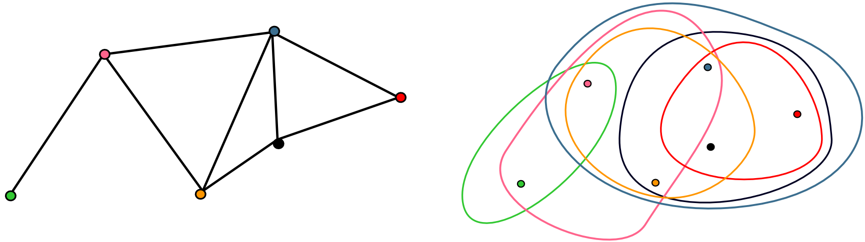
Typically bounded for geometric hypergraphs: 

Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph



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Corollary

G graph of order n, $LD(G) = k$, V-C dimension $\leq d$. Then $n = O(k^d)$.

Conclusion

Some open problems:

- Conjecture: $LD(G) \leq n/2$ in the absence of twins

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THANKS FOR YOUR ATTENTION