Covering a graph using shortest paths

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joint work with:

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AND

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PARIS
CLERMONT-FERRAND
LYON
BORDEAUX
MARSEILLE


MICHELIN


Pascal

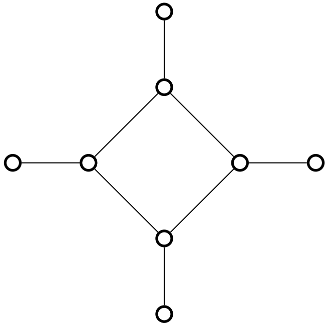

Saint Nectaire

Definitions

isometric path = shortest path between its endpoints

Isometric Path Cover

A set of shortest paths covering every vertex from a graph.



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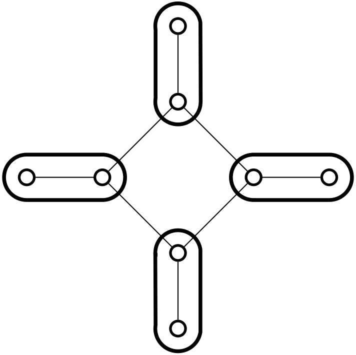

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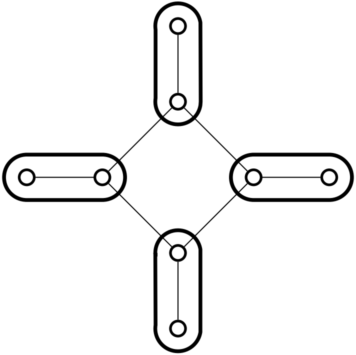

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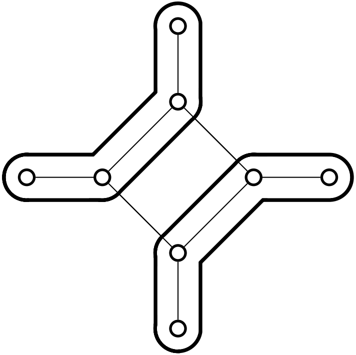

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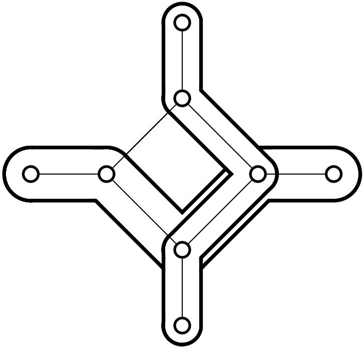
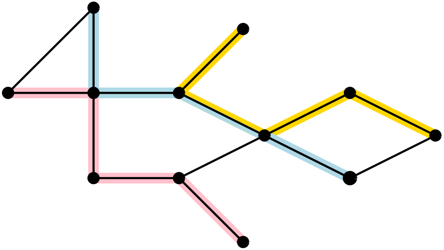


Covering a city by bus routes



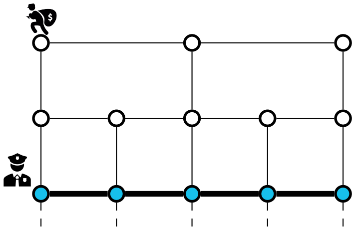
→ The shortest paths represent optimal bus routes

Context: pursuit-evasion problems

Cops and robber game: k cops and one robber are placed on a graph, and alternate their moves (along edges of the graph). The cops win if they can eventually catch the robber.

Lemma [Aigner & Fromme, 1983]

In *cops and robber*, one cop can "protect" a shortest path.



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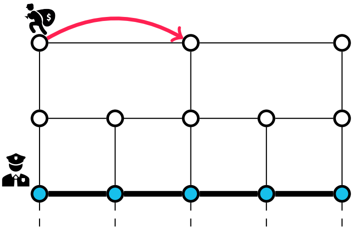

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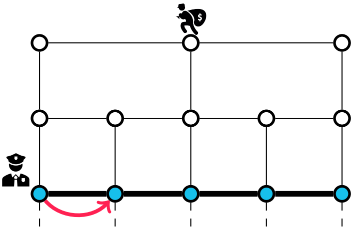

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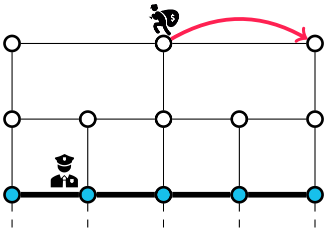

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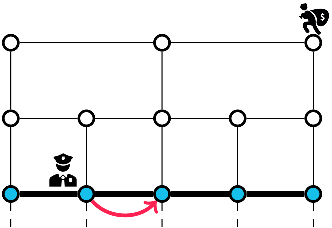

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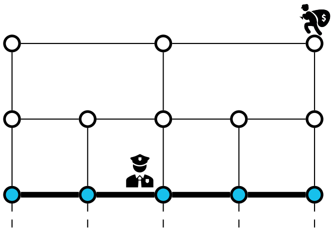

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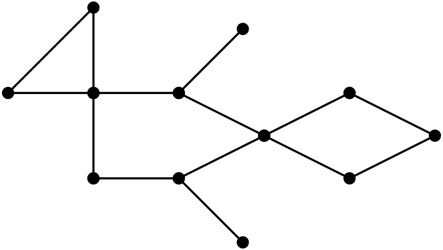
$\Rightarrow$ The minimum size of an Isometric Path Cover is an upper bound for the number of cops required to catch the robber

Formal problem statement



ISOMETRIC PATH COVER (IPC)

Input : A graph G and an integer k.

Question : Is there a set of k shortest paths of G, such that each vertex of G belongs to at least one of the shortest paths?

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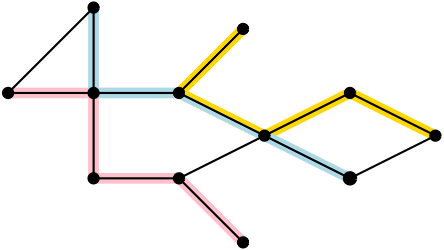

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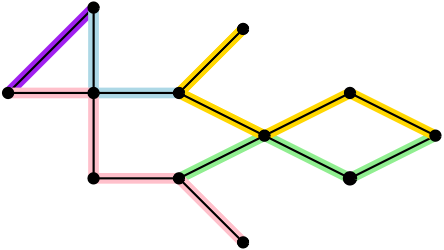

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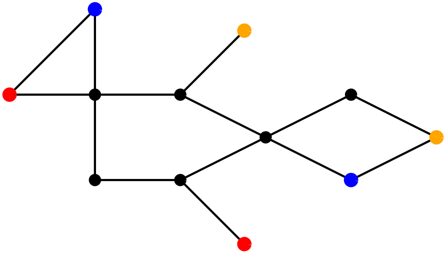

Isometric Path Cover with Terminals (IPC with Terminals)

Input :A graph G, and k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$, the terminals.

Question : Is there a set of k shortest paths of G, the ith path being an s_i-t_i shortest path, such that each vertex of G belongs to at least one of the shortest paths ?

Algorithmic questions

A c-**approximation algorithm** for a given problem is a polynomial-time algorithm producing a feasible solution whose value is at most c times the optimum.

A problem with parameter k is called **FPT** (fixed-parameter tractable) if it has an algorithm of complexity $f(k) \cdot n^{O(1)}$.

It is called **XP** if it has an algorithm with running time $n^{f(k)}$.

Questions

- Is IPC polynomial-time solvable?
- If not, is it approximable?
- Are IPC and IPC WITH TERMINALS FPT? Or at least XP ?

Related problems

Covering:

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Packing (with terminals):

- k DISJOINT PATHS (NP-c [Karp, 1975],
  FPT algorithm: $f(k)n^3$ [Robertson & Seymour, 1995])
- k DISJOINT SHORTEST PATHS (W[1]-hard,
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Partitioning:

- Isometric Path Partition (NP-c [Manuel, 2021])

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Exact values

- Trees, cycles, complete bipartite graphs, several cartesian products of paths [Fitzpatrick, 1997 & 1999]
- Some hypercubes [Fitzpatrick et al, 2001]
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Algorithms

- Linear-time algorithm for block graphs [Pan & Chang, 2005]
- poly-time $\log(d)$-approximation for graphs of diameter d [Thiessen & Gaertner, 2021]

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Exact algorithm in $2^{k2^{\mathcal{O}(w)}}n$ and $2^{2^{\mathcal{O}(k)}}n$ on chordal graphs (k = solution size, w = treewidth).

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XP algorithm in general

Structural result: $IPC = k \Rightarrow w \leq f(k)$. This implies a $g(k)n^k$ XP algorithm for Isometric Path Cover.

NP-completeness

chordal graph: every cycle of length at least 4 has a *chord*

Theorem [Chakraborty, Dailly, Das, F, Gahlawat, Ghosh 2022]

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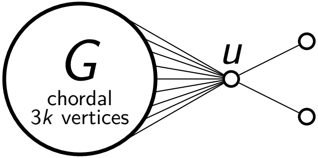

We are looking for an ISOMETRIC PATH COVER of size $k + 1$.

Part 1: approximation algorithm for chordal graphs and beyond

Approximation algorithm for chordal graphs

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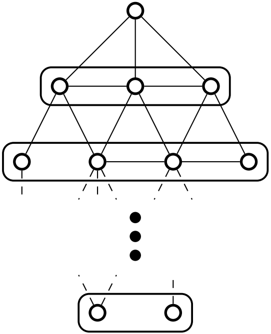
1. Do a Breadth-First Search (BFS) of G



Approximation algorithm for chordal graphs

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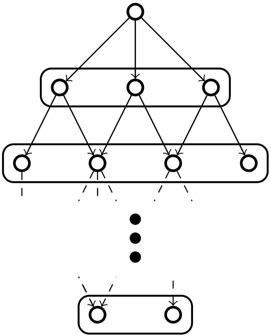

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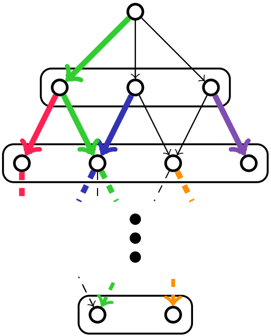

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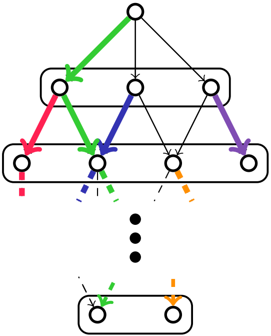

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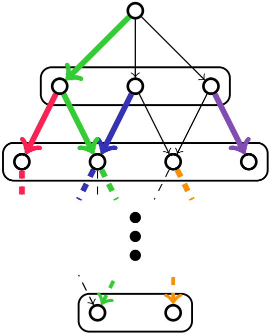
4. **Main Idea:** If any isometric path can contain at most ℓ vertices of any antichain, then the algorithm gives an ℓ-approximation.



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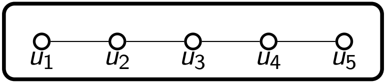

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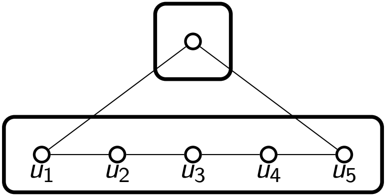

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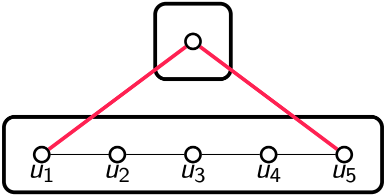

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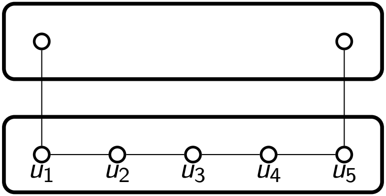

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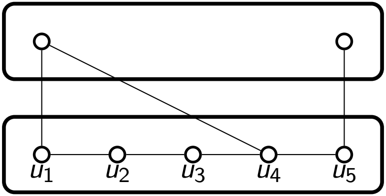

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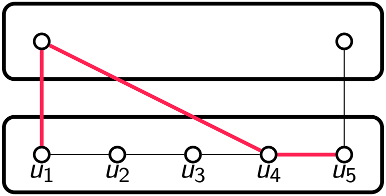

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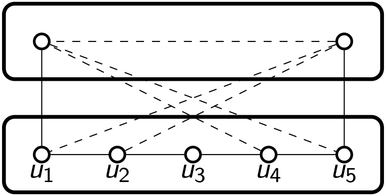

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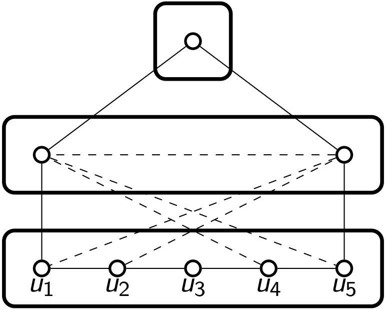

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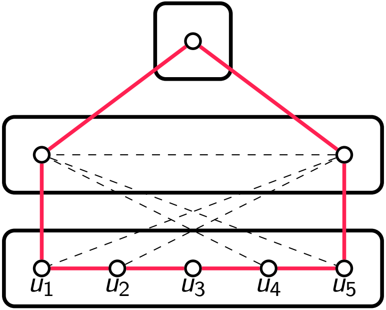

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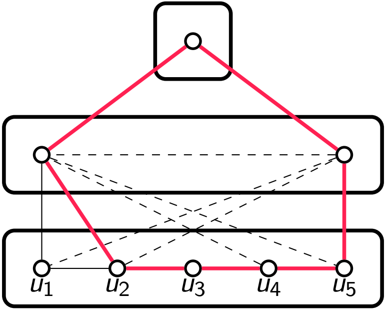

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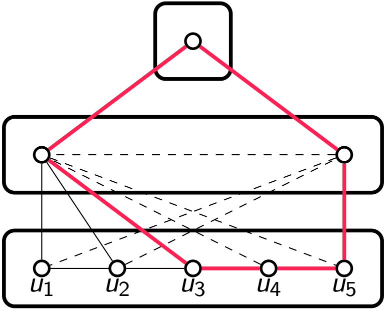

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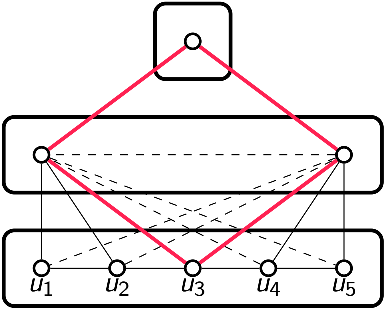

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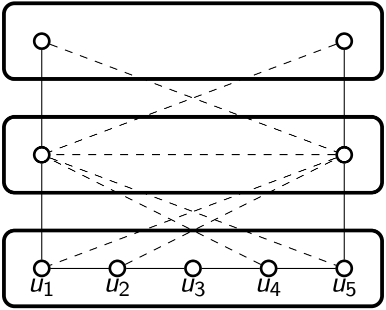

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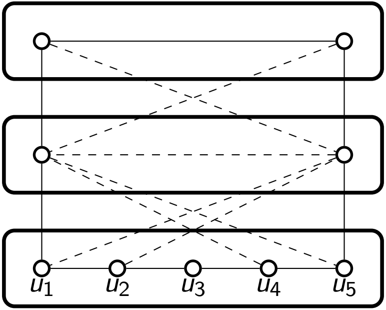

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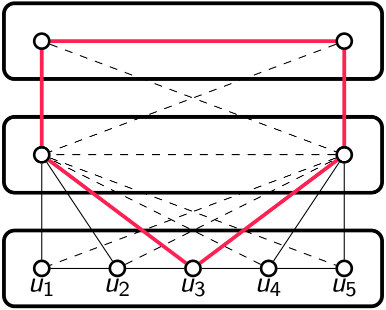

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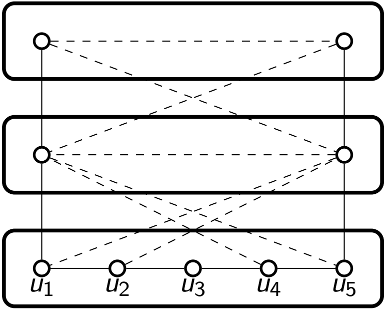

Approximation for chordal graphs: proof idea

No shortest path can contain 5 antichain vertices

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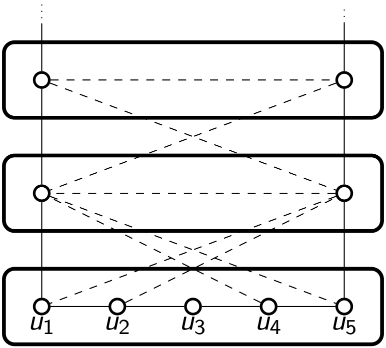

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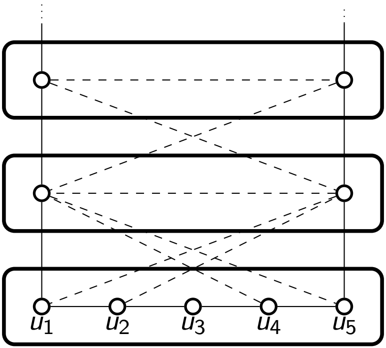

Impossible to have a common ancestor ⇒ contradiction

→ Other cases follow a similar reasoning

Approximation: Extension

Theorem [Chakraborty, Dailly, Das, F, Gahlawat, Ghosh 2022]

The algorithm yields the following approximation ratios:

- 4 on chordal graphs

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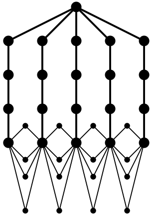
General graphs (n vertices): the algorithm can yield $\Omega(\sqrt{n})$-factor



A new graph parameter: IPCO

isometric path complexity of graph G

Minimum integer k such that there exists $v \in V(G)$ s.t: the vertices of any isometric path P of G can be covered by k many v-rooted isometric paths. $\rightarrow$ Denoted $ipco(G)$

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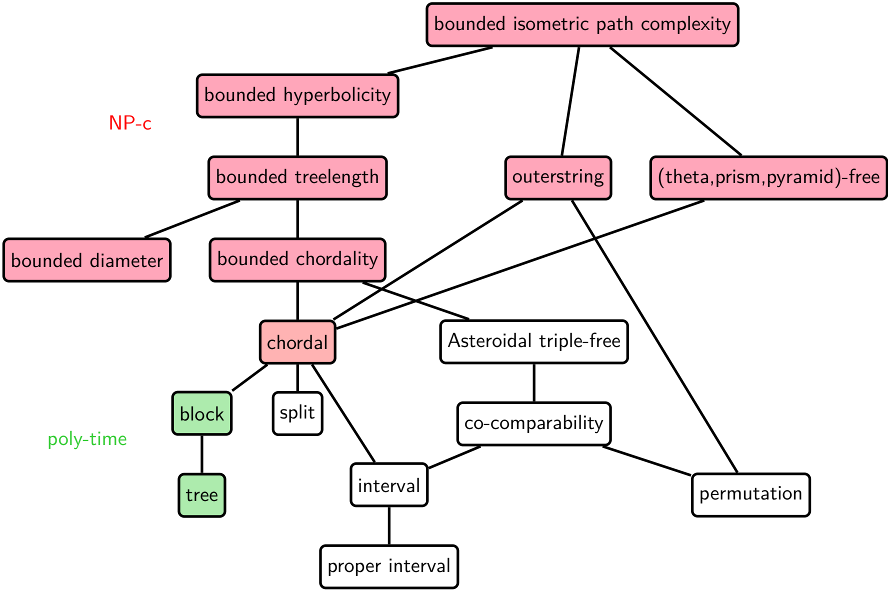
Theorem [Chakraborty, Chalopin, F, Vaxès 2023]

$ipco(G)$ is bounded for:

- graphs of bounded hyperbolicity
- outerstring graphs
- (theta, prism, pyramid)-free graphs

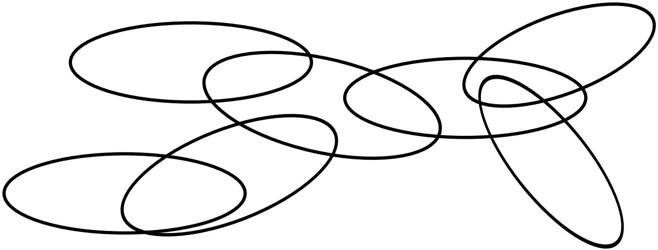
Graph classes where IPC is constant-factor approximable



Part 2: relation with tree-width

Graphs of bounded tree-width

Graphs of tree-width k: look like a tree where each edge is replaced by a set of k vertices



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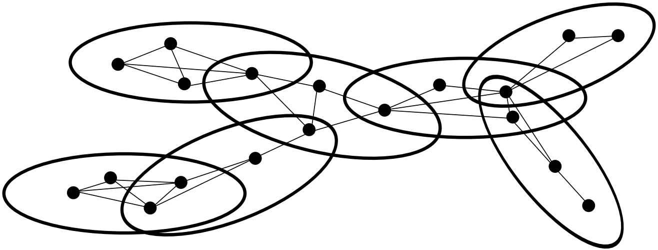

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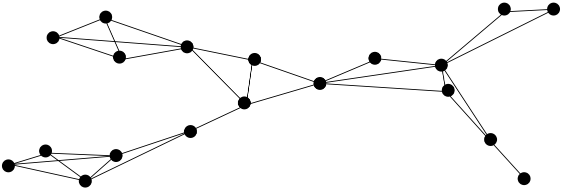

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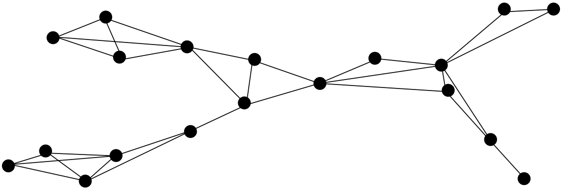

For graphs of treewidth k and n vertices, many problems can be solved in time $f(k)poly(n)$, for some (potentially exponential) function k.

$\rightarrow$ dynamic programming on the tree-like structure

$\rightarrow$ Courcelle's theorem (1990)

FPT algorithm for chordal graphs using tree-width

Theorem [Chakraborty, Dailly, Das, F, Gahlawat, Ghosh 2022]

IPC can be solved in time $2^{k2^{\mathcal{O}(w)}}n$ and $2^{2^{\mathcal{O}(k)}}n$ on chordal graphs ($k =$ solution size, $w =$ treewidth).

- Dynamic programming on tree decomposition. For chordal graphs: $TW =$ Clique Number-1.
- Each solution path can contain at most 2 vertices from each clique. So $k \geq \frac{TW+1}{2}$.
- Hence, FPT algorithm by TW implies FPT algorithm by solution size.

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- *close*(X, y): vertices in X which are closer to y.
- Main idea: P is isometric $\iff$ for every vertex v of P and clique X intersecting P, P contains exactly one vertex from $V(P) \cap X$ in *close*(X, v)

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- For each bag X and vertex y, the DP table maintains:
  - the intersection between each path and X (at most 2 vertices per path)
  - for each path intersecting X, if it continues "above" or "below" X
  - for each path, if it has already been used entierly "below" or not
  - for each path P_i and each subset S of X, if there is a vertex y "above" and "below" with $close(X, y) = S$

Combinatorial result

Theorem [Dumas, F, Perez, Todinca 2022]

If G is coverable by k shortest paths then, for any vertex a and any fixed distance D, the number of vertices at distance exactly D from a is upper bounded by some function $g(k)$.

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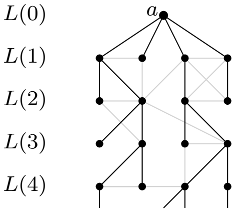

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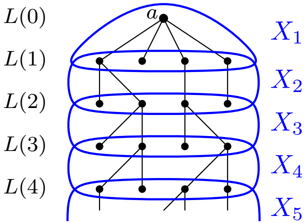
- Decomposition based on a breadth-first search (BFS).
- Each bag: two consecutive layers.



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Theorem [Dumas, F, Perez, Todinca 2022]

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Corollary

IPC is **XP** for parameter k: $f(k) \cdot n^{O(k)}$.

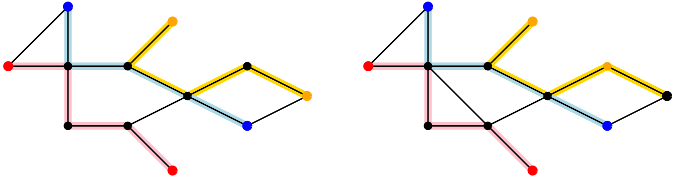
- Brute force: try all possible pairs of k terminals + above theorem

IPC with Terminals is FPT

Theorem [Courcelle, 1990]

Every problem expressible in MSOL_2 can be solved in $f(w) \cdot n$ time on graphs of treewidth at most w.

- Compute a tree decomposition by BFS. If $w > 2g(k)$ return `false`.
- Express in MSOL_2: "find paths $P_1, \ldots, P_k$ s.t. $\forall i, s_i, t_i \in P_i$, each vertex of G is covered". Minimize sum of paths' lengths.

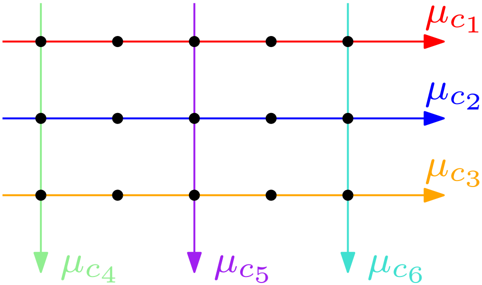


- Use Courcelle's theorem to solve the problem (optimization version).
- **If** $\forall i, |P_i| = dist(s_i, t_i)$ **then** answer `true`, **else** answer `false`.

Proof sketch: base paths colouring

Base paths : the k shortest paths $\mu_1, \ldots, \mu_k$ that cover the graph
To each base path μ_c we give :

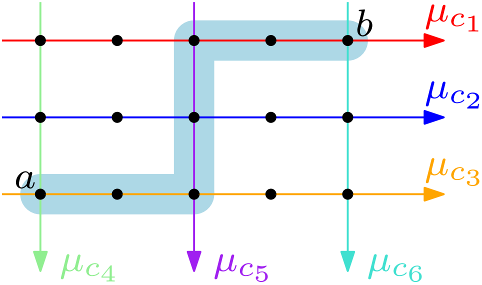
- A colour c, $1 \leq c \leq k$,
- An arbitrary direction.



Each edge of the graph receives a set of colours
colours(e) $\subseteq \{1, \ldots, k\}$.

Colouring of a path

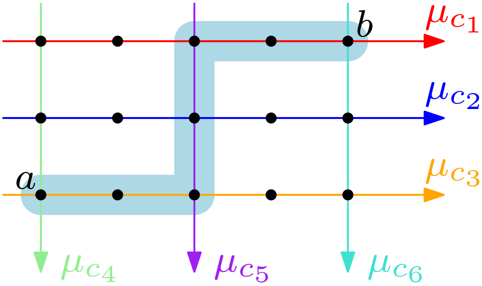
Given a path P from a to b, we can pick for each edge one of its possible colours, it is a colouring col of P.
(P, col) is a coloured path



Colours-signs word

(P, col) can be divided in monochromatic subpaths. These subpaths induce either a "$+$" sign or a "$-$" sign w.r.t. the direction of the base path.

We associate to (P, col) with colours $\{c_1, \ldots, c_\ell\}$ a colours-signs word $\omega = ((c_1, s_1), \ldots, (c_\ell, s_\ell))$ on the alphabet $\{1, \ldots, k\} \times \{+, -\}$.



$$\omega = ((c_3, +), (c_5, -), (c_1, +))$$

Good colouring of a path

(P, col) is well-coloured if the set of edges using a colour c form a **connected subpath** of P.

A path well-coloured :

A path not well-coloured :

Good colouring of a path

Good colouring Lemma

For every pair of vertices a, b of G, there exists a **shortest well-coloured** a-b path.

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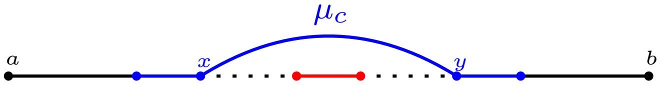

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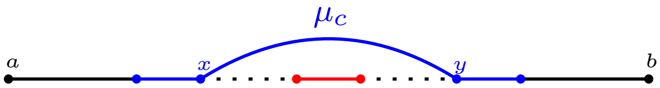

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The constructed path is still a shortest path
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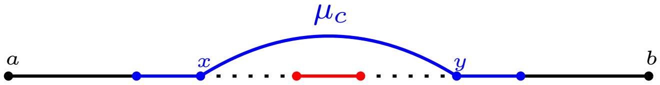

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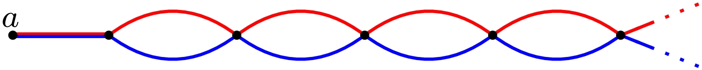
The number of possible colours-signs words is upper-bounded by $g(k) = O(3^k)$.

The bound for isometric path edge-covers

Multiple shortest paths of same length may have the same colours-signs word :



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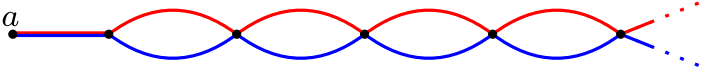

Colours-signs word Lemma

The shortest paths starting at a vertex a, of length D and colours-signs word ω all end at the same vertex b.

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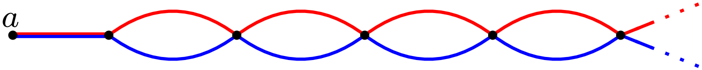

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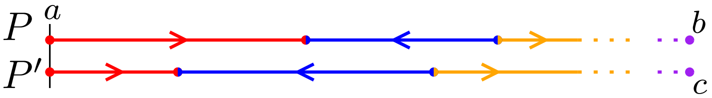
Theorem [Dumas, F, Perez, Todinca 2022]

For any vertex a and any fixed distance D, the number of vertices at distance exactly D from a is upper bounded by $g(k)$ (the number of colours-signs words).

Proof of the Colours-signs word Lemma

Let b and c be vertices at distance D from vertex a. Let (P, col), (P', col') be two well-coloured shortest a-b and a-c paths.

Claim : If they have the same colours-signs word, then $b = c$.



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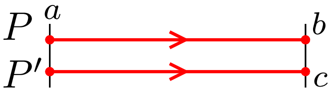
If $\ell = 1$



$dist(a, b) = dist(a, c)$ thus $b = c$.

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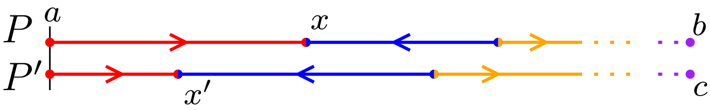

Take P the path with the longest subpath of the colour c_1.

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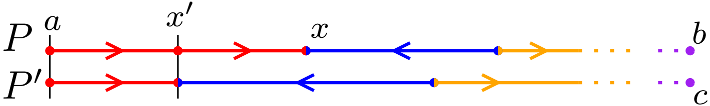

The vertex x' is in the path $\mu_{c_1}[a, x]$, thus in the path P

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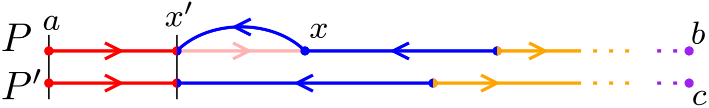
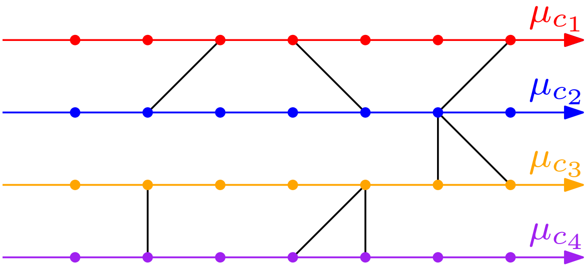


Replace $P[x', x]$ by $\mu_{c_2}[x', x]$.
$P[x', b]$ and $P'[x', c]$ have $\ell - 1$ colours, by the induction hypothesis, $b = c$.

Vertex-covering case: colouring of a path

A colour and a direction given to each base path.



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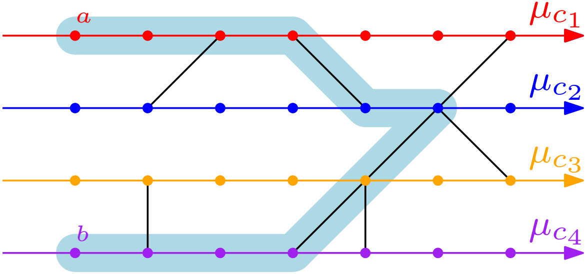

A colouring **col** associate to each vertex $v \in P$ a colour $c \in \text{colours}(v)$.

Colours-signs words are defined the same way as in the edge case.

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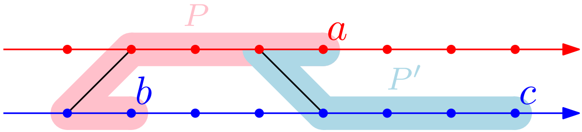

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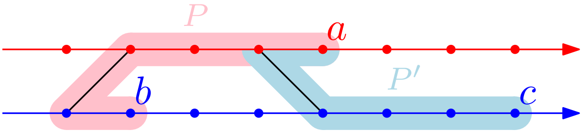

- BUT: number of vertices at distance D from a that share the same colours-signs word is at most $2k$.

Problem

Colours-signs word Lemma

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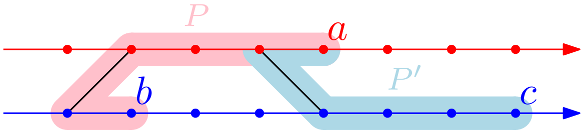
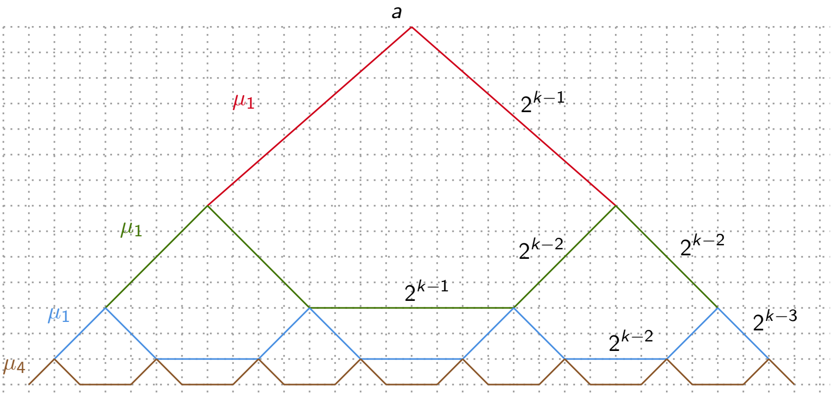


- BUT: number of vertices at distance D from a that share the same colours-signs word is at most $2k$.
- There are at most $g(k) = O(k \cdot 3^k)$ vertices at a given distance of vertex a.

Lower bound

In a graph vertex-coverable by k shortest paths, the number of vertices at same distance of a source is upper bounded by $g(k)$.

- We have shown that $g(k) = O(k \cdot 3^k)$.
- Lower bound : $g(k) \geq 2^k$



Conclusion

- NP-completeness on chordal graphs
- Approximation algorithm on several classes
- FPT algorithm by solution size on chordal graphs
- Bounding the treewidth by a function of solution size
- FPT algorithm by solution size when terminals fixed
- XP algorithm by solution size when terminals free

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Open problems

- Complexity on interval graphs, split graphs... Planar graphs?
- Constant factor approximation algorithm for all graphs?
- Is Isometric Path Cover FPT or W[1]-hard? (sol. size)
- Can we improve the bound on the treewidth? Polynomial?
- Approximation for Isometric Path Partition?

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THANK YOU!