Identification problems in graphs

selected topics

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L'IMOS

UNIVERSITÉ Clermont Auvergne

IPM, April 2024

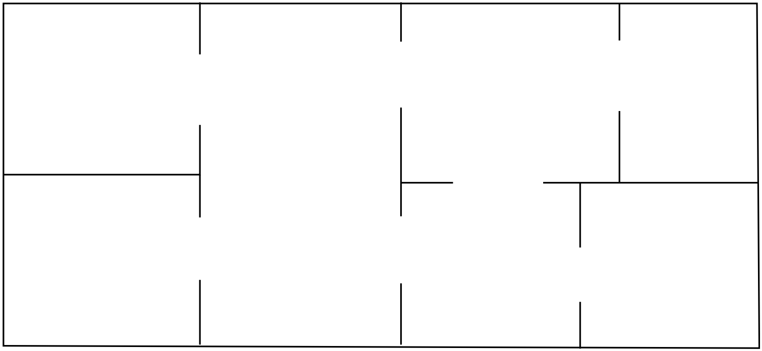

PARIS
CLERMONT-FERRAND
LYON
BORDEAUX
MARSEILLE


MICHELIN


Pascal


Saint Nectaire

Locating a burglar



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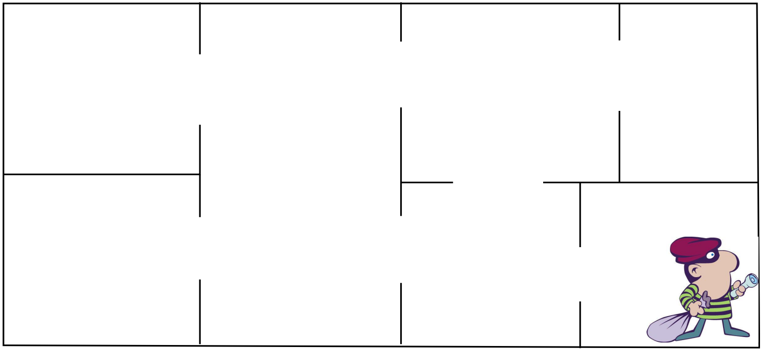

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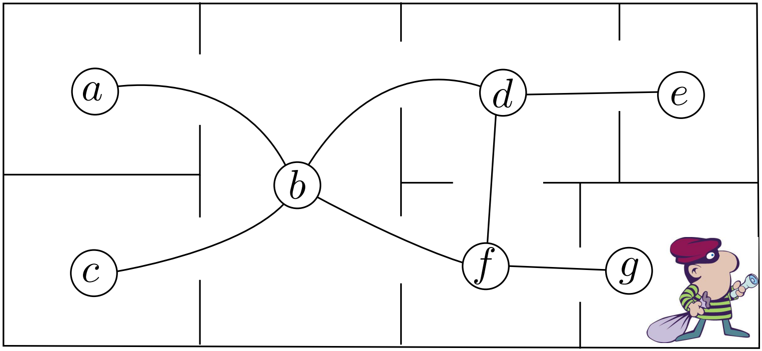

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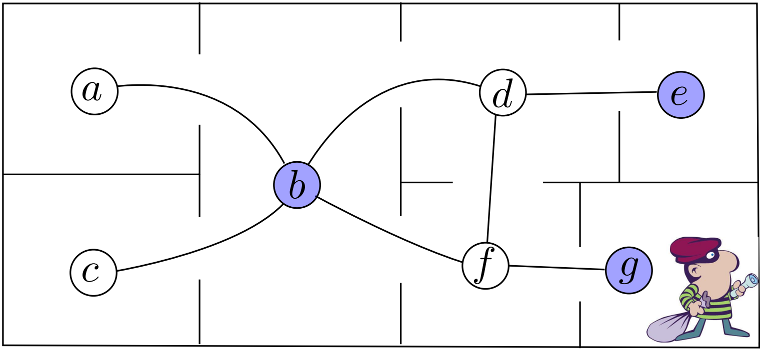
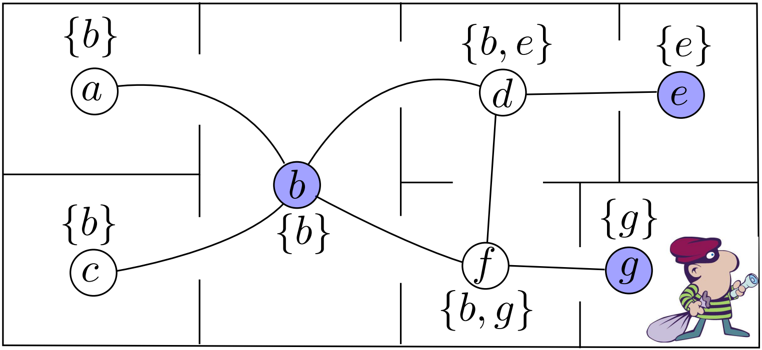


Detectors can detect movement in their room and adjacent rooms



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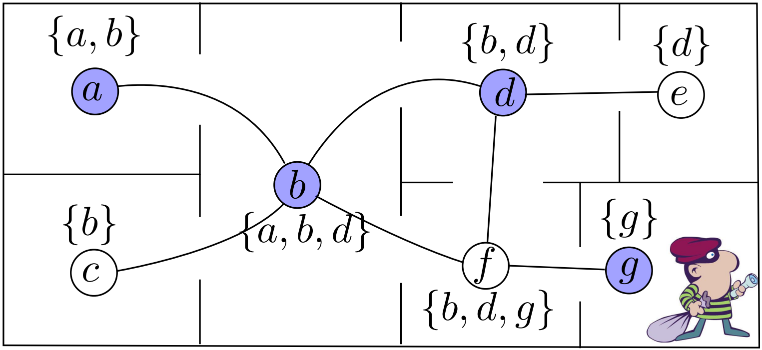

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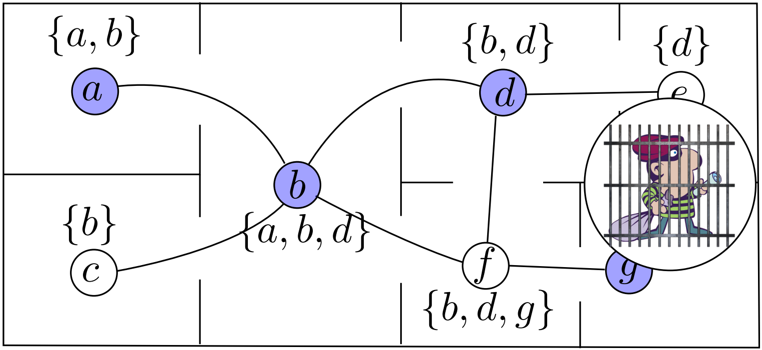

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Separating sets in hypergraphs

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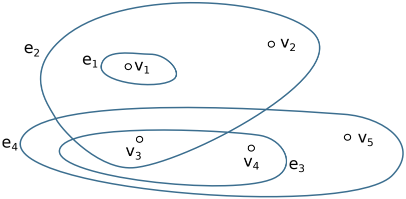
Hypergraph $(X,\mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.



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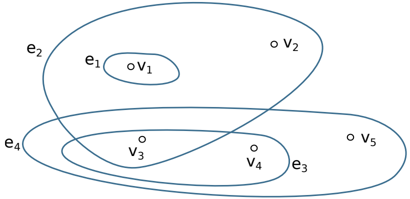

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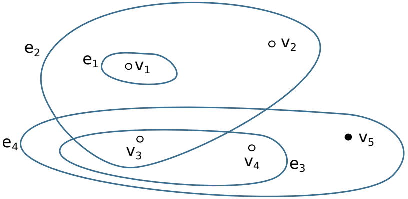

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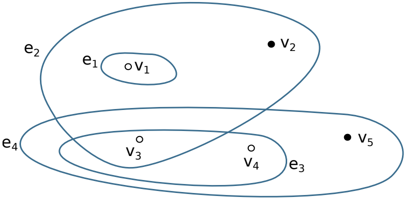

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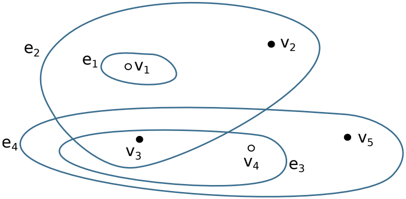

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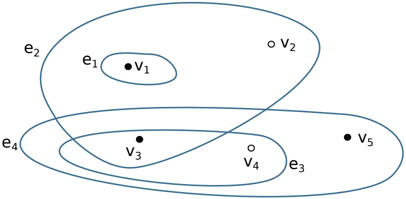

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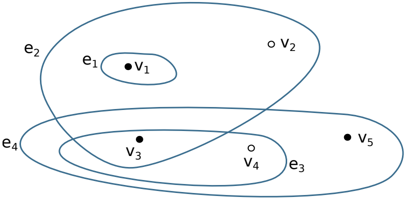

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Also known as Separating system, Distinguishing set, Test cover, Distinguishing transversal, Discriminating code...

Applications

- network-monitoring, fault detection (burglar)
- medical diagnostics: testing samples for diseases (*test cover*)
- biological identification (attributes of individuals)
- learning theory: teaching dimension
- machine learning: V-C dimension (Vapnik, Červonenkis, 1971)
- graph isomorphism: canonical representation of graphs (Babai, 1982)
- logic definability of graphs (Kim, Pikhurko, Spencer, Verbitsky, 2005)

Proposition

For a hypergraph $(X,\mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

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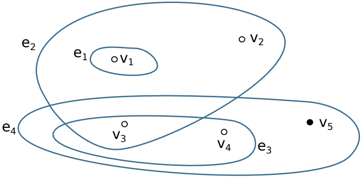

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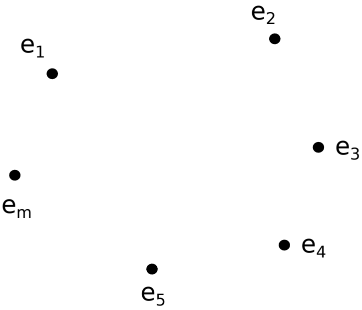

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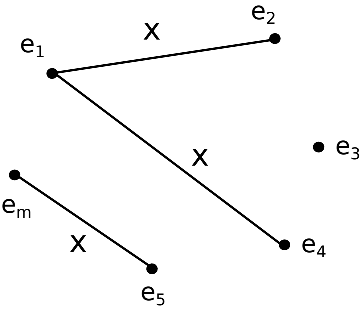

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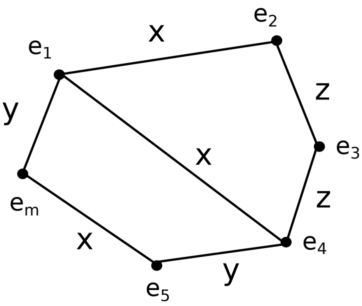

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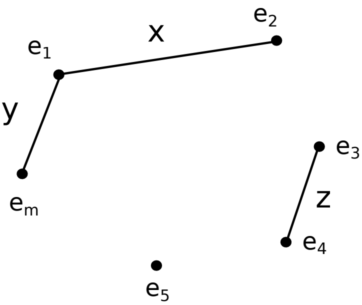

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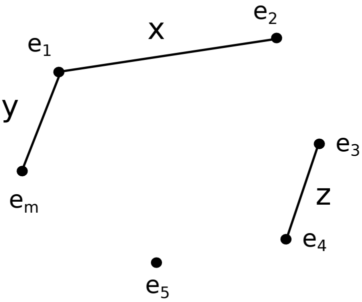

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So, at most $|\mathscr{E}|-1$ "problematic" vertices.
→ Find "non-problematic vertex", omit it. □

Some example problems

Special cases of separating sets in hypergraphs (graph-based):

- identifying codes
- **open identifying codes**
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Coloring-based identification

- Adjacent vertex-distinguishing edge-coloring
- locally identifying coloring
- locating coloring
- neighbor-locating coloring

Open identifying codes in graphs

(a.k.a. open locating-dominating sets)

G: undirected graph $\quad N(u)$: set of neighbours of v

Definition - open identifying code (Seo, Slater, 2010 )

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
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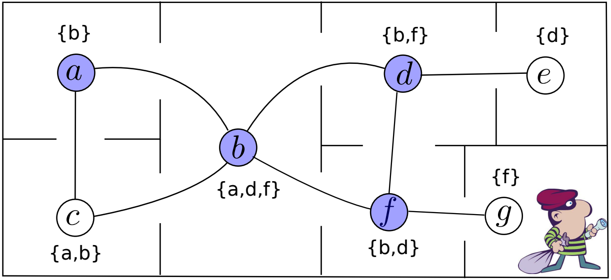
Notation. $OID(G)$: open identifying code number of G, minimum size of an open identifying code in G



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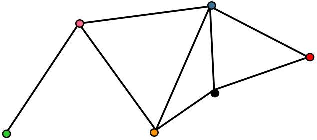

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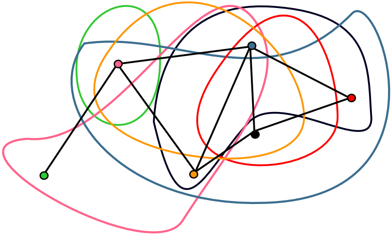

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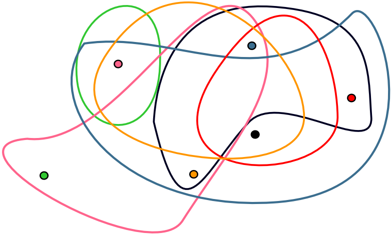

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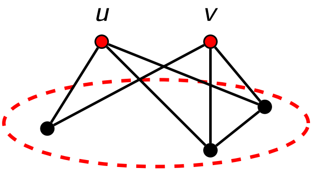

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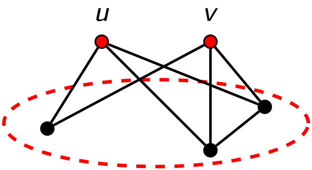

Proposition

A graph is **locatable** if and only if it has no **isolated vertices** and **open twins**.

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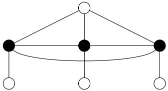

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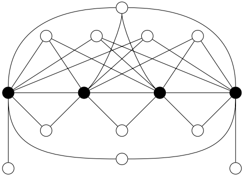
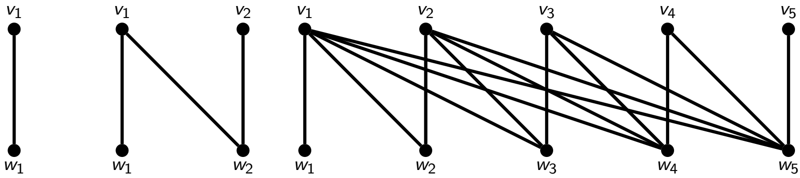


□

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.



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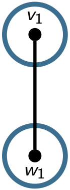

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Some vertices **forced** in any open identifying code because of domination

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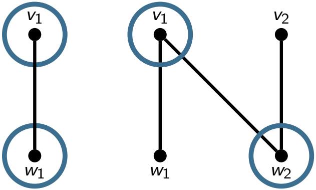

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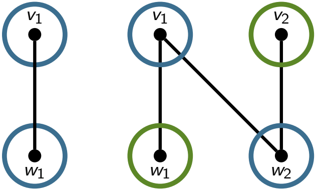

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Upper bound on $OID(G)$?

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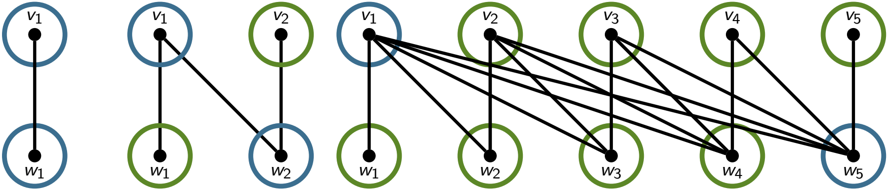

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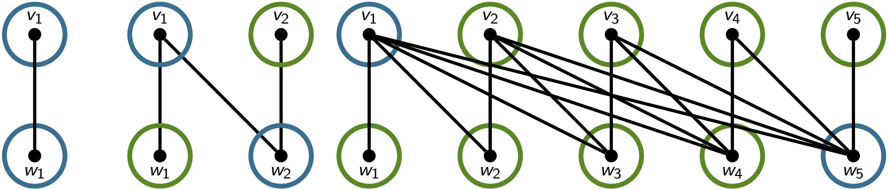

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Proposition

For every half-graph H_k of order $n = 2k$, $OID(H_k) = n$.

Characterizing “bad graphs” for open identifying codes

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.

Then, $OID(G) = n$ if and only if G is a half-graph.

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• By induction, G' is a half-graph. We can conclude that G is a half-graph too, after some case analysis. □

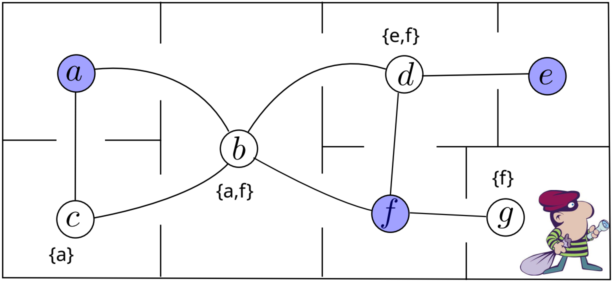
Location-domination in graphs

Definition - Locating-dominating set (Slater, 1980's) 

$D \subseteq V(G)$ locating-dominating set of G:

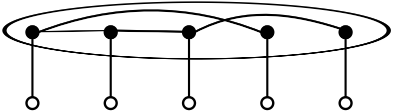
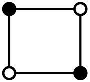
- for every $u \in V$, $N[v] \cap D \neq \emptyset$ (domination).
- $\forall u \neq v$ of $V(G) \setminus D$, $N(u) \cap D \neq N(v) \cap D$ (location).

Notation. location-domination number $LD(G)$,
smallest size of a locating-dominating set of G



Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Tight examples:  

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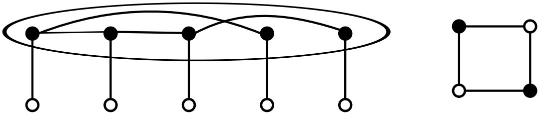
Tight examples: 

Proof: Consider an *inclusionwise minimal* dominating set D of G.

$\rightarrow$ its complement set $V(G) \setminus D$ is also a dominating set!

Thus, either D or $V(G) \setminus D$ has size at most $\frac{n}{2}$. $\square$

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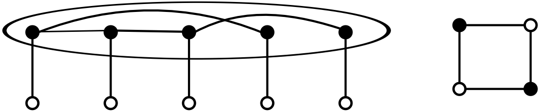

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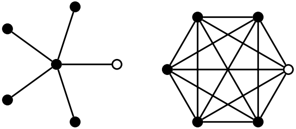

Remark: tight examples contain many twin-vertices!!

Upper bound: a conjecture

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Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

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Remark:

- twins are easy to detect
- twins have a trivial behaviour w.r.t. location-domination

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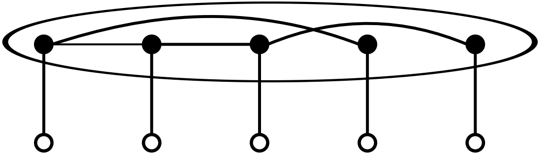
If true, tight: 1. domination-extremal graphs



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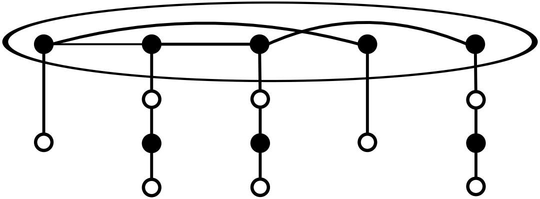
If true, tight: 2. a similar construction



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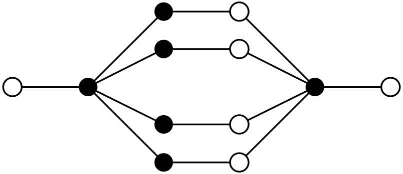
If true, tight: 3. a family with domination number 2



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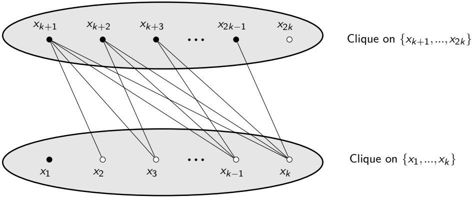
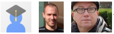
If true, tight: 4. family with dom. number 2: complements of half-graphs



Conjecture (Garijo, González & Márquez, 2014 )

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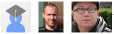
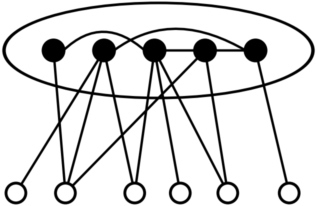
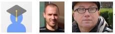
Theorem (Garijo, González & Márquez, 2014 )

Conjecture true if G has independence number $\geq n/2$. (e.g. bipartite)

Proof: every vertex cover of a twin-free graph is a locating-dominating set



Conjecture (Garijo, González & Márquez, 2014 )

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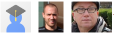
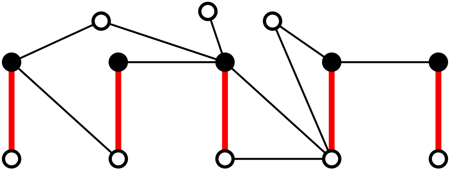
$\alpha'(G)$: matching number of G

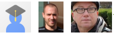
Theorem (Garijo, González & Márquez, 2014 )

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Proof:

- Consider special maximum matching M
- Select one vertex in each edge of M



Conjecture (Garijo, González & Márquez, 2014 )

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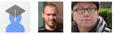
Theorem (Garijo, González & Márquez, 2014 )

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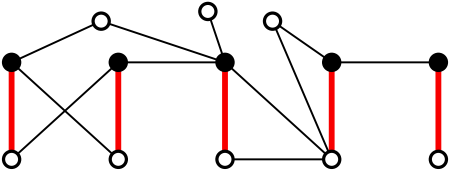
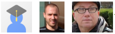


Conjecture (Garijo, González & Márquez, 2014 )

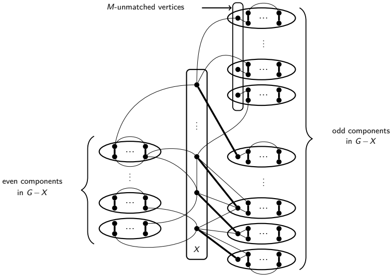
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

Conjecture true if G is cubic.

Proof: Involved argument using maximum matching and Tutte-Berge theorem.

$$\alpha'(G) = \min_{X \subseteq V(G)} \frac{1}{2}\left(|V(G)| + |X| - oc(G - X)\right)$$



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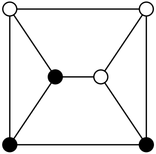
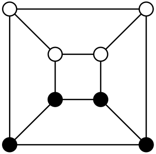
Bound is tight for cubic graphs:





Question

Do we have $LD(G) = \frac{n}{2}$ for other cubic graphs?

Upper bound: a conjecture - special graph classes

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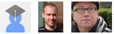
Question

Are there twin-free (cubic) graphs with $LD(G) > \alpha'(G)$?

(if not, conjecture is true)

Theorem (Garijo, González & Márquez, 2014 )

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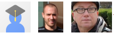
Theorem (Garijo, González & Márquez, 2014 )

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

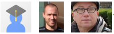
Conjecture true if G is cubic.

Theorem (F., Henning, Löwenstein, Sasse, 2016 )

Conjecture true if G is split graph or complement of bipartite graph.

Theorem (Chakraborty, F., Parreau, Wagler, 2023 )

Conjecture true if G is a block graph.

Conjecture (Garijo, González & Márquez, 2014 )

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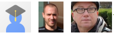
Conjecture (Garijo, González & Márquez, 2014 )

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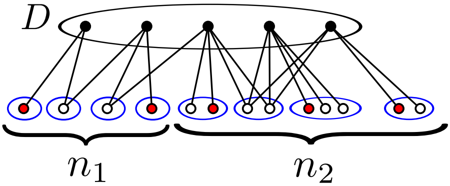
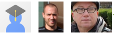


Conjecture (Garijo, González & Márquez, 2014 )

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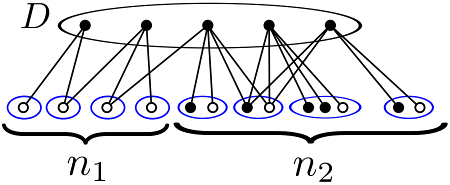
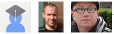
• there is a LD-set of size $n - n_1 - n_2$



Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2016 )

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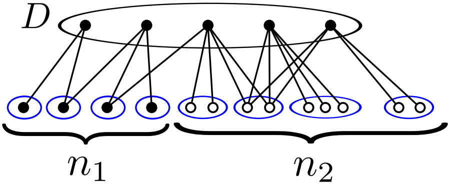
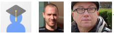
Proof: • There exists a dominating set D such that each vertex has a private neighbour, thus $|D| \leq n_1 + n_2$. Take such D that is inclusionwise maximal.

• there is a LD-set of size $n - n_1 - n_2$

• there is a LD-set of size $|D| + n_1$ because D is maximal



Upper bound: a conjecture - general bound

Conjecture (Garijo, González & Márquez, 2014 )

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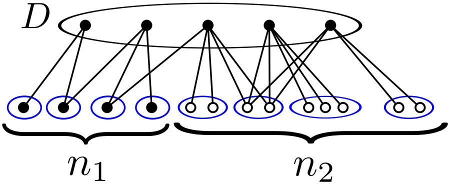
- there is a LD-set of size $n - n_1 - n_2$
- there is a LD-set of size $|D| + n_1$ because D is maximal
- $\min\{|D| + n_1, n - n_1 - n_2\} \leq \frac{2}{3}n$



Lower bounds (neighbourhood complexity)

Lower bounds

Proposition

G graph, n vertices, $LD(G) = k$. Then, $n \leq 2^k + k - 1$.

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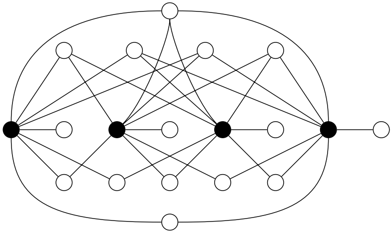
Tight example ($k = 4$):



Lower bounds

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G tree of order n, $LD(G)=k$. Then $n \leq 3k-1 \rightarrow LD(G) \geq \frac{n+1}{3}$.

Tight examples:

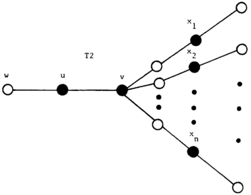

FIG. 2. Tree T2

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Theorem (Rall & Slater, 1980's  )

G planar graph, order n, $LD(G)=k$. Then $n \leq 7k-10 \rightarrow LD(G) \geq \frac{n+10}{7}$.

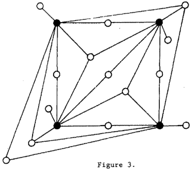
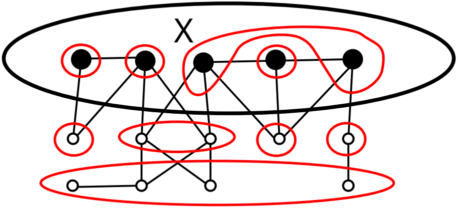
Tight examples: 

Figure 3.

Neighbourhood complexity of a graph G:

maximum number $|\{N(v) \cap X\}|$ of neighbourhoods inside any set X of k vertices, as a function of k



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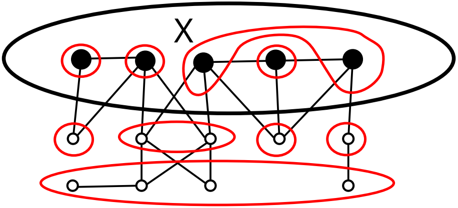
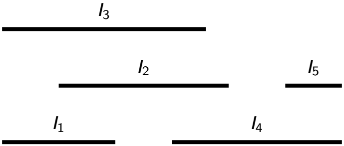


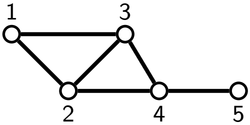
$|\{N(v) \cap X\}| = 9$

- General graphs : exponential neighbourhood complexity 2^k
- Trees/planar graphs : linear neighbourhood complexity $O(k)$

Definition - Interval graph

Intersection graph of intervals of the real line.





Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

G interval graph of order n, $LD(G) = k$.

Then $n \leq \frac{k(k+1)}{2}$, i.e. $LD(G) = \Omega(\sqrt{n})$.

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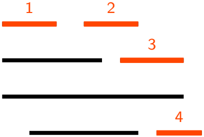

- Identifying code D of size k.
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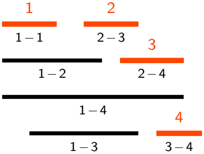

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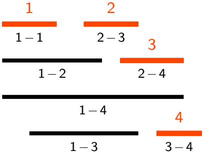

- Identifying code D of size k.
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$$\rightarrow n \leq \sum_{i=1}^{k}(k-i) = \frac{k(k+1)}{2}.$$

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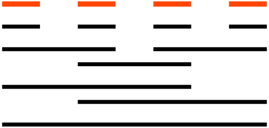
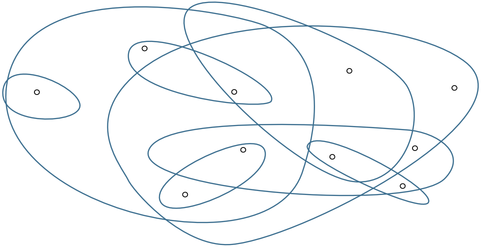
Tight:




Measure of intersection complexity of sets in a hypergraph $(X, \mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered:

for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.



V-C dimension of H: maximum size of a shattered set in H

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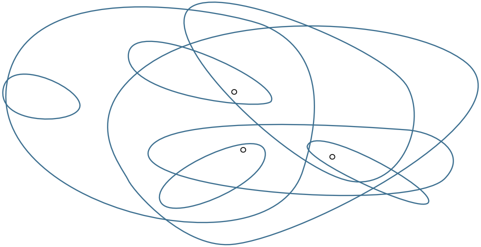

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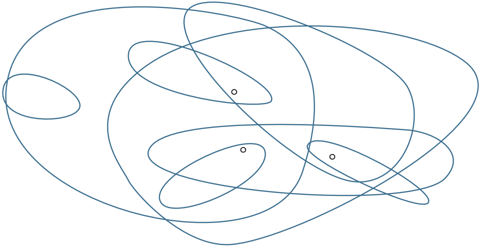

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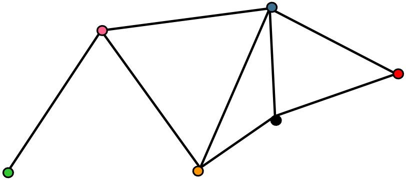
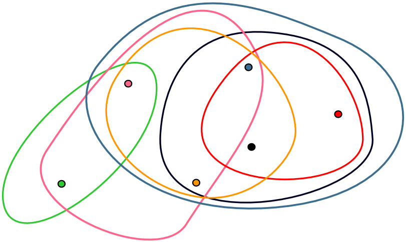
Typically bounded for geometric hypergraphs: 

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph





Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph

Typically bounded for geometric intersection graphs:

→ interval graphs ($d = 2$), C_4-free graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

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Let H be a hypergraph of V-C dimension at most d. Then, any set S of vertices has at most $|S|^d$ distinct traces.

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$O(k^2)$: interval, permutation, line...

$O(k)$: cographs, unit interval, bipartite permutation, block...

Sparse/structured graphs

Graph classes of bounded expansion: all shallow minors of its members have bounded average degree → e.g. planar graphs, minor-closed classes, bounded degree...

Theorem (Reidl, Sánchez-Villaamil, Stavropoulos, 2019   )

Let $\mathscr{C}$ be a graph class of bounded expansion. Let G in $\mathscr{C}$, order n, and $LD(G) = k$.
Then, $n \leq f(\mathscr{C})k$.

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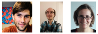
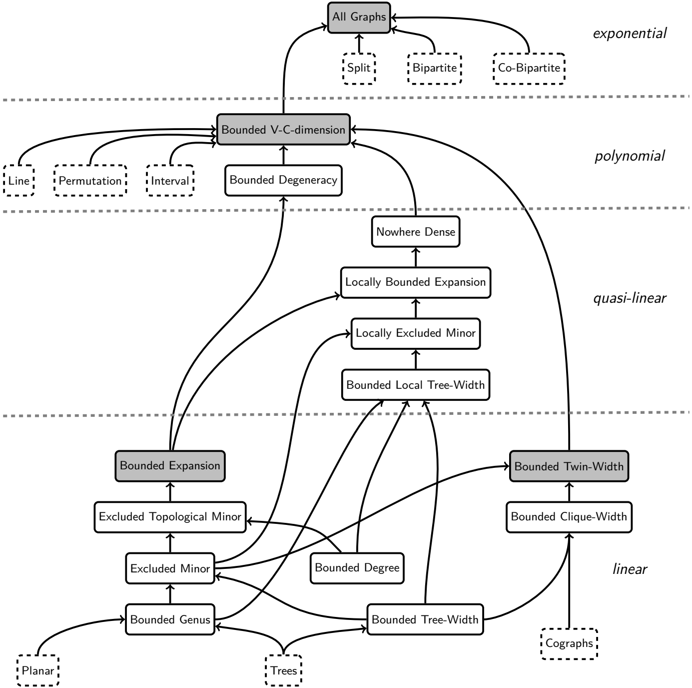
Recently introduced structural measure: twin-width.

Theorem (Bonnet, F., Lehtilä, Parreau, 2024 )

Let G be a graph of twin-width at most d and order n, and $LD(G) = k$.

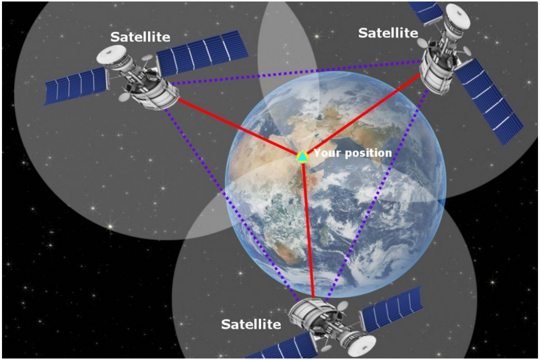
Then, $n \leq (d+2)2^{d+1}k$.


All Graphs
Split
Bipartite
Co-Bipartite
exponential
Bounded V-C-dimension
Line
Permutation
Interval
Bounded Degeneracy
polynomial
Nowhere Dense
Locally Bounded Expansion
quasi-linear
Locally Excluded Minor
Bounded Local Tree-Width
Bounded Expansion
Bounded Twin-Width
Excluded Topological Minor
Bounded Clique-Width
Excluded Minor
Bounded Degree
linear
Bounded Genus
Bounded Tree-Width
Cographs
Planar
Trees

Metric dimension

Determination of Position in 3D euclidean space

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



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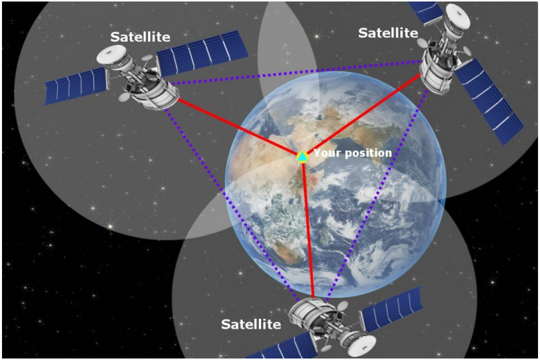

Question

Does the “GPS” approach also work in undirected unweighted graphs?

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

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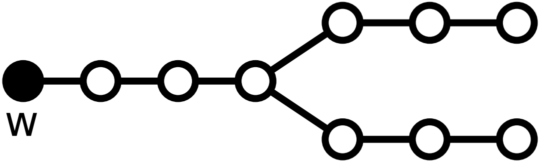

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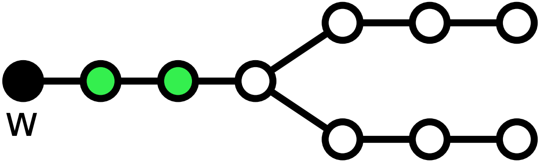

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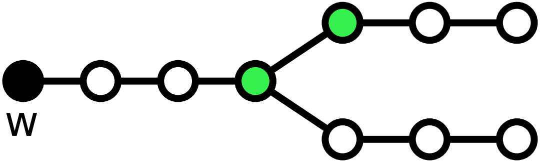

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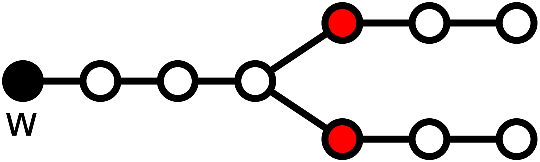

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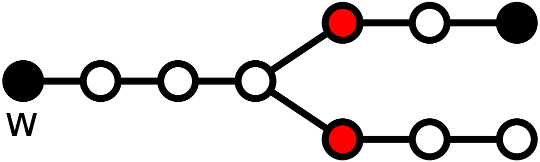

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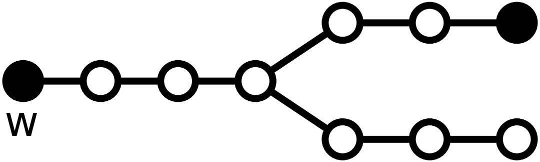

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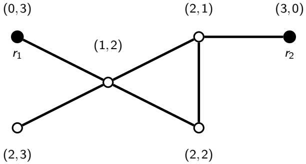

$R = \{r_1, r_2\}$

$MD(G) = 2$

Every vertex receives a unique distance-vector w.r.t. to the solution vertices.

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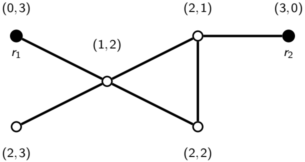

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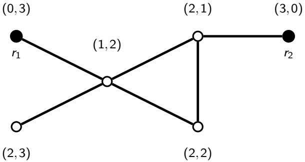

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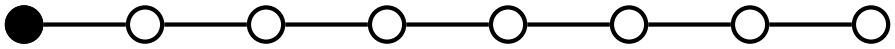
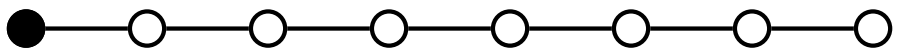
Remark

- Any locating-dominating set is a resolving set, hence $MD(G) \leq LD(G)$.
- A locating-dominating set can be seen as a "distance-1-resolving set".

Examples

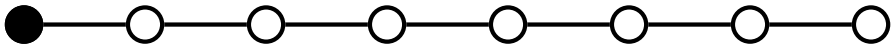




Proposition

$MD(G) = 1 \Leftrightarrow G$ is a path

Examples



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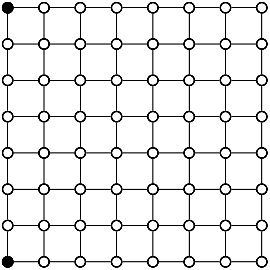
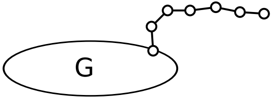


Proposition

For any square grid G, $MD(G) = 2$.

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.



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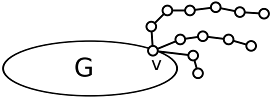

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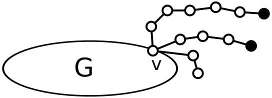

Observation

R resolving set. If v has k legs, at least $k-1$ legs contain a vertex of R.

Simple leg rule: if v has $k \geq 2$ legs, select $k-1$ leg endpoints.

Trees

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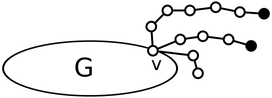

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For any tree, the simple leg rule produces an optimal resolving set.

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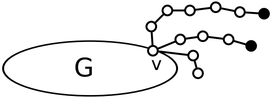

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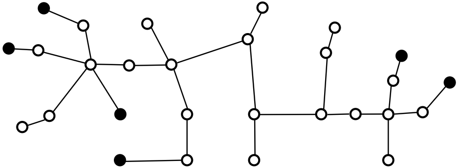

Bounds with diameter

Example of path: no bound $n \leq f(MD(G))$ possible.

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Theorem (Khuller, Raghavachari & Rosenfeld, 2002   )

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter D: maximum distance between two vertices)

Proof: Every vertex not in the solution R is assigned to a unique vector of length k, with values in $\{1, \ldots, D\}$: D^k possibilities, plus the k ones in R. □

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→ Proof is similar as that for locating-dominating sets.

Trees

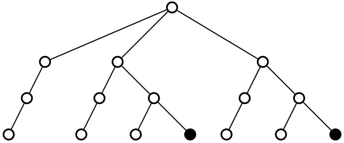
Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

T a tree with diameter D and $MD(T) = k$, then

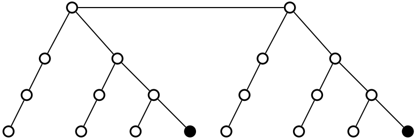
$$n \leq \begin{cases} \frac{1}{8}(kD+4)(D+2) & \text{if } D \text{ even,} \\ \frac{1}{8}(kD-k+8)(D+1) & \text{if } D \text{ odd.} \end{cases} = \Theta(kD^2)$$

Bounds are tight.

$k = 2,\ D = 6$



$k = 2,\ D = 7$



Using the concept of distance-VC-dimension:

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

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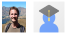
Theorem (Joret, Rambaud, 2023+ )

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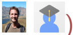
Theorem (Joret, Rambaud, 2023+ )

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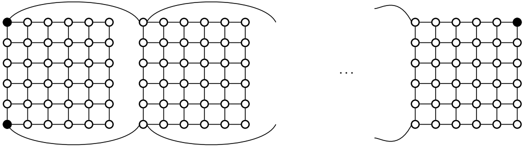
Tight? Planar example with $k = 3$ and $n = \Theta(D^3)$:



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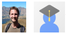
Theorem (Joret, Rambaud, 2023+ )

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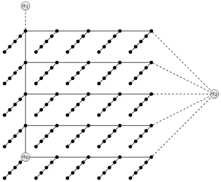
Tight? Planar example with treewidth 2 and $n = \Theta(kD^3)$:



Selected open questions

- Characterize graphs G of order n with $OID(G) = n - 1$?
- Conjecture: $LD(G) \leq n/2$ in the absence of twins
- Analogue of $LD(G) \leq n/2$ conjecture for digraphs?
- Find tight bounds for Metric Dimension of planar graphs of diameter D
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- Neighbourhood complexity at distance r
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THANKS FOR YOUR ATTENTION!