Identification problems in graphs

selected topics

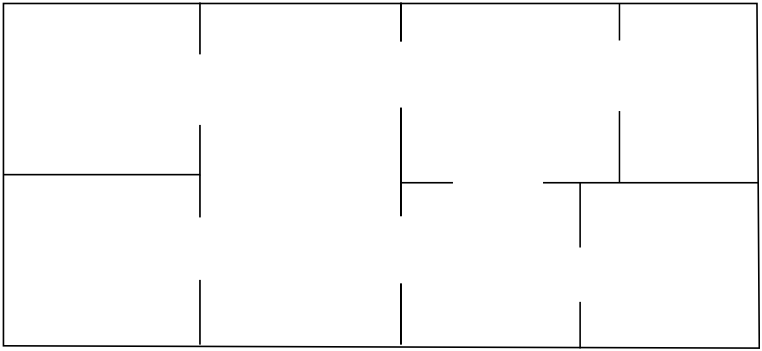
Florent Foucaud

LIMOS

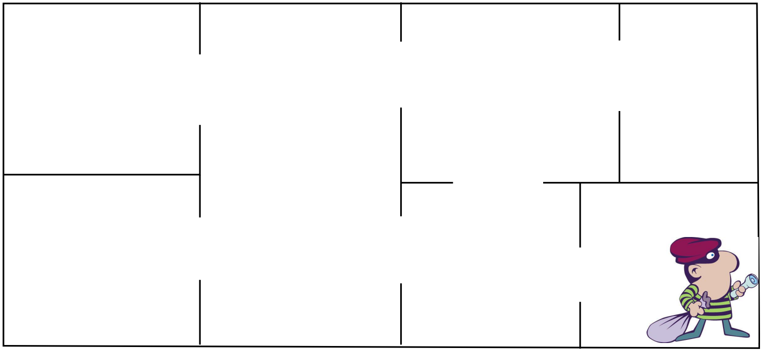
UNIVERSITÉ Clermont Auvergne

JGA 2023, Lyon

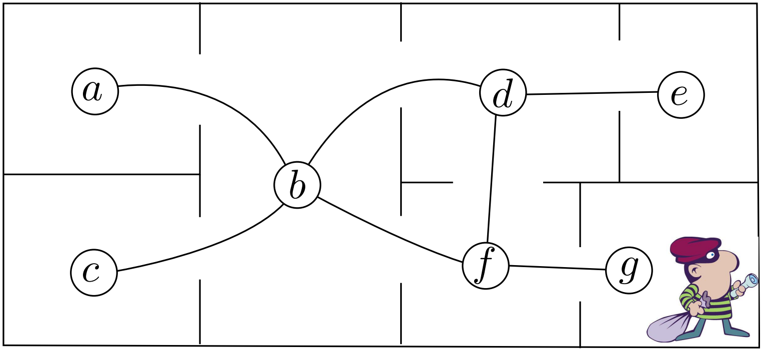
Locating a burglar

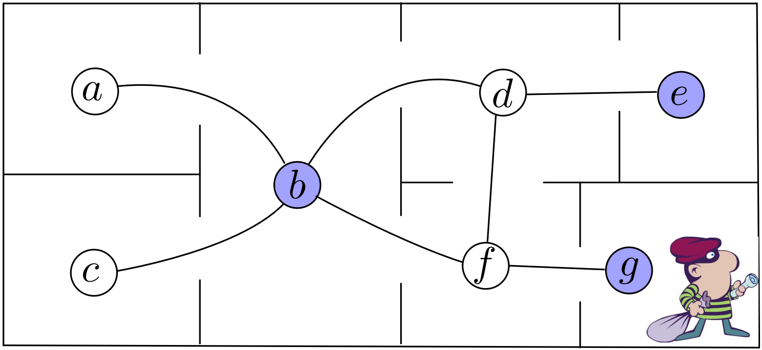


Locating a burglar



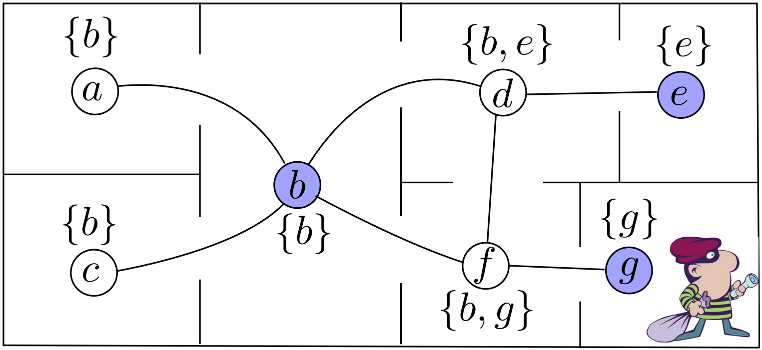
Locating a burglar



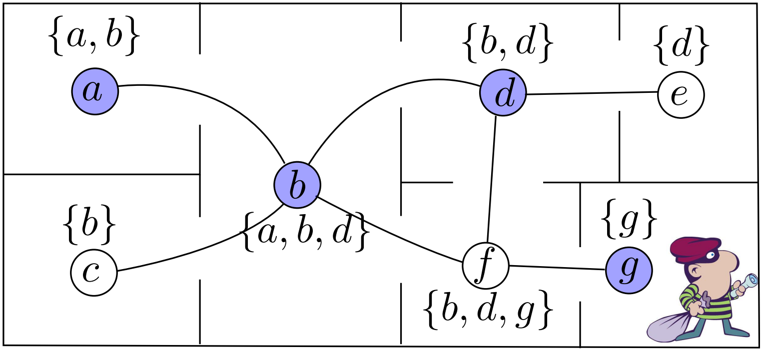


Detectors can detect movement in their room and adjacent rooms

Locating a burglar

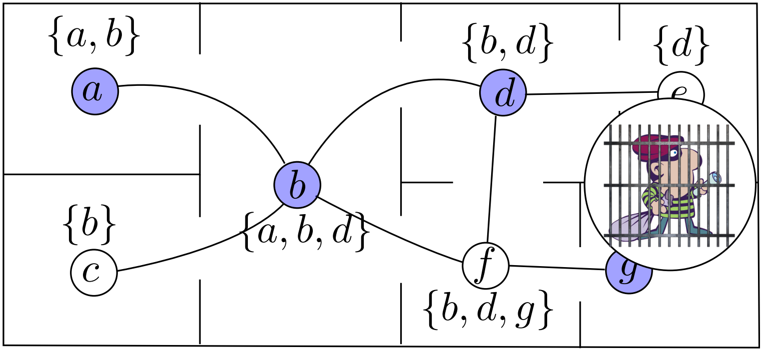


Detectors can detect movement in their room and adjacent rooms



Detectors can detect movement in their room and adjacent rooms

Locating a burglar



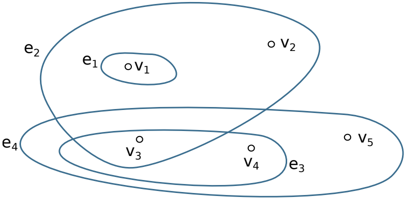
Detectors can detect movement in their room and adjacent rooms

Separating sets in hypergraphs

Separating sets in hypergraphs

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X, \mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.



$$X = \{v_1, v_2, v_3, v_4, v_5\}$$
$$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$$

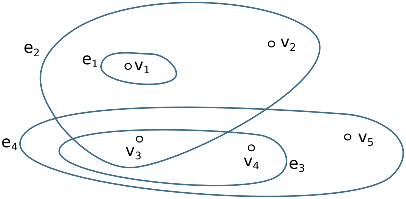
Separating sets in hypergraphs

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X,\mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

Equivalently:

for any pair e, f of edges, there is a vertex in C contained in **exactly** one of e, f.



$X = \{v_1, v_2, v_3, v_4, v_5\}$
$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$

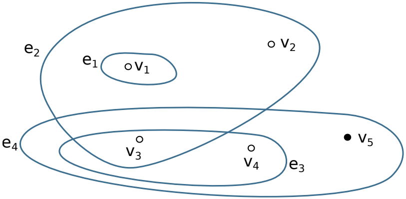
Separating sets in hypergraphs

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X,\mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

Equivalently:

for any pair e,f of edges, there is a vertex in C contained in **exactly** one of e,f.



$X = \{v_1, v_2, v_3, v_4, v_5\}$
$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$

$C = \{v_5\}$

$e_1 \cap C = \emptyset$
$e_2 \cap C = \emptyset$
$e_3 \cap C = \emptyset$
$e_4 \cap C = \{v_5\}$

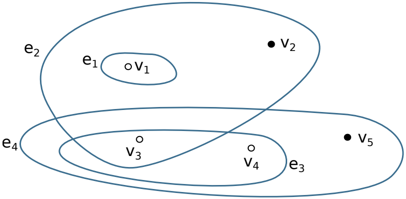
Separating sets in hypergraphs

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X,\mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

Equivalently:

for any pair e, f of edges, there is a vertex in C contained in **exactly** one of e, f.



$X = \{v_1, v_2, v_3, v_4, v_5\}$
$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$

$C = \{v_5, v_2\}$

$e_1 \cap C = \emptyset$
$e_2 \cap C = \{v_2\}$
$e_3 \cap C = \emptyset$
$e_4 \cap C = \{v_5\}$

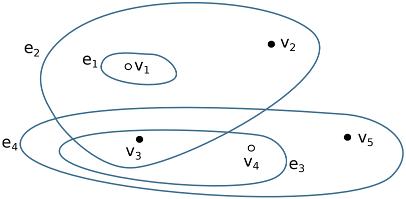
Separating sets in hypergraphs

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X, \mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

Equivalently:

for any pair e, f of edges, there is a vertex in C contained in **exactly** one of e, f.



$$X = \{v_1, v_2, v_3, v_4, v_5\}$$
$$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$$

$$C = \{v_5, v_2, v_3\}$$

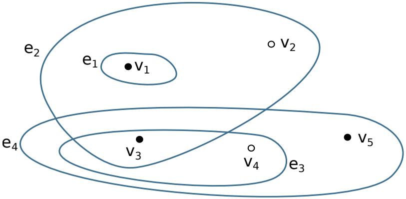
$$e_1 \cap C = \emptyset$$
$$e_2 \cap C = \{v_2, v_3\}$$
$$e_3 \cap C = \{v_3\}$$
$$e_4 \cap C = \{v_3, v_5\}$$

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X, \mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

Equivalently:

for any pair e, f of edges, there is a vertex in C contained in **exactly** one of e, f.



$$X = \{v_1, v_2, v_3, v_4, v_5\}$$
$$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$$

$$C = \{v_5, v_1, v_3\}$$

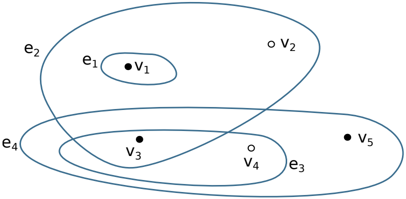
$$e_1 \cap C = \{v_1\}$$
$$e_2 \cap C = \{v_1, v_3\}$$
$$e_3 \cap C = \{v_3\}$$
$$e_4 \cap C = \{v_3, v_5\}$$

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X, \mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.

Equivalently:

for any pair e, f of edges, there is a vertex in C contained in **exactly** one of e, f.



$$X = \{v_1, v_2, v_3, v_4, v_5\}$$
$$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$$
$$C = \{v_5, v_1, v_3\}$$
$$e_1 \cap C = \{v_1\}$$
$$e_2 \cap C = \{v_1, v_3\}$$
$$e_3 \cap C = \{v_3\}$$
$$e_4 \cap C = \{v_3, v_5\}$$

Also known as Separating system, Distinguishing set, Test cover, Distinguishing transversal, Discriminating code...

Applications

- network-monitoring, fault detection (burglar)
- medical diagnostics: testing samples for diseases (*test cover*)
- biological identification (attributes of individuals)
- learning theory: teaching dimension
- machine learning: V-C dimension (Vapnik, Červonenkis, 1971)
- graph isomorphism: canonical representation of graphs (Babai, 1982)
- logic definability of graphs (Kim, Pikhurko, Spencer, Verbitsky, 2005)

General bounds, Bondy's theorem

Proposition

For a hypergraph $(X,\mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

General bounds, Bondy's theorem

Proposition

For a hypergraph $(X,\mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X,\mathscr{E})$ has size at most $|\mathscr{E}|-1$.

Proposition

For a hypergraph $(X, \mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X, \mathscr{E})$ has size at most $|\mathscr{E}| - 1$.

Proof:

Proposition

For a hypergraph $(X, \mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

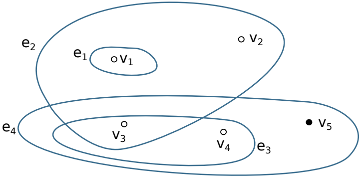
Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X, \mathscr{E})$ has size at most $|\mathscr{E}| - 1$.

Proof:

Which are the "problematic" vertices?



Proposition

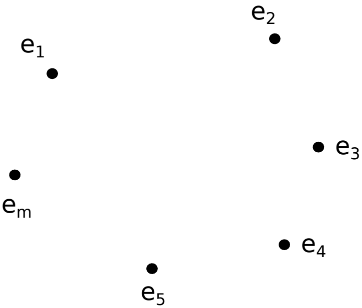
For a hypergraph $(X, \mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X, \mathscr{E})$ has size at most $|\mathscr{E}| - 1$.

Proof:



Build graph G on vertex set $V(G) = \mathscr{E}$.

General bounds, Bondy's theorem

Proposition

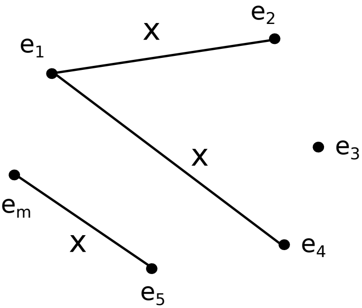
For a hypergraph $(X, \mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X, \mathscr{E})$ has size at most $|\mathscr{E}| - 1$.

Proof:



Build graph G on vertex set $V(G) = \mathscr{E}$.

Join e_i to e_j iff $e_i = e_j \cup \{x\}$ for some $x \in X$, label it "x"

General bounds, Bondy's theorem

Proposition

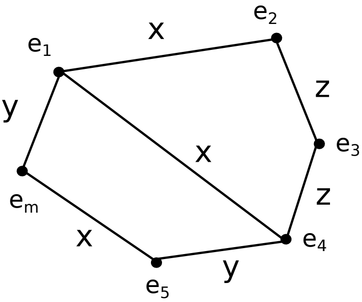
For a hypergraph $(X,\mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X,\mathscr{E})$ has size at most $|\mathscr{E}|-1$.

Proof:



Build graph G on vertex set $V(G)=\mathscr{E}$.

Join e_i to e_j iff $e_i = e_j \cup \{x\}$ for some $x \in X$, label it "x"

General bounds, Bondy's theorem

Proposition

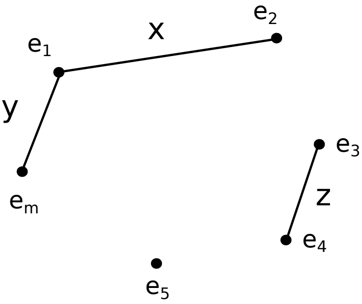
For a hypergraph $(X,\mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X,\mathscr{E})$ has size at most $|\mathscr{E}|-1$.

Proof:



Build graph G on vertex set $V(G)=\mathscr{E}$.

Join e_i to e_j iff $e_i = e_j \cup \{x\}$ for some $x \in X$, label it "x"

If an edge labeled x appears multiple times, keep only one of them.

This destroys all cycles in G! $\rightarrow$ forest

Proposition

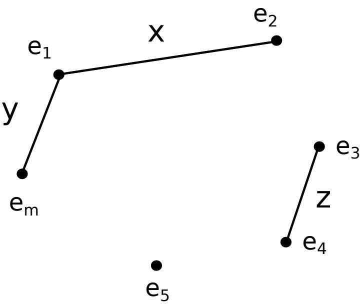
For a hypergraph $(X, \mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

Theorem (Bondy's theorem, 1972 )

A minimal separating set of hypergraph $(X, \mathscr{E})$ has size at most $|\mathscr{E}| - 1$.

Proof:



Build graph G on vertex set $V(G) = \mathscr{E}$.

Join e_i to e_j iff $e_i = e_j \cup \{x\}$ for some $x \in X$, label it "x"

If an edge labeled x appears multiple times, keep only one of them.

This destroys all cycles in G! → forest

So, at most $|\mathscr{E}| - 1$ "problematic" vertices.
→ Find "non-problematic vertex", omit it. □

Some example problems

Special cases of separating sets in hypergraphs (graph-based):

- identifying codes
- **open identifying codes**
- path/cycle identifying covers, separating path systems

Some example problems

Special cases of separating sets in hypergraphs (graph-based):

- identifying codes
- **open identifying codes**
- path/cycle identifying covers, separating path systems

A variation:

- **locating-dominating sets**
- locating-total dominating sets

Some example problems

Special cases of separating sets in hypergraphs (graph-based):

- identifying codes
- **open identifying codes**
- path/cycle identifying covers, separating path systems

A variation:

- **locating-dominating sets**
- locating-total dominating sets

Geometric versions: e.g. seperating points using disks in Euclidean space

Some example problems

Special cases of separating sets in hypergraphs (graph-based):

- identifying codes
- **open identifying codes**
- path/cycle identifying covers, separating path systems

A variation:

- **locating-dominating sets**
- locating-total dominating sets

Geometric versions: e.g. seperating points using disks in Euclidean space

Distance-based identification:

- **resolving sets (metric dimension)**
- strongly resolving sets
- centroidal locating sets
- tracking paths problem

Some example problems

Special cases of separating sets in hypergraphs (graph-based):

- identifying codes
- **open identifying codes**
- path/cycle identifying covers, separating path systems

A variation:

- **locating-dominating sets**
- locating-total dominating sets

Geometric versions: e.g. seperating points using disks in Euclidean space

Distance-based identification:

- **resolving sets (metric dimension)**
- strongly resolving sets
- centroidal locating sets
- tracking paths problem

Coloring-based identification

- Adjacent vertex-distinguishing edge-coloring
- locally identifying coloring
- locating coloring
- neighbor-locating coloring

Open identifying codes in graphs

(a.k.a. open locating-dominating sets)

Open identifying codes

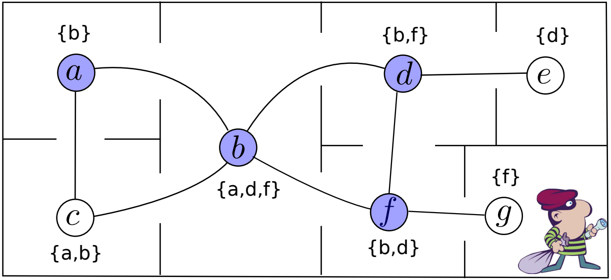
G: undirected graph $\quad$ $N(u)$: set of neighbours of v

Definition - open identifying code (Seo, Slater, 2010  )

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Notation. $OID(G)$: open identifying code number of G, minimum size of an open identifying code in G



Open identifying codes

G: undirected graph $N(u)$: set of neighbours of v

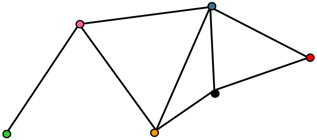
Definition - open identifying code (Seo, Slater, 2010  )

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Notation. $OID(G)$: open identifying code number of G, minimum size of an open identifying code in G

Separating code of G = separating set of open neighbourhood hypergraph of G



G: undirected graph $N(u)$: set of neighbours of v

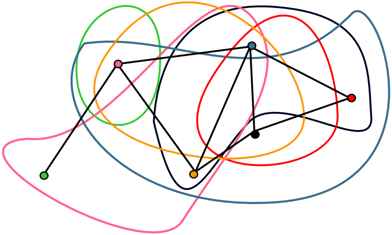
Definition - open identifying code (Seo, Slater, 2010  )

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Notation. $OID(G)$: open identifying code number of G, minimum size of an open identifying code in G

Separating code of G = separating set of open neighbourhood hypergraph of G



Open identifying codes

G: undirected graph $\quad N(u)$: set of neighbours of v

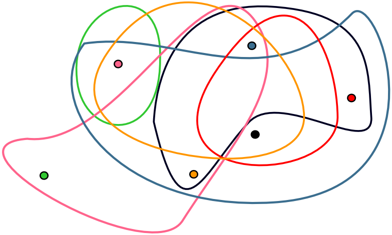
Definition - open identifying code (Seo, Slater, 2010  **)**

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Notation. $OID(G)$: open identifying code number of G,
minimum size of an open identifying code in G

Separating code of G = separating set of open neighbourhood hypergraph of G



Remark

Not all graphs have an open identifying code!

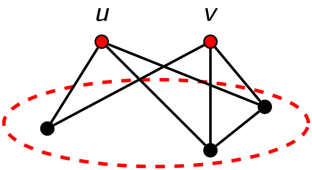
An **isolated vertex** cannot be totally dominated.

Remark

Not all graphs have an open identifying code!

An **isolated vertex** cannot be totally dominated.

Open twins = pair u, v such that $N(u) = N(v)$.

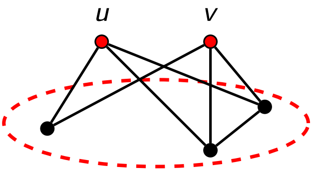


Remark

Not all graphs have an open identifying code!

An **isolated vertex** cannot be totally dominated.

Open twins = pair u, v such that $N(u) = N(v)$.



Proposition

A graph is **locatable** if and only if it has no **isolated vertices** and **open twins**.

Lower bound on $OID(G)$

Definition - open identifying code

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Proposition

G locatable graph on n vertices: $\lceil \log_2(n+1) \rceil \leq OID(G)$. (Tight.)

Lower bound on $OID(G)$

Definition - open identifying code

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Proposition

G locatable graph on n vertices: $\lceil \log_2(n+1) \rceil \leq OID(G)$. (Tight.)

Proof: For any open identifying code D, we must assign to each vertex, a distinct non-empty subset of D: $n \leq 2^{|D|} - 1$.

Definition - open identifying code

Subset D of $V(G)$ such that:

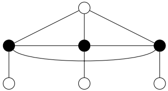
- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is a **separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Proposition

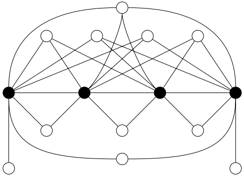
G locatable graph on n vertices: $\lceil \log_2(n+1) \rceil \leq OID(G)$. (Tight.)

Proof: For any open identifying code D, we must assign to each vertex, a distinct non-empty subset of D: $n \leq 2^{|D|} - 1$.

$OID(G) = \log_2(n+1)$



$OID(G) = \log_2(n+1)$

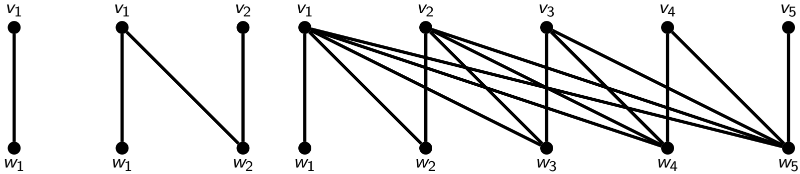


□

Upper bound on $OID(G)$?

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.

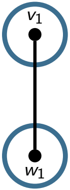


$H_1 = P_2$ $H_2 = P_4$ H_5

Upper bound on $OID(G)$?

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.

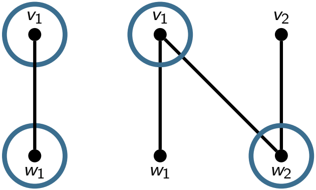


$H_1 = P_2$

Some vertices **forced** in any open identifying code because of domination

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.

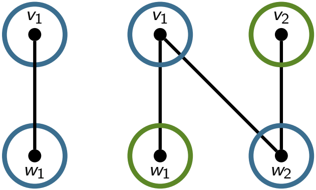


$H_1 = P_2$ $H_2 = P_4$

Some vertices **forced** in any open identifying code because of domination

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.

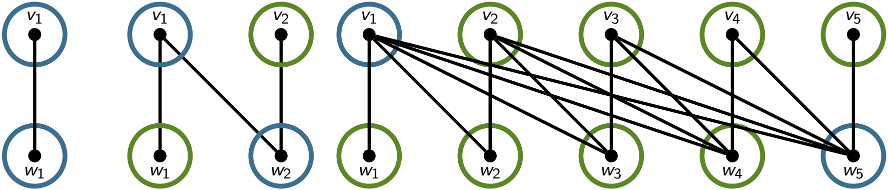


$H_1 = P_2$ $\quad$ $H_2 = P_4$

Some vertices **forced** in any open identifying code because of domination or location

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.



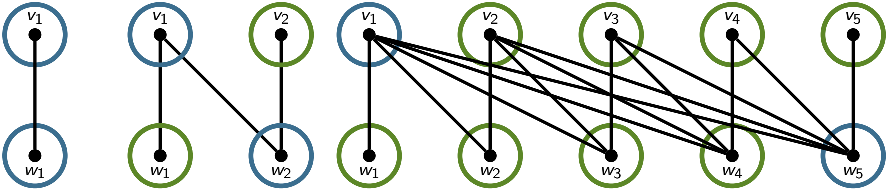
$H_1 = P_2$ $H_2 = P_4$ H_5

Some vertices **forced** in any open identifying code because of domination or location

Upper bound on $OID(G)$?

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.



$H_1 = P_2$ $H_2 = P_4$ H_5

Some vertices **forced** in any open identifying code because of domination or location

Proposition

For every half-graph H_k of order $n = 2k$, $OID(H_k) = n$.

Characterizing “bad graphs” for open identifying codes

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.
Then, $OID(G) = n$ if and only if G is a half-graph.

Characterizing “bad graphs” for open identifying codes

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.
Then, $OID(G) = n$ if and only if G is a half-graph.

Proof:

- Such a graph has only *forced* vertices: location-forced or domination-forced.

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.

Then, $OID(G) = n$ if and only if G is a half-graph.

Proof:

• Such a graph has only *forced* vertices: location-forced or domination-forced.

• By Bondy's theorem, there is at least one vertex x that is not location-forced: it is domination-forced. $\rightarrow$ Its neighbour y is of degree 1.

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.
Then, $OID(G) = n$ if and only if G is a half-graph.

Proof:

- Such a graph has only *forced* vertices: location-forced or domination-forced.
- By Bondy's theorem, there is at least one vertex x that is not location-forced: it is domination-forced. $\rightarrow$ Its neighbour y is of degree 1.
- $G' = G - \{x, y\}$ is locatable, connected.

Characterizing “bad graphs” for open identifying codes

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.
Then, $OID(G) = n$ if and only if G is a half-graph.

Proof:

• Such a graph has only *forced* vertices: location-forced or domination-forced.

• By Bondy's theorem, there is at least one vertex x that is not location-forced: it is domination-forced. $\rightarrow$ Its neighbour y is of degree 1.

• $G' = G - \{x, y\}$ is locatable, connected.

• We have $OID(G') = n-2$: By contradiction, if $OID(G') < n-2$, we could add two vertices to a solution and obtain $OID(G) < n$, a contradiction.

Characterizing "bad graphs" for open identifying codes

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.
Then, $OID(G) = n$ if and only if G is a half-graph.

Proof:

- Such a graph has only *forced* vertices: location-forced or domination-forced.
- By Bondy's theorem, there is at least one vertex x that is not location-forced: it is domination-forced. $\rightarrow$ Its neighbour y is of degree 1.
- $G' = G - \{x, y\}$ is locatable, connected.
- We have $OID(G') = n - 2$: By contradiction, if $OID(G') < n - 2$, we could add two vertices to a solution and obtain $OID(G) < n$, a contradiction.
- By induction, G' is a half-graph. We can conclude that G is a half-graph too, after some case analysis. □

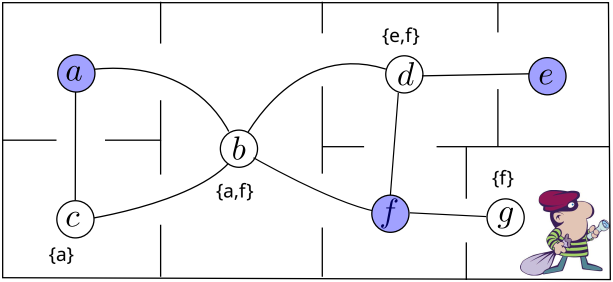
Location-domination in graphs

Definition - Locating-dominating set (Slater, 1980's) 

$D \subseteq V(G)$ locating-dominating set of G:

- for every $u \in V$, $N[v] \cap D \neq \emptyset$ (domination).
- $\forall u \neq v$ of $V(G) \setminus D$, $N(u) \cap D \neq N(v) \cap D$ (location).

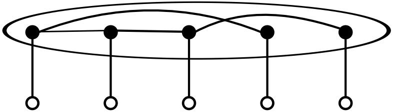
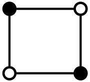
Notation. location-domination number $LD(G)$, smallest size of a locating-dominating set of G



Upper bounds

Theorem (Domination bound, Ore, 1960's )

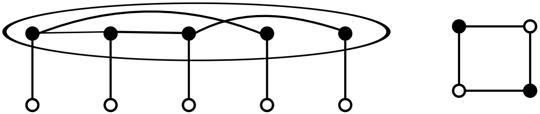
G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Tight examples:  

Upper bounds

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Tight examples: 

Proof: Consider an *inclusionwise minimal* dominating set D of G.

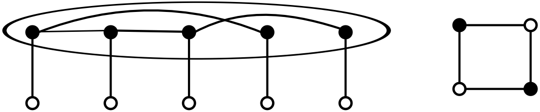
$\rightarrow$ its complement set $V(G) \setminus D$ is also a dominating set!

Thus, either D or $V(G) \setminus D$ has size at most $\frac{n}{2}$. □

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

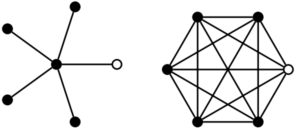
Tight examples:



Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Tight examples:



Remark: tight examples contain many twin-vertices!!

Upper bound: a conjecture

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Upper bound: a conjecture

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Remark:

- twins are easy to detect
- twins have a trivial behaviour w.r.t. location-domination

Upper bound: a conjecture

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

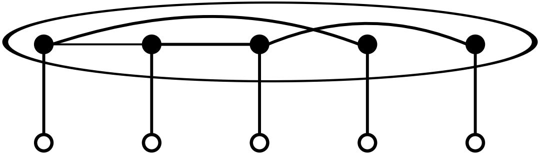
Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 1. domination-extremal graphs

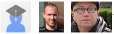


Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

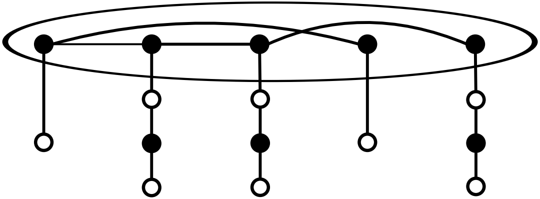
Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 2. a similar construction



Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

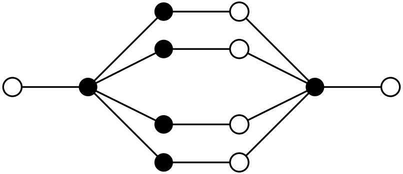
Theorem (Location-domination bound, Slater, 1980's )

G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 3. a family with domination number 2



Upper bound: a conjecture

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

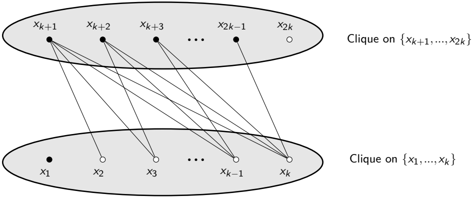
Theorem (Location-domination bound, Slater, 1980's )

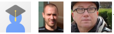
G graph of order n, no isolated vertices. Then $LD(G) \leq n-1$.

Conjecture (Garijo, González & Márquez, 2014   )

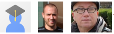
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

If true, tight: 4. family with dom. number 2: complements of half-graphs



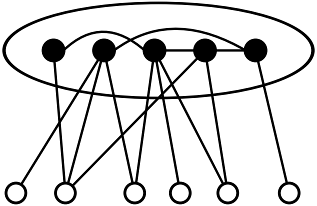
Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

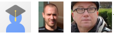
Theorem (Garijo, González & Márquez, 2014 )

Conjecture true if G has independence number $\geq n/2$. (e.g. bipartite)

Proof: every vertex cover of a twin-free graph is a locating-dominating set

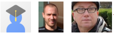


Upper bound: a conjecture - special graph classes

Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

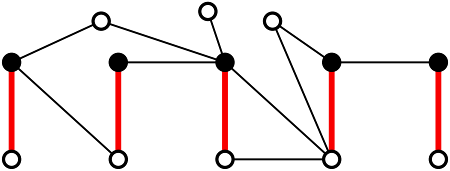
$\alpha'(G)$: matching number of G

Theorem (Garijo, González & Márquez, 2014 )

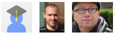
If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Proof:

- Consider special maximum matching M
- Select one vertex in each edge of M

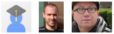


Upper bound: a conjecture - special graph classes

Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

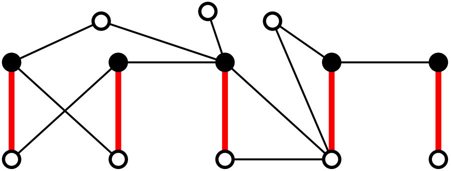
$\alpha'(G)$: matching number of G

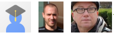
Theorem (Garijo, González & Márquez, 2014 )

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Proof:

- Consider special maximum matching M
- Select one vertex in each edge of M



Conjecture (Garijo, González & Márquez, 2014 )

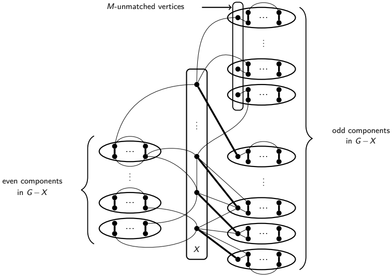
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

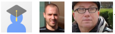
Theorem (F., Henning, 2016 )

Conjecture true if G is cubic.

Proof: Involved argument using maximum matching and Tutte-Berge theorem.

$$\alpha'(G) = \min_{X \subseteq V(G)} \frac{1}{2}\Big(|V(G)| + |X| - oc(G - X)\Big)$$



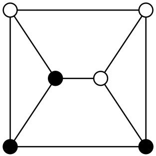
Conjecture (Garijo, González & Márquez, 2014 )

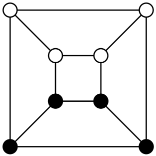
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

Conjecture true if G is cubic.

Bound is tight for cubic graphs:





Question

Do we have $LD(G) = \frac{n}{2}$ for other cubic graphs?

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

Conjecture true if G is cubic.

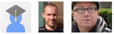
$\alpha'(G)$: matching number of G

Question

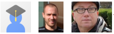
Are there twin-free (cubic) graphs with $LD(G) > \alpha'(G)$?

(if not, conjecture is true)

Upper bound: a conjecture - special graph classes

Theorem (Garijo, González & Márquez, 2014 )

Conjecture true if G has independence number $\geq n/2$. (e.g. bipartite)

Theorem (Garijo, González & Márquez, 2014 )

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

Conjecture true if G is cubic.

Theorem (F., Henning, Löwenstein, Sasse, 2016 )

Conjecture true if G is split graph or complement of bipartite graph.

Theorem (Chakraborty, F., Parreau, Wagler, 2023 )

Conjecture true if G is a block graph.

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2016   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

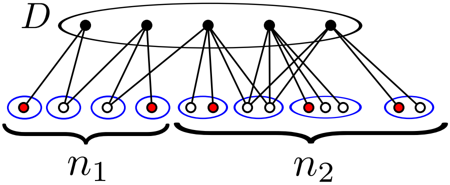
Conjecture (Garijo, González & Márquez, 2014   )

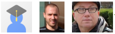
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2016   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

Proof: • There exists a dominating set D such that each vertex has a private neighbour, thus $|D| \leq n_1 + n_2$. Take such D that is inclusionwise maximal.



Conjecture (Garijo, González & Márquez, 2014 )

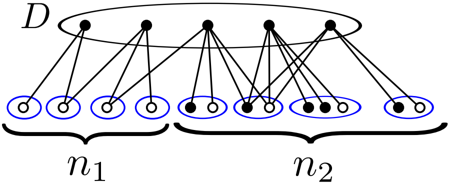
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

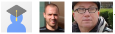
Theorem (F., Henning, Löwenstein, Sasse, 2016 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

Proof: • There exists a dominating set D such that each vertex has a private neighbour, thus $|D| \leq n_1 + n_2$. Take such D that is inclusionwise maximal.

• there is a LD-set of size $n - n_1 - n_2$



Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

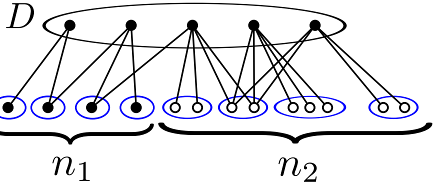
Theorem (F., Henning, Löwenstein, Sasse, 2016 )

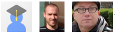
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

Proof: • There exists a dominating set D such that each vertex has a private neighbour, thus $|D| \leq n_1 + n_2$. Take such D that is inclusionwise maximal.

• there is a LD-set of size $n - n_1 - n_2$

• there is a LD-set of size $|D| + n_1$ because D is maximal



Conjecture (Garijo, González & Márquez, 2014 )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, Löwenstein, Sasse, 2016 )

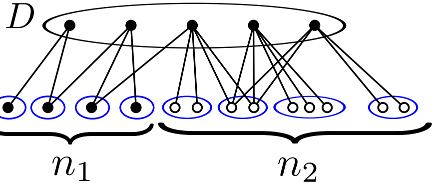
G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{2}{3}n$.

Proof: • There exists a dominating set D such that each vertex has a private neighbour, thus $|D| \leq n_1 + n_2$. Take such D that is inclusionwise maximal.

• there is a LD-set of size $n - n_1 - n_2$

• there is a LD-set of size $|D| + n_1$ because D is maximal

• $\min\{|D| + n_1, n - n_1 - n_2\} \leq \frac{2}{3}n$



Lower bounds
(neighbourhood complexity)

Proposition

G graph, n vertices, $LD(G) = k$. Then, $n \leq 2^k + k - 1$.

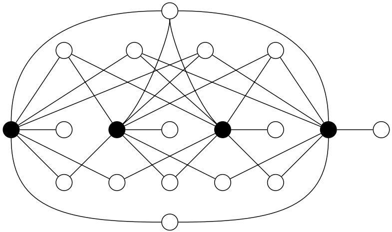
Proposition

G graph, n vertices, $LD(G) = k$. Then, $n \leq 2^k + k - 1$. $\rightarrow LD(G) \geq \lceil \log_2(n+1) - 1 \rceil$

Proposition

G graph, n vertices, $LD(G) = k$. Then, $n \leq 2^k + k - 1$. $\rightarrow LD(G) \geq \lceil \log_2(n+1) - 1 \rceil$

Tight example ($k = 4$):



Lower bounds

Proposition

G graph, n vertices, $LD(G)=k$. Then, $n \leq 2^k+k-1$. $\rightarrow LD(G) \geq \lceil \log_2(n+1)-1 \rceil$

Theorem (Slater, 1980's )

G tree of order n, $LD(G)=k$. Then $n \leq 3k-1 \rightarrow LD(G) \geq \frac{n+1}{3}$.

Tight examples:

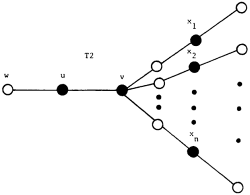


FIG. 2. Tree T2

Proposition

G graph, n vertices, $LD(G)=k$. Then, $n \leq 2^k+k-1$. $\rightarrow LD(G) \geq \lceil \log_2(n+1)-1 \rceil$

Theorem (Slater, 1980's )

G tree of order n, $LD(G)=k$. Then $n \leq 3k-1 \rightarrow LD(G) \geq \frac{n+1}{3}$.

Theorem (Rall & Slater, 1980's  )

G planar graph, order n, $LD(G)=k$. Then $n \leq 7k-10 \rightarrow LD(G) \geq \frac{n+10}{7}$.

Tight examples:

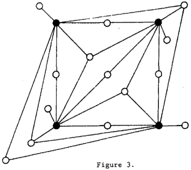
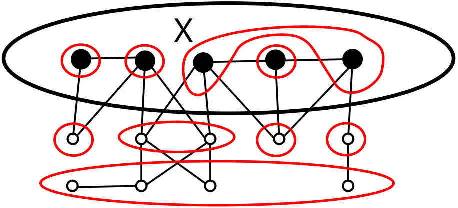


Figure 3.

Neighbourhood complexity of a graph G:

maximum number $|\{N(v) \cap X\}|$ of neighbourhoods inside any set X of k vertices, as a function of k

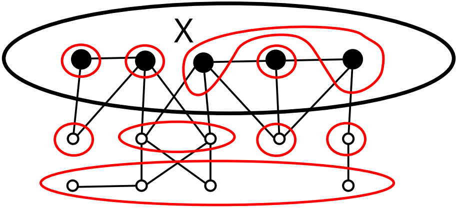


$|\{N(v) \cap X\}| = 9$

Neighbourhood complexity

Neighbourhood complexity of a graph G:

maximum number $|\{N(v) \cap X\}|$ of neighbourhoods inside any set X of k vertices, as a function of k

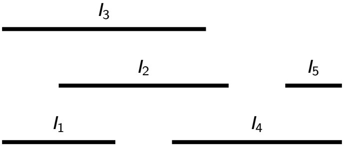


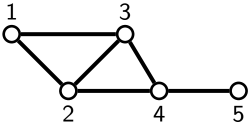
$|\{N(v) \cap X\}| = 9$

- General graphs : exponential neighbourhood complexity 2^k
- Trees/planar graphs : linear neighbourhood complexity $O(k)$

Definition - Interval graph

Intersection graph of intervals of the real line.





Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

G interval graph of order n, $LD(G) = k$.

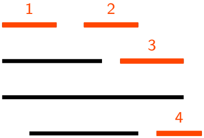
Then $n \leq \frac{k(k+1)}{2}$, i.e. $LD(G) = \Omega(\sqrt{n})$.

Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017 )

G interval graph of order n, $LD(G)=k$.

Then $n \leq \frac{k(k+1)}{2}$, i.e. $LD(G)=\Omega(\sqrt{n})$.

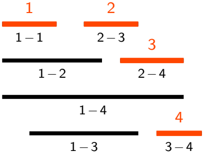


- Identifying code D of size k.
- Define zones using the right points of intervals in D.

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017 )

G interval graph of order n, $LD(G) = k$.

$$\text{Then } n \leq \frac{k(k+1)}{2}, \text{ i.e. } LD(G) = \Omega(\sqrt{n}).$$



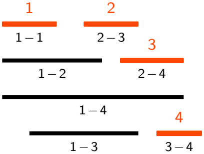
- Identifying code D of size k.
- Define zones using the right points of intervals in D.
- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

G interval graph of order n, $LD(G) = k$.

Then $n \leq \frac{k(k+1)}{2}$, i.e. $LD(G) = \Omega(\sqrt{n})$.



- Identifying code D of size k.
- Define zones using the right points of intervals in D.
- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

$\rightarrow n \leq \sum_{i=1}^{k}(k-i) = \frac{k(k+1)}{2}$.

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

G interval graph of order n, $LD(G) = k$.

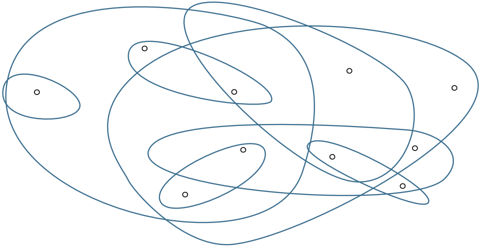
Then $n \leq \frac{k(k+1)}{2}$, i.e. $LD(G) = \Omega(\sqrt{n})$.

Tight:

Measure of intersection complexity of sets in a hypergraph $(X, \mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered:

for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.

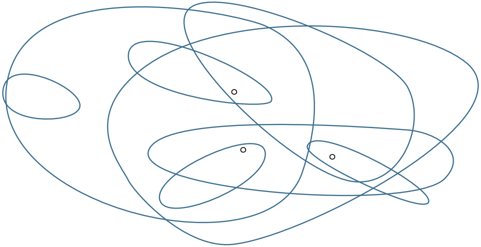


V-C dimension of H: maximum size of a shattered set in H

Measure of intersection complexity of sets in a hypergraph $(X,\mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered: for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.

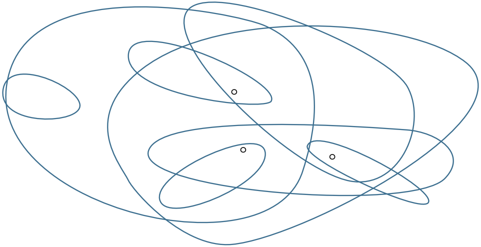


V-C dimension of H: maximum size of a shattered set in H

Measure of intersection complexity of sets in a hypergraph $(X,\mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered:

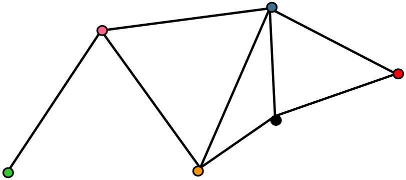
for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.

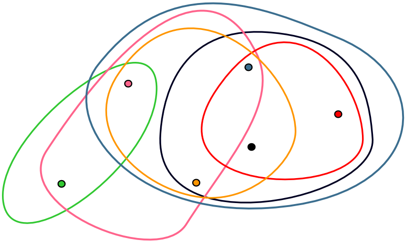


V-C dimension of H: maximum size of a shattered set in H

Typically bounded for geometric hypergraphs: 

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph





Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph

Typically bounded for geometric intersection graphs:

$\rightarrow$ interval graphs ($d = 2$), C_4-free graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph

Typically bounded for geometric intersection graphs:

$\rightarrow$ interval graphs ($d = 2$), C_4-free graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

Theorem (Sauer-Shelah Lemma, 1972  )

Let H be a hypergraph of V-C dimension at most d. Then, any set S of vertices has at most $|S|^d$ distinct traces.

Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph

Typically bounded for geometric intersection graphs:

→ interval graphs ($d = 2$), C_4-free graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

Theorem (Sauer-Shelah Lemma, 1972  )

Let H be a hypergraph of V-C dimension at most d. Then, any set S of vertices has at most $|S|^d$ distinct traces.

Corollary

G graph of order n, $LD(G) = k$, V-C dimension $\leq d$. Then $n = O(k^d)$.

Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph

Typically bounded for geometric intersection graphs:

→ interval graphs ($d=2$), C_4-free graphs ($d=2$), line graphs ($d=4$), permutation graphs ($d=3$), unit disk graphs ($d=3$), planar graphs ($d=4$)...

Theorem (Sauer-Shelah Lemma, 1972  )

Let H be a hypergraph of V-C dimension at most d. Then, any set S of vertices has at most $|S|^d$ distinct traces.

Corollary

G graph of order n, $LD(G)=k$, V-C dimension $\leq d$. Then $n=O(k^d)$.

$O(k^2)$: interval, permutation, line...

$O(k)$: cographs, unit interval, bipartite permutation, block...

Sparse/structured graphs

Graph classes of bounded expansion: all shallow minors of its members have bounded average degree → e.g. planar graphs, minor-closed classes, bounded degree...

Theorem (Reidl, Sánchez-Villaamil, Stavropoulos, 2019   )

Let $\mathscr{C}$ be a graph class of bounded expansion. Let G in $\mathscr{C}$, order n, and $LD(G) = k$. Then, $n \leq f(\mathscr{C})k$.

Graph classes of bounded expansion: all shallow minors of its members have bounded average degree → e.g. planar graphs, minor-closed classes, bounded degree...

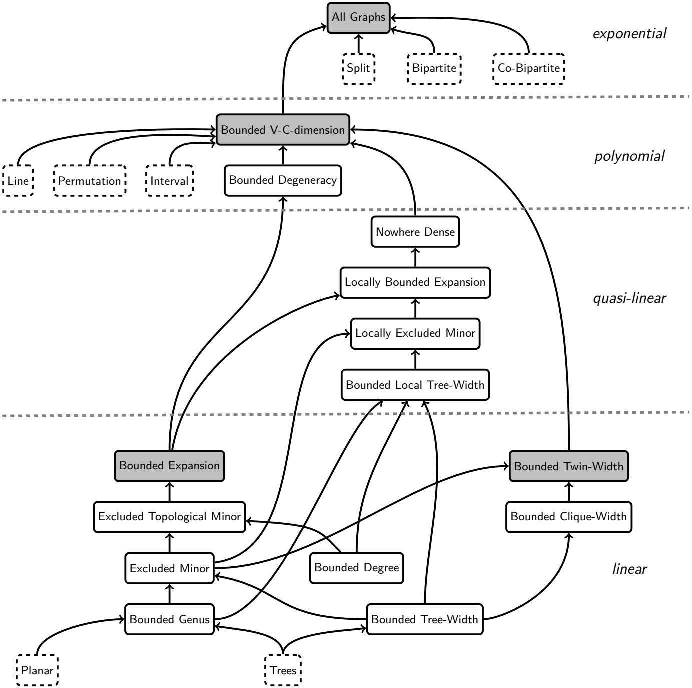
Theorem (Reidl, Sánchez-Villaamil, Stavropoulos, 2019   )

Let $\mathscr{C}$ be a graph class of bounded expansion. Let G in $\mathscr{C}$, order n, and $LD(G) = k$.
Then, $n \leq f(\mathscr{C})k$.

Recently introduced structural measure: twin-width.

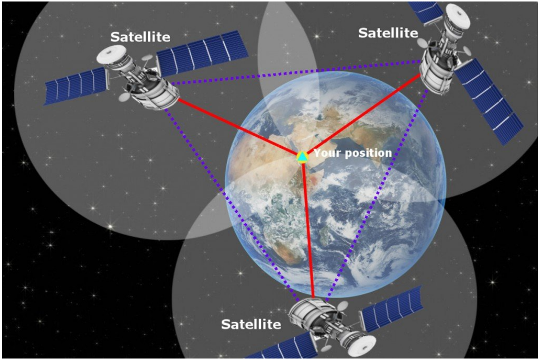
Theorem (Bonnet, F., Lehtilä, Parreau, 2024   )

Let G be a graph of twin-width at most d and order n, and $LD(G) = k$.
Then, $n \leq (d+2)2^{d+1}k$.


All Graphs
Split
Bipartite
Co-Bipartite
exponential
Bounded V-C-dimension
Line
Permutation
Interval
Bounded Degeneracy
polynomial
Nowhere Dense
Locally Bounded Expansion
quasi-linear
Locally Excluded Minor
Bounded Local Tree-Width
Bounded Expansion
Bounded Twin-Width
Excluded Topological Minor
Bounded Clique-Width
Excluded Minor
Bounded Degree
linear
Bounded Genus
Bounded Tree-Width
Planar
Trees

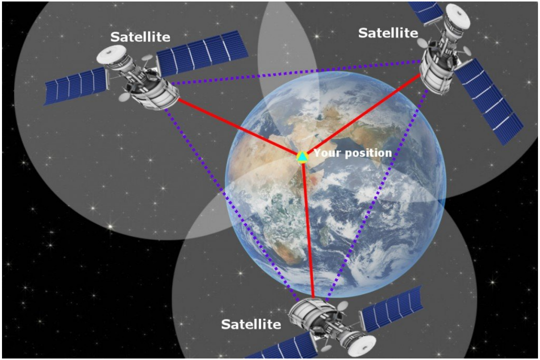
Metric dimension

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



Determination of Position in 3D euclidean space

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



Question

Does the "GPS" approach also work in undirected unweighted graphs?

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w,u) \neq dist(w,v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

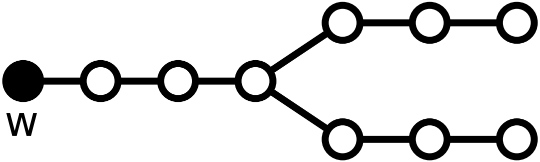
$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$

Now, $w \in V(G)$ distinguishes $\{u,v\}$ if $dist(w,u) \neq dist(w,v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

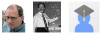
$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u,v\}.$$



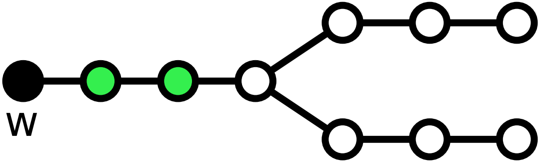
Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)



$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$

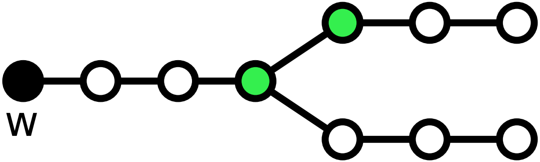


Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976) 

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$

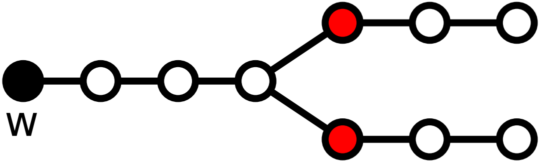


Now, $w \in V(G)$ distinguishes $\{u,v\}$ if $dist(w,u) \neq dist(w,v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u,v\}.$$



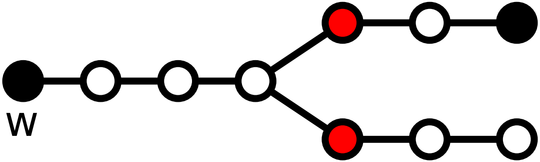
Metric dimension

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$

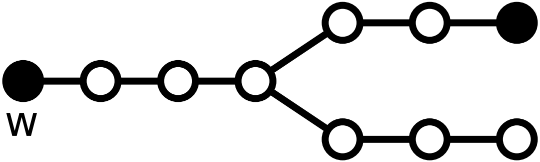


Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$

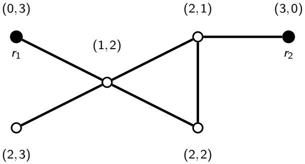


Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976) 

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$



$R = \{r_1, r_2\}$

$MD(G) = 2$

Every vertex receives a unique distance-vector w.r.t. to the solution vertices.

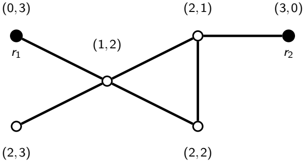
Metric dimension

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w,u) \neq dist(w,v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$



$R = \{r_1, r_2\}$

$MD(G) = 2$

Every vertex receives a unique distance-vector w.r.t. to the solution vertices.

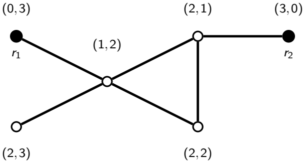
$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that distinguishes } \{u, v\}.$$



$R = \{r_1, r_2\}$

$MD(G) = 2$

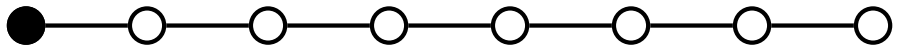
Every vertex receives a unique distance-vector w.r.t. to the solution vertices.

$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Remark

- Any locating-dominating set is a resolving set, hence $MD(G) \leq LD(G)$.
- A locating-dominating set can be seen as a “distance-1-resolving set”.

Examples

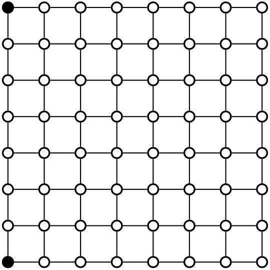


Proposition

$MD(G) = 1 \Leftrightarrow G$ is a path

Proposition

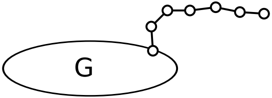
$MD(G) = 1 \Leftrightarrow G$ is a path



Proposition

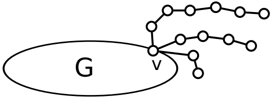
For any square grid G, $MD(G) = 2$.

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.

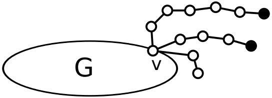


Trees

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.



Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.

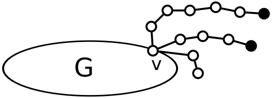


Observation

R resolving set. If v has k legs, at least $k-1$ legs contain a vertex of R.

Simple leg rule: if v has $k \geq 2$ legs, select $k-1$ leg endpoints.

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.



Observation

R resolving set. If v has k legs, at least $k-1$ legs contain a vertex of R.

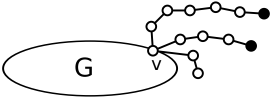
Simple leg rule: if v has $k \geq 2$ legs, select $k-1$ leg endpoints.

Theorem (Slater, 1975 )

For any tree, the simple leg rule produces an optimal resolving set.

Trees

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.



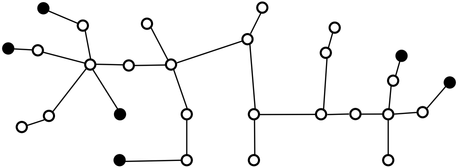
Observation

R resolving set. If v has k legs, at least $k-1$ legs contain a vertex of R.

Simple leg rule: if v has $k \geq 2$ legs, select $k-1$ leg endpoints.

Theorem (Slater, 1975 )

For any tree, the simple leg rule produces an optimal resolving set.



Bounds with diameter

Example of path: no bound $n \leq f(MD(G))$ possible.

Bounds with diameter

Example of path: no bound $n \leq f(MD(G))$ possible.

Theorem (Khuller, Raghavachari & Rosenfeld, 2002   )

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter D: maximum distance between two vertices)

Proof: Every vertex not in the solution R is assigned to a unique vector of length k, with values in $\{1, \ldots, D\}$: D^k possibilities, plus the k ones in R. □

Example of path: no bound $n \leq f(MD(G))$ possible.

Theorem (Khuller, Raghavachari & Rosenfeld, 2002   )

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter D: maximum distance between two vertices)

Proof: Every vertex not in the solution R is assigned to a unique vector of length k, with values in $\{1, \ldots, D\}$: D^k possibilities, plus the k ones in R. □

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

G interval graph of order n, $MD(G) = k$, diameter D. Then $n = O(Dk^2)$ i.e. $k = \Omega\left(\sqrt{\frac{n}{D}}\right)$. (Tight.)

Example of path: no bound $n \leq f(MD(G))$ possible.

Theorem (Khuller, Raghavachari & Rosenfeld, 2002   )

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter D: maximum distance between two vertices)

Proof: Every vertex not in the solution R is assigned to a unique vector of length k, with values in $\{1, \ldots, D\}$: D^k possibilities, plus the k ones in R. □

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

G interval graph of order n, $MD(G) = k$, diameter D. Then $n = O(Dk^2)$ i.e. $k = \Omega\left(\sqrt{\frac{n}{D}}\right)$. (Tight.)

→ Proof is similar as that for locating-dominating sets.

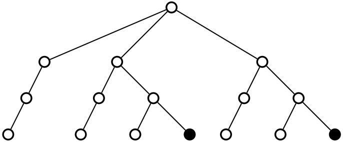
Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

T a tree with diameter D and $MD(T) = k$, then

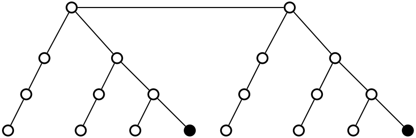
$$n \leq \begin{cases} \frac{1}{8}(kD+4)(D+2) & \text{if } D \text{ even,} \\ \frac{1}{8}(kD-k+8)(D+1) & \text{if } D \text{ odd.} \end{cases} = \Theta(kD^2)$$

Bounds are tight.

$k = 2,\ D = 6$



$k = 2,\ D = 7$



Using the concept of distance-VC-dimension:

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

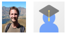
G planar with diameter D and $MD(G) = k$, then $n = O(k^4 D^4)$.

Using the concept of distance-VC-dimension:

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

G planar with diameter D and $MD(G) = k$, then $n = O(k^4 D^4)$.

Using the concept of profiles and r-neighbourhood complexity:

Theorem (Joret, Rambaud, 2023+ )

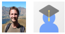
G planar with diameter D and $MD(G) = k$, then $n = O(kD^4)$.

Using the concept of distance-VC-dimension:

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

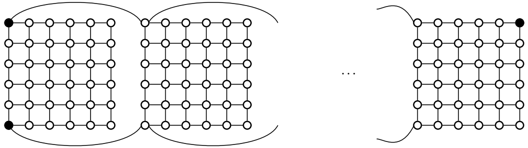
G planar with diameter D and $MD(G) = k$, then $n = O(k^4 D^4)$.

Using the concept of profiles and r-neighbourhood complexity:

Theorem (Joret, Rambaud, 2023+ )

G planar with diameter D and $MD(G) = k$, then $n = O(kD^4)$.

Tight? Planar example with $k = 3$ and $n = \Theta(D^3)$:

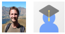


Using the concept of distance-VC-dimension:

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

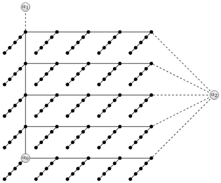
G planar with diameter D and $MD(G)=k$, then $n=O(k^4D^4)$.

Using the concept of profiles and r-neighbourhood complexity:

Theorem (Joret, Rambaud, 2023+ )

G planar with diameter D and $MD(G)=k$, then $n=O(kD^4)$.

Tight? Planar example with treewidth 2 and $n=\Theta(kD^3)$:



Selected open questions

- Graphs G of order n with $OID(G) = n - 1$?
- Conjecture: $LD(G) \leq n/2$ in the absence of twins
- Find tight bounds for Metric Dimension of planar graphs of diameter D
  (and other classes)
- Neighbourhood complexity at distance r
  $\rightarrow$ graphs of bounded twin-width, planar graphs...
- Algorithms : efficient algorithms for unit interval graphs?

Selected open questions

- Graphs G of order n with $OID(G) = n - 1$?
- Conjecture: $LD(G) \leq n/2$ in the absence of twins
- Find tight bounds for Metric Dimension of planar graphs of diameter D
  (and other classes)
- Neighbourhood complexity at distance r
  $\rightarrow$ graphs of bounded twin-width, planar graphs...
- Algorithms : efficient algorithms for unit interval graphs?

THANKS FOR YOUR ATTENTION!