Identification problems in graphs and other discrete structures

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LIMOS

UNIVERSITÉ Clermont Auvergne

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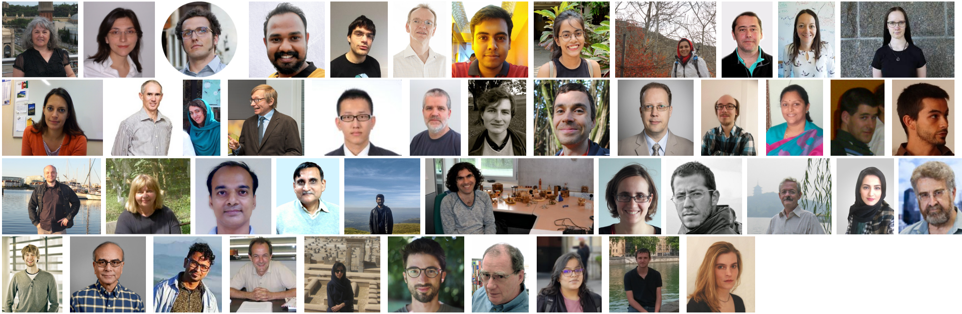

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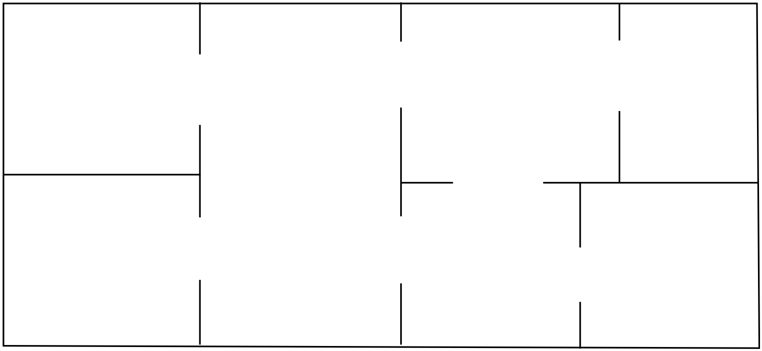

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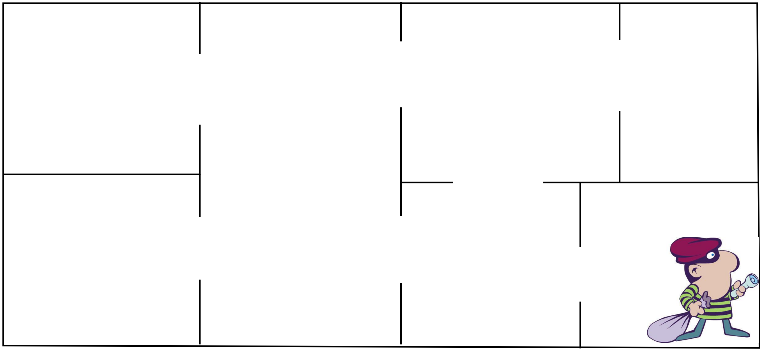

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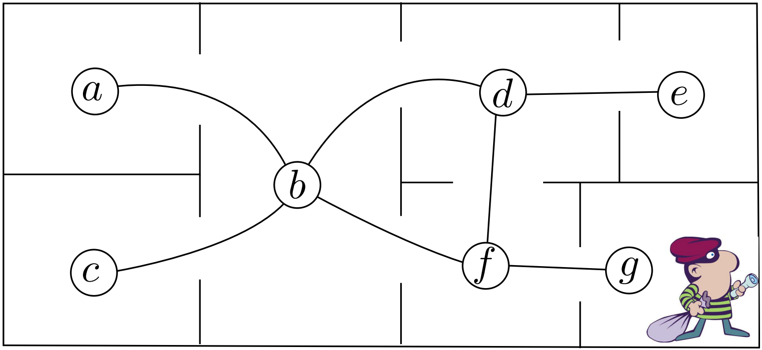
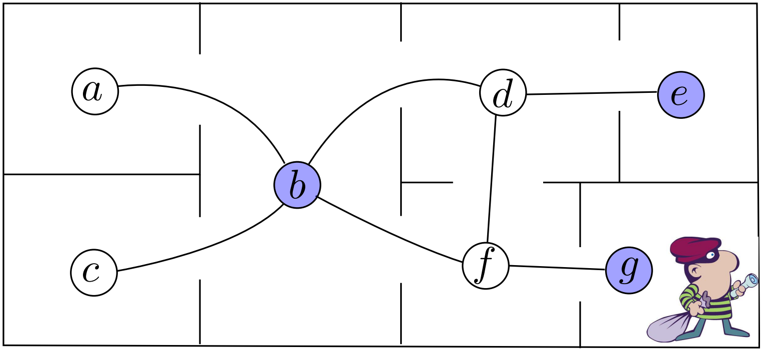
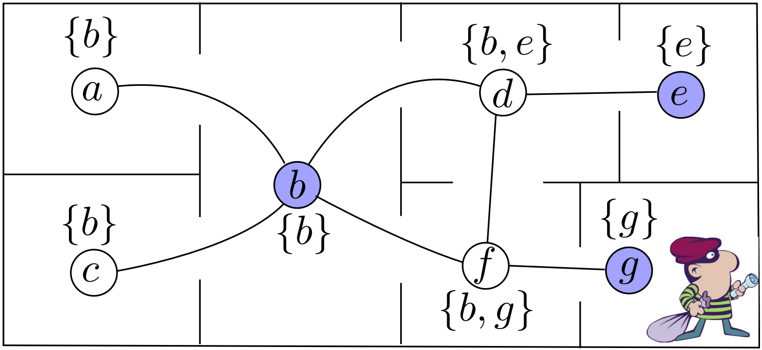




Detectors can detect movement in their room and adjacent rooms



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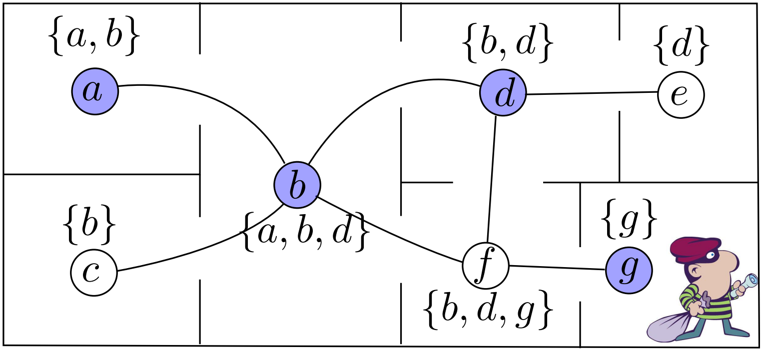

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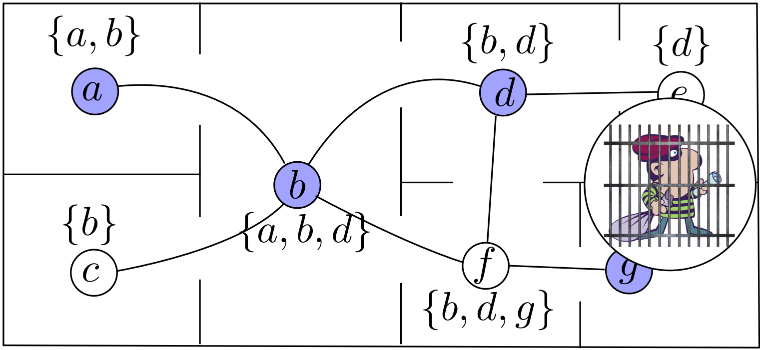

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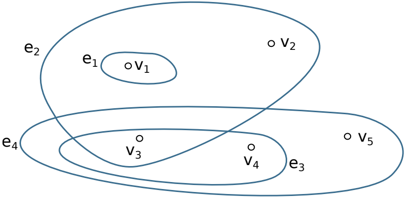
Separating sets in hypergraphs

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X, \mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.



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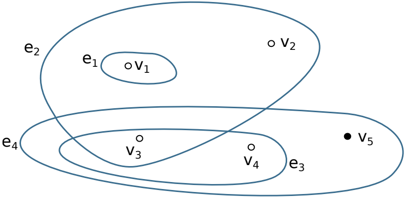

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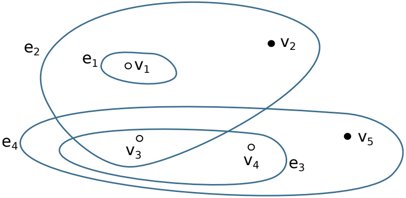

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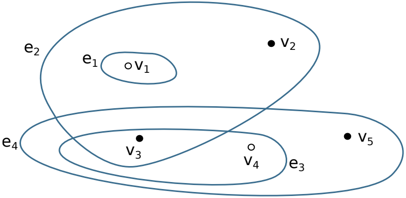

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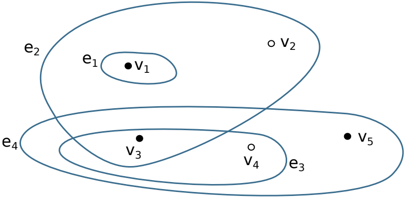

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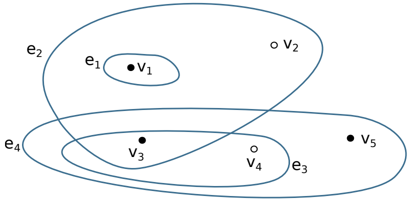

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Equivalently: for any pair e, f of edges, there is a vertex of C in exactly one of e, f.

$\rightarrow$ hitting set of the symmetric differences of all pairs of hyperedges

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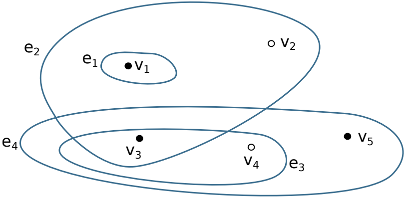

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Also known as Separating system, Distinguishing set, Test cover, Distinguishing transversal, Discriminating code...

Applications

- network-monitoring, fault detection (burglar)
- medical diagnostics: testing samples for diseases (*test cover*)
- biological identification (attributes of individuals)
- learning theory: teaching dimension
- machine learning: V-C dimension (Vapnik, Červonenkis, 1971)
- graph isomorphism: canonical representation of graphs (Babai, 1982)
- logic definability of graphs (Kim, Pikhurko, Spencer, Verbitsky, 2005)

Proposition

For a hypergraph $(X,\mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. □

General bounds, Bondy's theorem

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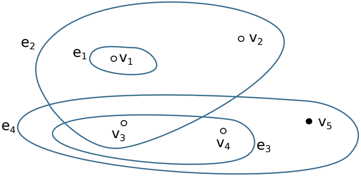
Which are the "problematic" vertices?



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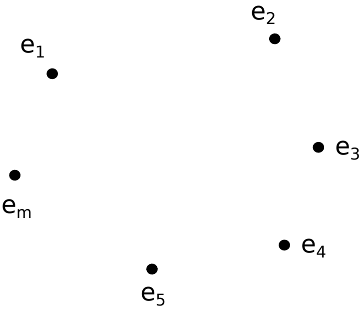

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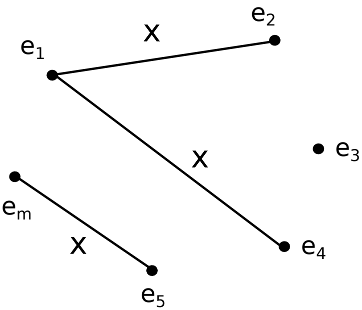

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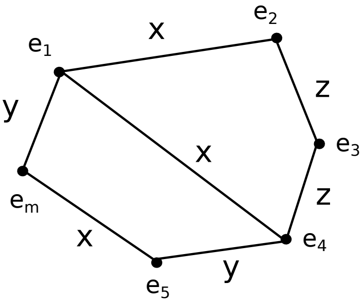

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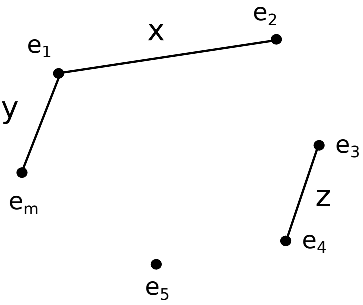

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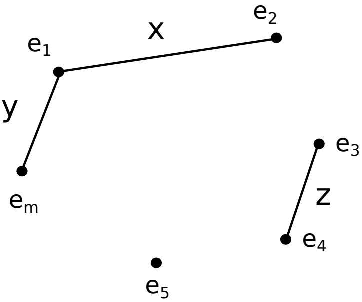

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So, there are at most $|\mathscr{E}|-1$ "problematic" vertices. $\rightarrow$ Find one "non-problematic vertex" and omit it. □

Some example problems

- Special graph-based cases of separating sets in hypergraphs:
  - **identifying codes**
  - open neighbourhood locating-dominating sets
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- Colouring-based identification:
  - Locally identifying colourings
  - Locating-colourings
  - Neighbour-locating colourings

Identifying codes in graphs

G: undirected graph
$N[u]$: set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that:

- C is a **dominating set**: $\forall u \in V(G)$, $N[u] \cap C \neq \emptyset$, and
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$ID(G)$: identifying code number of G,
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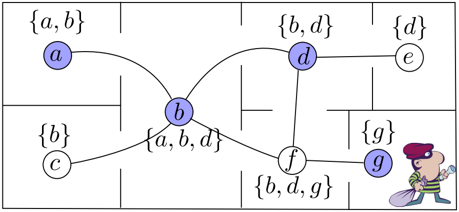

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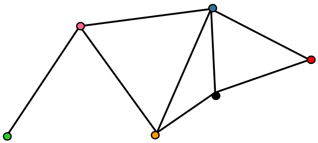

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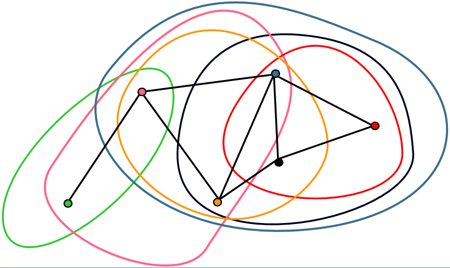

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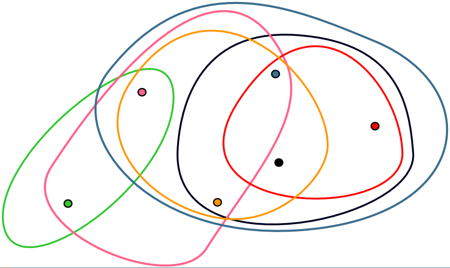

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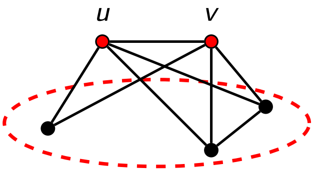
Identifying code number: $ID(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$

Remark

Not all graphs have an identifying code!

Closed twins = pair u, v such that $N[u] = N[v]$.



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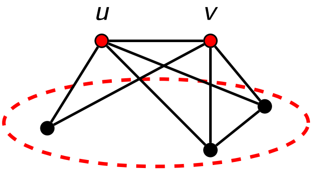

Proposition

A graph is **identifiable** if and only if it is **closed twin-free** (i.e. has no twins).

Bounds on $ID(G)$

n: number of vertices

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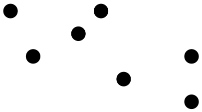
$ID(G) = n \Leftrightarrow G$ has no edges



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Theorem

G identifiable, n vertices, some edges: $\lceil \log_2(n+1) \rceil \leq ID(G) \leq n-1$

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G identifiable, n vertices, some edges: $\lceil \log_2(n+1) \rceil \leq ID(G) \leq n-1$

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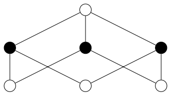

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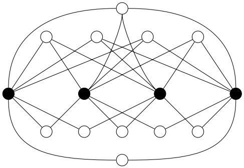

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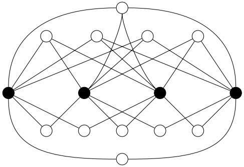

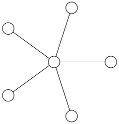

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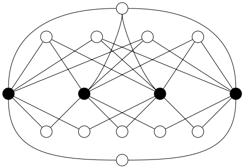

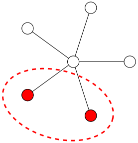

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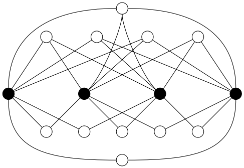

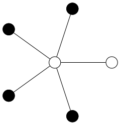

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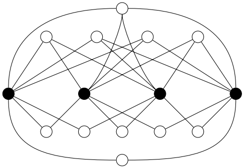

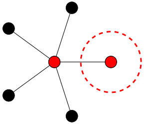

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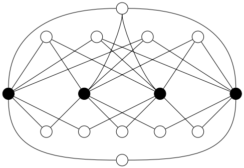

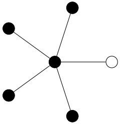

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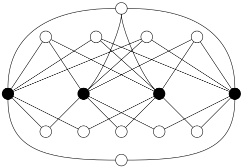

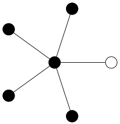

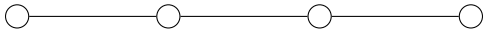

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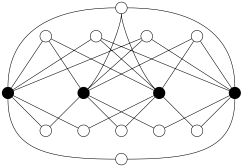

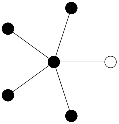

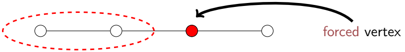

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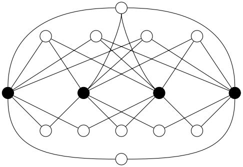
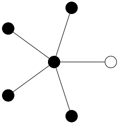
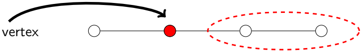




forced vertex



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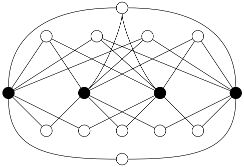

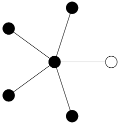

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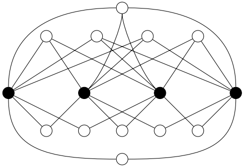
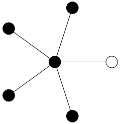
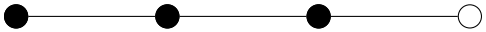






A question

Theorem (Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

G identifiable graph on n vertices with at least one edge:

$$ID(G) \leq n - 1$$

Question

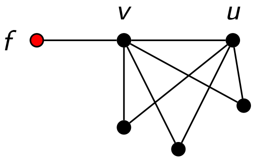
What are the graphs G with n vertices and $ID(G) = n - 1$?

Forced vertices

u, v such that $N[v] \ominus N[u] = \{f\}$:

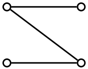
f belongs to **any identifying code**

$\rightarrow$ f forced by u, v.

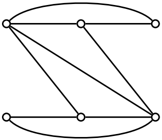


Graphs with many forced vertices

Special path powers: $A_k = P_{2k}^{k-1}$ (also called complements of half-graphs)



$A_2 = P_4$



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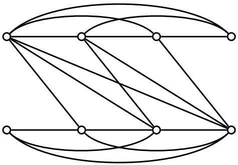

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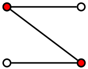

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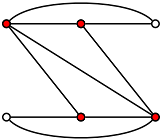

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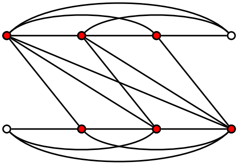

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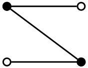

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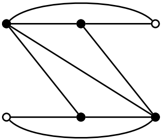

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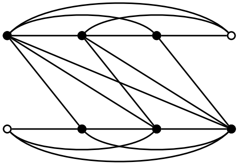

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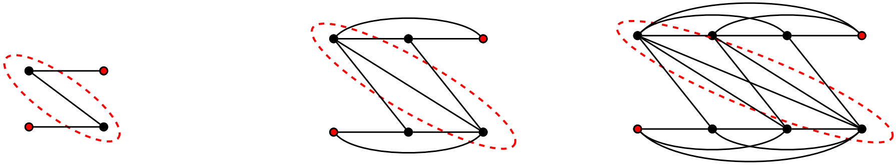

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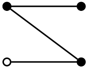

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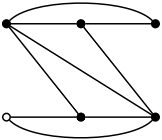

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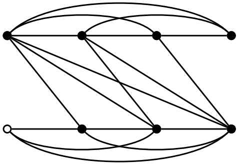

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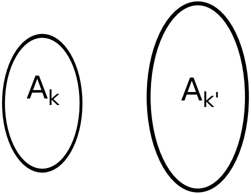
Proposition

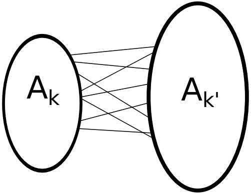
$$ID(A_k) = n - 1$$

Constructions using joins

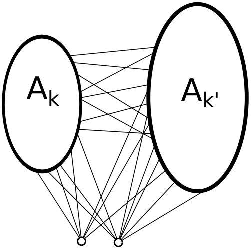


Two graphs A_k and $A_{k'}$

Constructions using joins

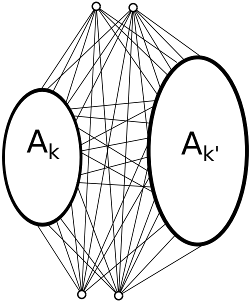


Join: add all edges between them



Join the new graph to two non-adjacent vertices ($\overline{K_2}$)

Constructions using joins



Join the new graph to two non-adjacent vertices, again

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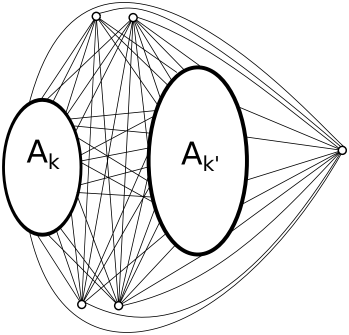
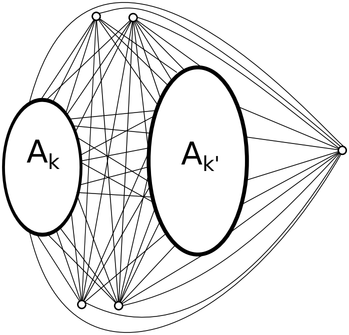


Finally, add a **universal vertex**



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Proposition

At each step, the constructed graph has $ID = n - 1$

A characterization

(1) stars
(2) $A_k = P_{2k}^{k-1}$
(3) joins between 0 or more members of (2) and 0 or more copies of $\overline{K_2}$
(4) (2) or (3) with a universal vertex

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

G connected identifiable graph, n vertices:

$$ID(G) = n - 1 \Leftrightarrow G \in (1), (2), (3) \text{ or } (4)$$

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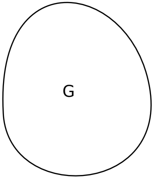

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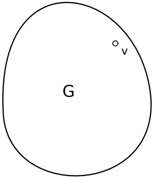

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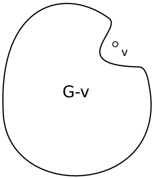
- Lemma: $ID(G-v) = n'-1$



A characterization

(1) stars
(2) $A_k = P_{2k}^{k-1}$
(3) joins between 0 or more members of (2) and 0 or more copies of $\overline{K_2}$
(4) (2) or (3) with a universal vertex

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

G connected identifiable graph, n vertices:

$$ID(G) = n-1 \Leftrightarrow G \in (1), (2), (3) \text{ or } (4)$$

- G: minimum counterexample
- v: vertex such that $G-v$ identifiable (exists)
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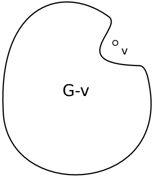
$\Rightarrow$ By minimality of G:

$$G - v \in (1), (2), (3) \text{ or } (4)$$



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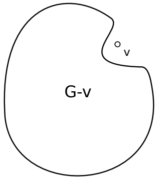
- Put v back $\Rightarrow$ **contradiction:** no counterexample exists!



Location-domination in graphs

Definition - Locating-dominating set (Slater, 1980's) 

$D \subseteq V(G)$ locating-dominating set of G:

- for every $u \in V$, $N[v] \cap D \neq \emptyset$ (domination).
- $\forall u \neq v$ of $\mathbf{V(G) \setminus D}$, $N(u) \cap D \neq N(v) \cap D$ (location).

Notation. location-domination number $LD(G)$,
smallest size of a locating-dominating set of G

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Domination number: $DOM(P_n) = \lceil \frac{n}{3} \rceil$

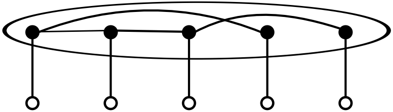
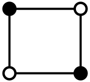
Identifying code number: $ID(P_n) = \lceil \frac{n+1}{2} \rceil$

Location-domination number: $LD(P_n) = \lceil \frac{2n}{5} \rceil$

Upper bounds

Theorem (Domination bound, Ore, 1960's )

G graph of order n, no isolated vertices. Then $DOM(G) \leq \frac{n}{2}$.

Tight examples:  

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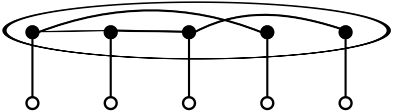
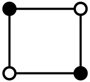
Tight examples:  

Theorem (Location-domination bound, Slater, 1980's )

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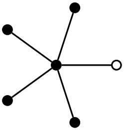
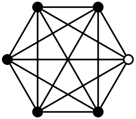
Tight examples:  

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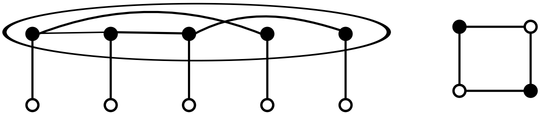

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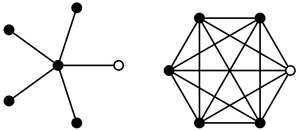

Remark: tight examples contain many twin-vertices!!

Upper bound: a conjecture

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Remark:

- twins are easy to detect
- twins have a trivial behaviour w.r.t. location-domination

Upper bound: a conjecture

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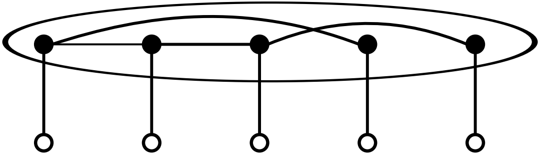
If true, tight: 1. domination-extremal graphs



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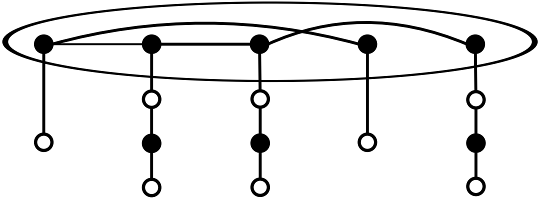
If true, tight: 2. a similar construction



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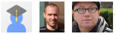
Conjecture (Garijo, González & Márquez, 2014 )

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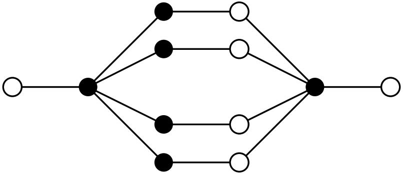
If true, tight: 3. a family with domination number 2



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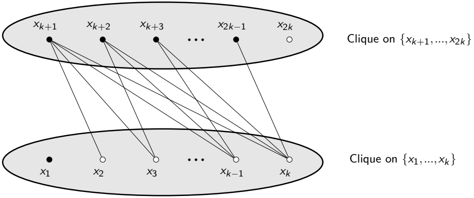
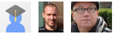
If true, tight: 4. family with dom. number 2: complements of half-graphs



Upper bound: a conjecture - special graph classes

Conjecture (Garijo, González & Márquez, 2014 )

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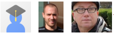
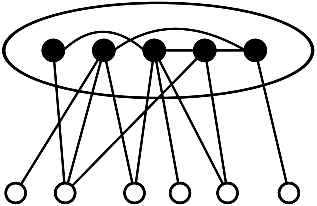
Theorem (Garijo, González & Márquez, 2014 )

Conjecture true if G has independence number $\geq n/2$.
(in particular, if bipartite)

Proof: every vertex cover of a twin-free graph is a locating-dominating set



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$\alpha'(G)$: matching number of G

Theorem (Garijo, González & Márquez, 2014   )

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (F., Henning, 2016 )

Conjecture true if G is cubic.

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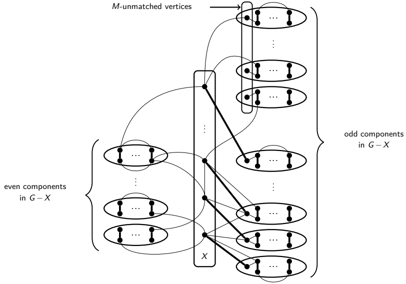
Proof: Involved argument using maximum matching and Tutte-Berge theorem.



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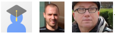
Conjecture (Garijo, González & Márquez, 2014 )

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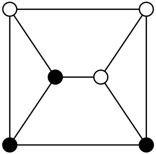
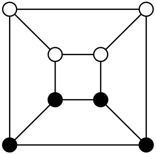
Bound is tight:





Question

Do we have $LD(G) = \frac{n}{2}$ for other cubic graphs?

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

True for bipartite, split, co-bipartite, cubic, line...

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But the conjecture remains open in the general case!

Lower bounds

Lower bounds

Proposition

G identifiable graph on n vertices: $\lceil \log_2(n+1) \rceil \leq ID(G) \leq LD(G)$.

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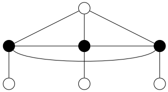
Tight examples:

$ID(G) = \log_2(n+1)$



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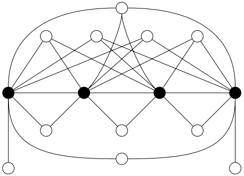

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Theorem (Rall & Slater, 1980's  )

G planar graph, order n, $LD(G) = k$. Then $n \leq 7k - 10 \rightarrow LD(G) \geq \frac{n+10}{7}$.

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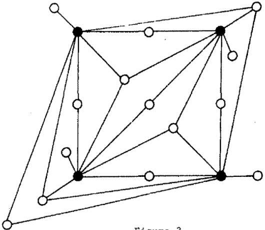
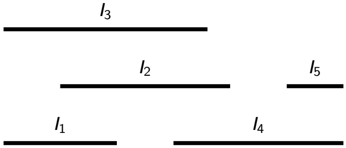
Tight examples: 

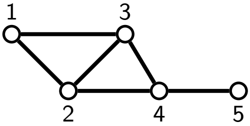
Figure 3.

Interval graphs

Definition - Interval graph

Intersection graph of intervals of the real line.





Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

G interval graph of order n, $LD(G) = k$.

Then $n \leq \frac{k(k+1)}{2}$, i.e. $LD(G) = \Omega(\sqrt{n})$.

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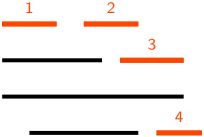

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- Define zones using the right points of intervals in D.

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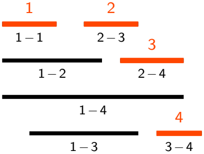

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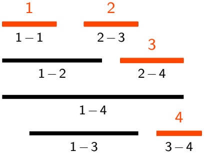

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- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

$$\rightarrow n \leq \sum_{i=1}^{k}(k-i) = \frac{k(k+1)}{2}.$$

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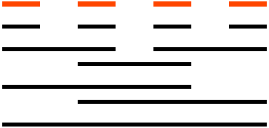
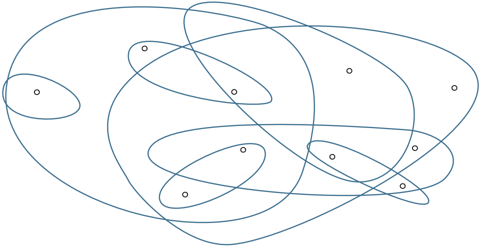
Tight:



Measure of intersection complexity of sets in a hypergraph $(X, \mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered:

for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.



V-C dimension of H: maximum size of a shattered set in H

Vapnik-Červonenkis dimension

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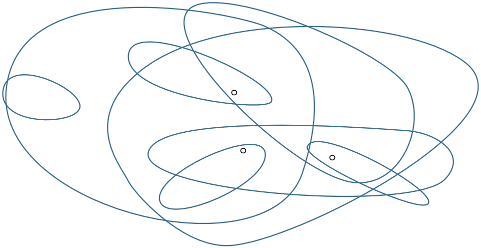

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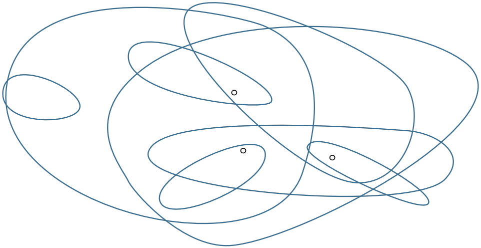
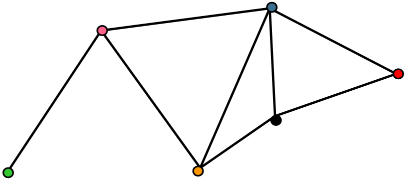
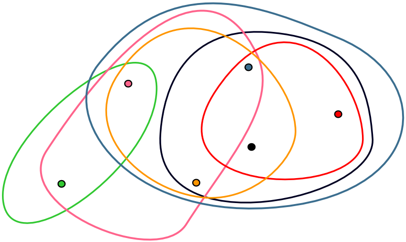


V-C dimension of H: maximum size of a shattered set in H

Typically bounded for geometric hypergraphs:

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph





Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph

Typically bounded for geometric intersection graphs:

→ interval graphs ($d = 2$), C_4-free graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

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Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph

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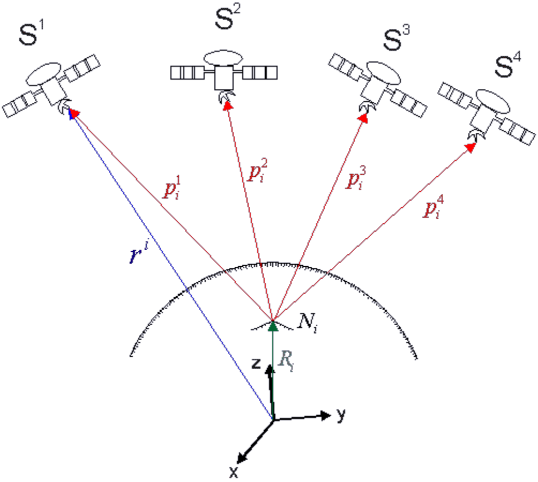
Corollary

G graph of order n, $LD(G) = k$, V-C dimension $\leq d$. Then $n = O(k^d)$.

Metric dimension

Determination of Position in 3D euclidean space

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



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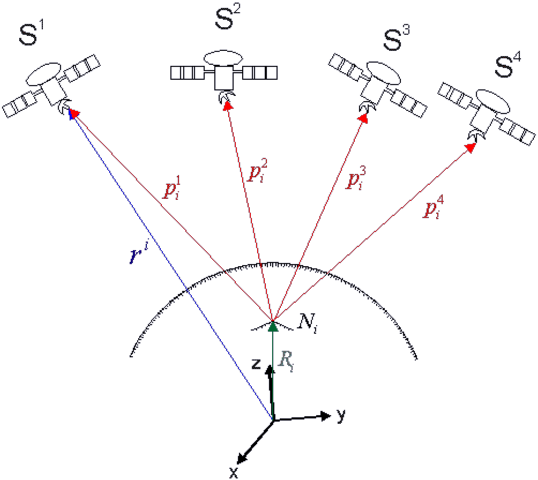

Question

Does the "GPS" approach also work in undirected unweighted graphs?

Metric dimension

Now, $w \in V(G)$ distinguishes $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)   

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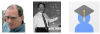
Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976) 

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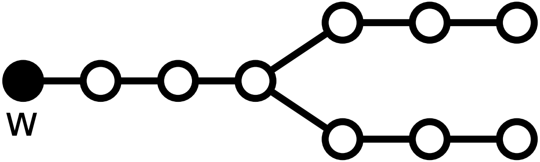

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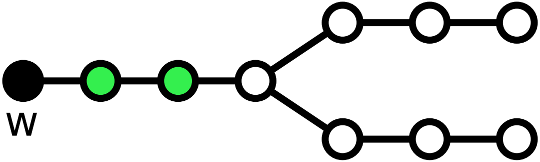

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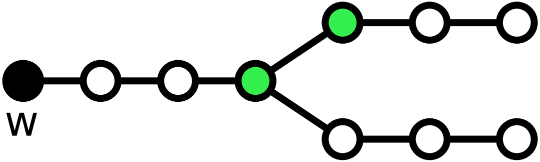

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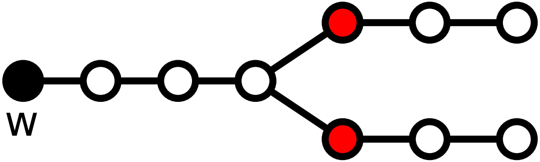

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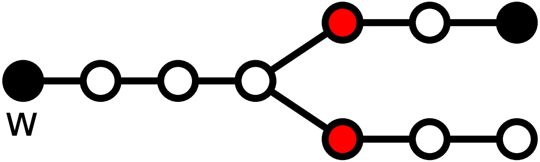

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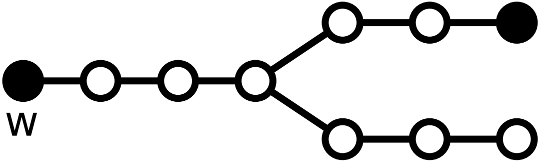

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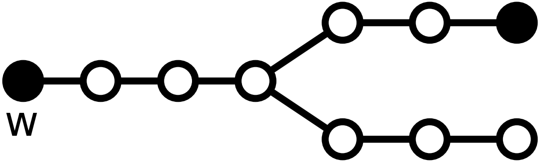

$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

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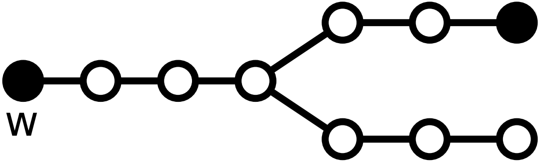

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Remark

- Any locating-dominating set is a resolving set, hence $MD(G) \leq LD(G)$.
- A locating-dominating set can be seen as a “distance-1-resolving set”.

Examples

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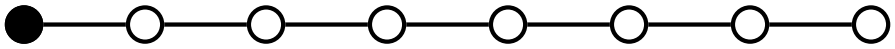
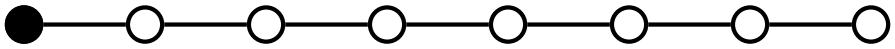
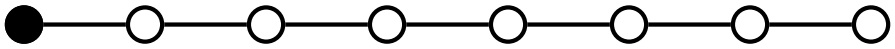




Proposition

$MD(G) = 1 \Leftrightarrow G$ is a path



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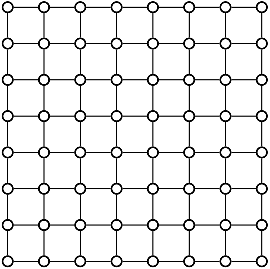

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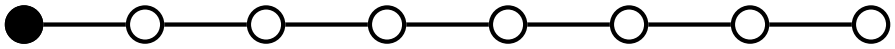

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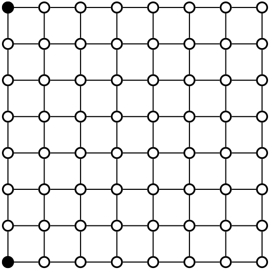
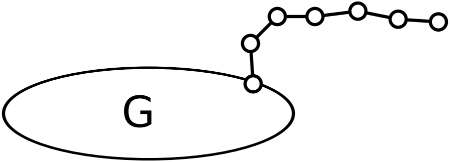


Proposition

For any square grid G, $MD(G) = 2$.

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.



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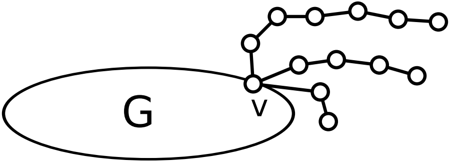

Trees

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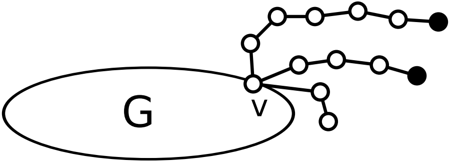

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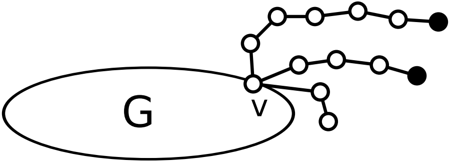

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Theorem (Slater, 1975 )

For any tree, the simple leg rule produces an optimal resolving set.

Bounds with diameter

Example of path: no bound $n \leq f(MD(G))$ possible.

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→ Proofs are similar as for identifying codes.

Trees

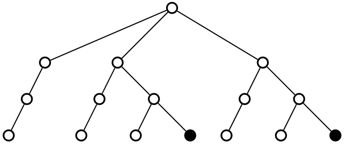
Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018 )

T a tree with diameter D and $MD(T) = k$, then

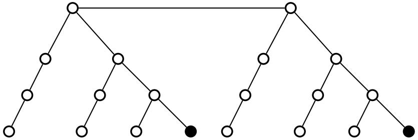
$$n \leq \begin{cases} \frac{1}{8}(kD+4)(D+2) & \text{if } D \text{ even,} \\ \frac{1}{8}(kD-k+8)(D+1) & \text{if } D \text{ odd.} \end{cases} = \Theta(kD^2)$$

Bounds are tight.

$k = 2,\ D = 6$



$k = 2,\ D = 7$



Planar graphs

Using the concept of distance-VC-dimension:

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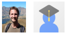
Theorem (Joret, Rambaud, 2023+ )

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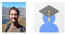
Theorem (Joret, Rambaud, 2023+ )

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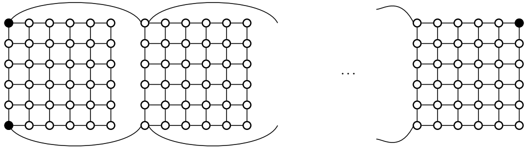
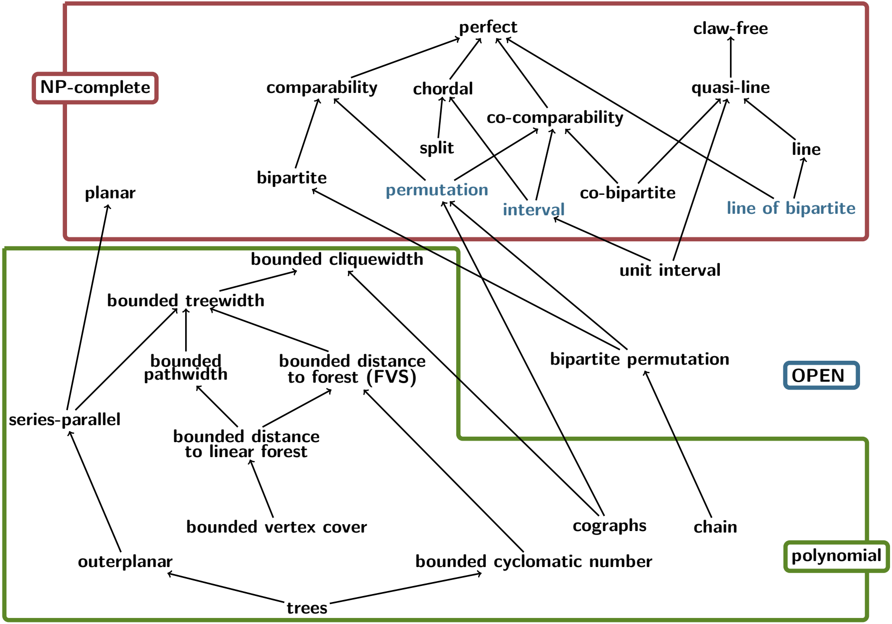
Tight? Example with $k = 3$ and $n = \Theta(D^3)$:

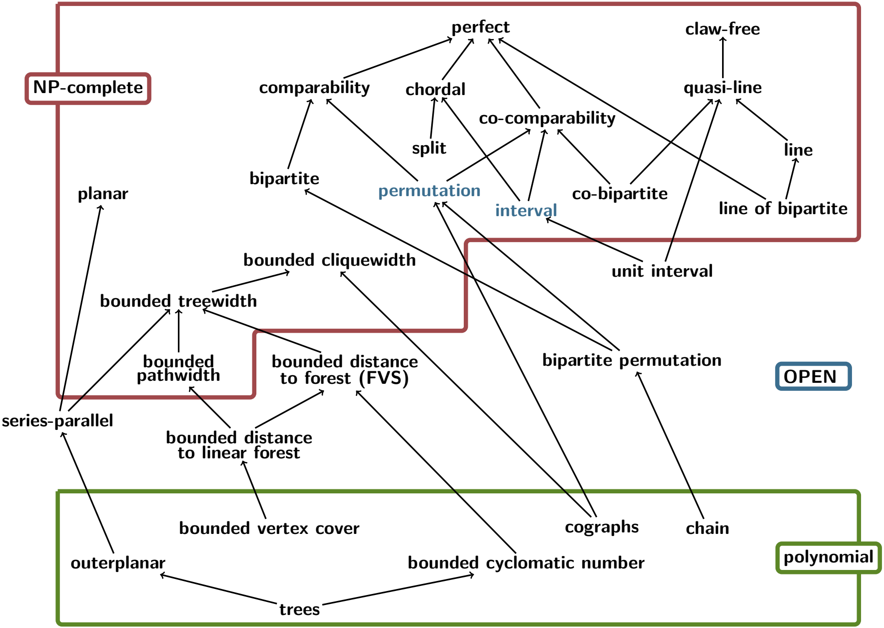


Two slides on complexity and algorithms

Complexity of IDENTIFYING CODE / LOCATING-DOMINATING SET



Complexity of METRIC DIMENSION



Conclusion

Some of my favorite open problems:

- Conjecture: $LD(G) \leq n/2$ in the absence of twins
- Find tight bounds for Metric Dimension in planar graphs of diameter D (and other classes, e.g. graphs of bounded twin-width)
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THANKS FOR YOUR ATTENTION