Domination-based identification problems in graphs

selected topics

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LIMOS

UNIVERSITÉ Clermont Auvergne

April 2025


PARIS
CLERMONT-FERRAND
LYON
BORDEAUX
MARSEILLE



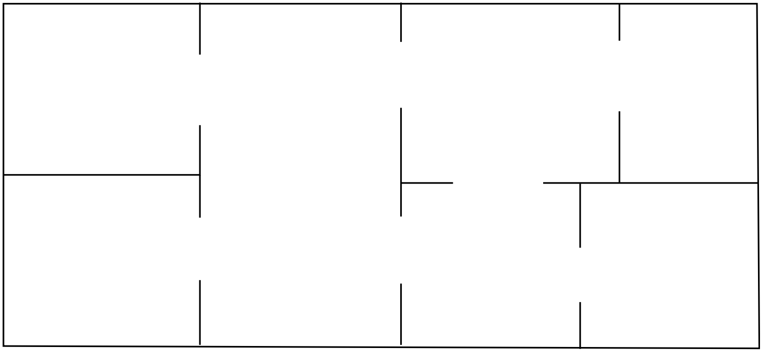

MICHELIN

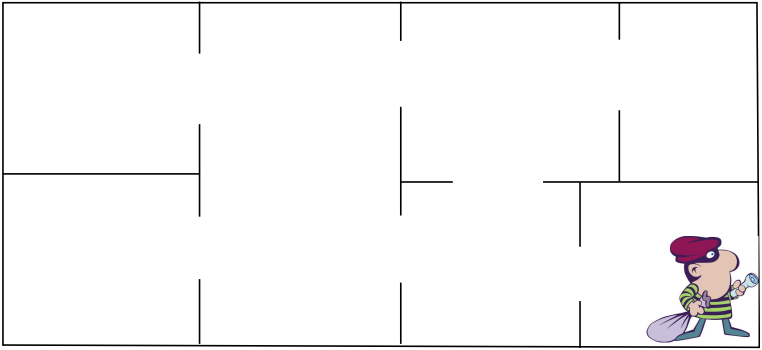


Pascal


Saint Nectaire

Locating a burglar in a building





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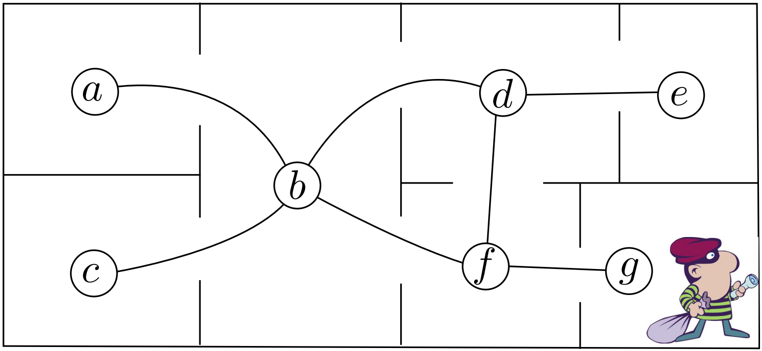
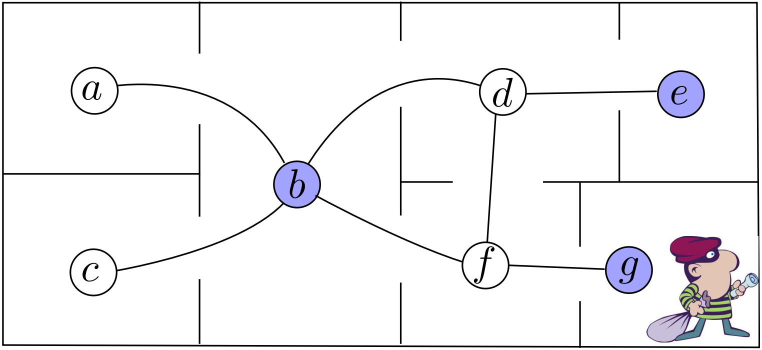


Building: undirected **graph** (rooms: *vertices*, doors: *edges*)

Detectors can detect movement in their room and adjacent rooms



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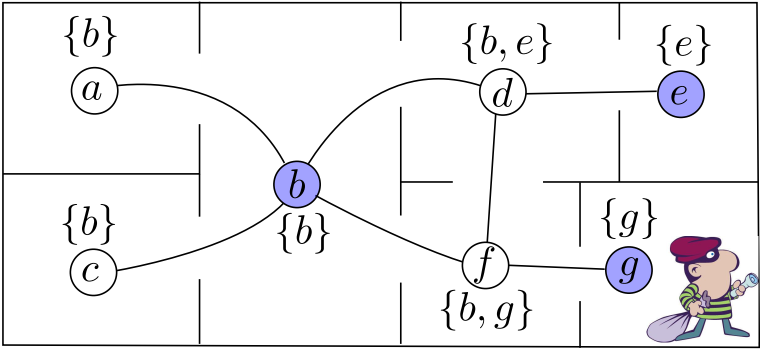

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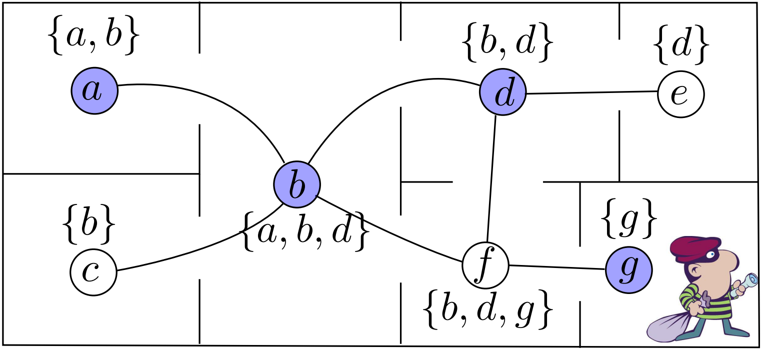

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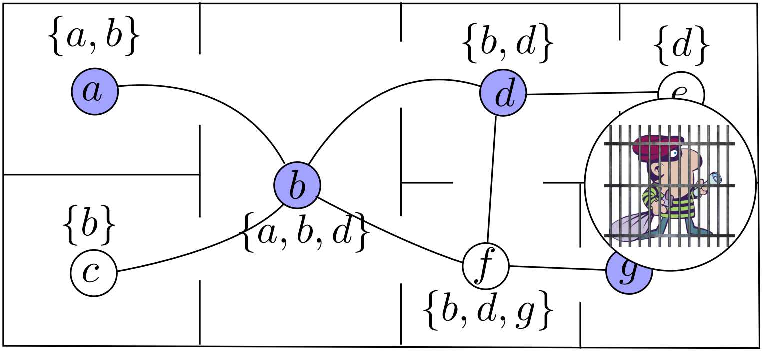

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Domination in graphs

Domination

$V(G)$: set of vertices of G

Definition - Dominating set (Ore, 1960's) 

$D \subseteq V(G)$ dominating set of G:

- every vertex not in D has a neighbour in D

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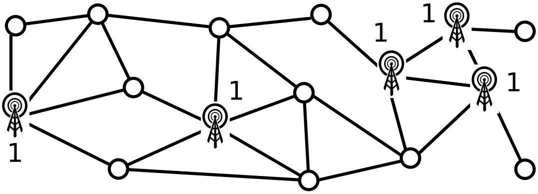
Motivation: covering problems in telecommunication networks



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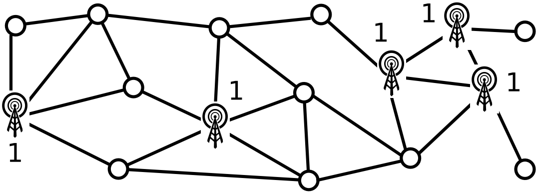

Notation: domination number $DOM(G)$: smallest size of a dominating set of G

Theorem (Domination bound, Ore, 1960's )

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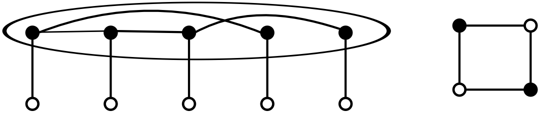
Tight examples: 

A classic upper bound

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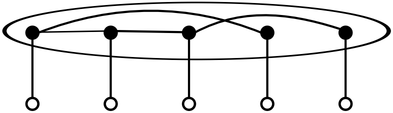
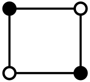
Tight examples:  

Proof: Consider an *inclusionwise minimal* dominating set D of G.

$\rightarrow$ its complement set $V(G) \setminus D$ is also a dominating set!

Thus, either D or $V(G) \setminus D$ has size at most $\frac{n}{2}$. $\square$

Location-domination in graphs

Location-domination

Definition - Locating-dominating set (Slater, 1980's) 

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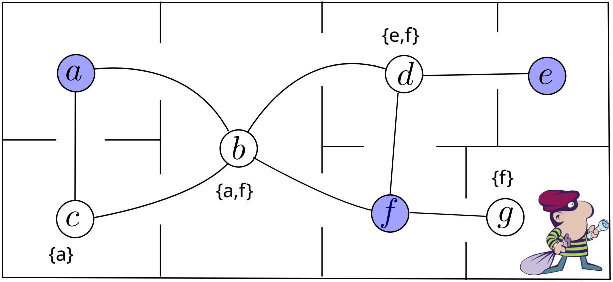
Notation. location-domination number $LD(G)$,
smallest size of a locating-dominating set of G



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Domination number: $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Location-domination number: $LD(P_n) = \left\lceil \frac{2n}{5} \right\rceil$

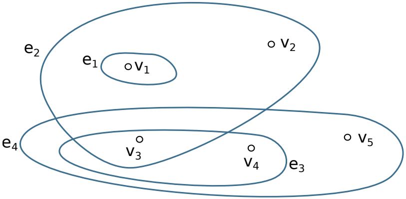
Separation type	Locating-Sep		Closed-Sep		Open-Sep		Full-Sep	
adj	$N(u) \triangle N(v)$		$N[u] \triangle N[v]$		$N(u) \triangle N(v)$		$N[u] \triangle N[v]$	
non-adj	$N[u] \triangle N[v]$						$N(u) \triangle N(v)$	
D/TD	$N[u]$	$N(u)$	$N[u]$	$N(u)$	$N[u]$	$N(u)$	$N[u]$	$N(u)$
	Locating-dominating sets	Locating total-dominating sets	Closed-separating dominating sets a.k.a Identifying codes	Closed-separating total-dominating sets a.k.a Total identifying codes	Open-separating dominating sets	Open-separating total-dominating sets a.k.a Identifying open codes a.k.a Open locating-dominating-sets	Full-separating dominating sets	Full-separating total-dominating sets

Some variants of identification problems

1988:	LD-sets	P. Slater
1998:	ID-codes	M. Karpovsky, K. Chakrabarty & L. Levitin
2002: **2010:**	Open ID-codes	I. Honkala, T. Laihonen, S Ranto S. Seo & P. Slater
2006:	LTD-sets ITD-sets	T. Haynes, M. Henning & J. Howard
2024:	OD-sets FD-sets FTD-sets	D. Chakraborty & A. Wagler

Definition - Separating set (Rényi, 1961 )

Hypergraph $(X,\mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.



$X = \{v_1, v_2, v_3, v_4, v_5\}$
$\mathscr{E} = \{\{v_1\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_4, v_5\}\}$

Separating sets in hypergraphs

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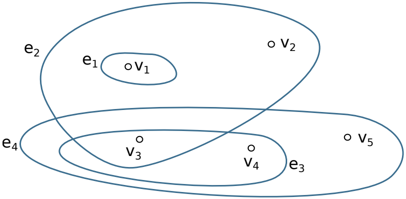

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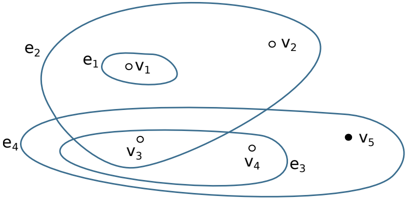

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$$C = \{v_5\}$$

$$e_1 \cap C = \emptyset$$
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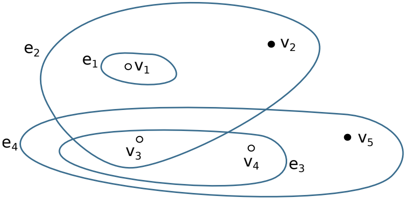

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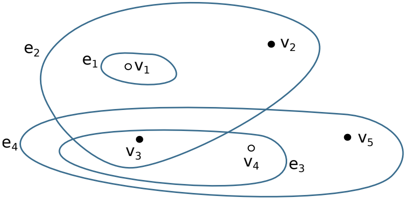

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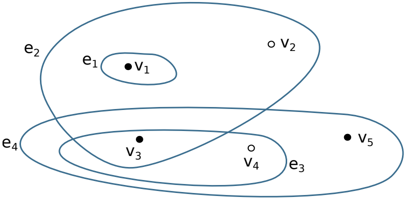

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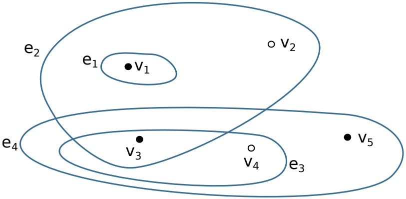

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Also known as Separating system, Distinguishing set, Test cover, Distinguishing transversal, Discriminating code...

Applications

- network-monitoring, fault detection (burglar)
- medical diagnostics: testing samples for diseases (*test cover*)
- biological identification (attributes of individuals)
- learning theory: teaching dimension
- machine learning: V-C dimension (Vapnik, Červonenkis, 1971)
- graph isomorphism: canonical representation of graphs (Babai, 1982)
- logic definability of graphs (Kim, Pikhurko, Spencer, Verbitsky, 2005)

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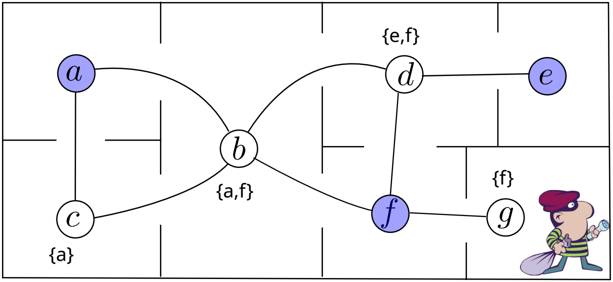

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Proof: Take $V(G) \setminus v$, for any vertex v of G. $\square$

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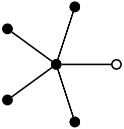
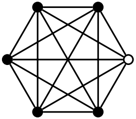
Tight examples:  

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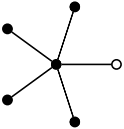
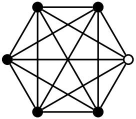
Tight examples:  

Remark: tight examples contain many twin-vertices!!

(Twins: vertices with the same sets of neighbours)

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Conjecture (Garijo, González & Márquez, 2014   )

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Remark:

- twins are easy to detect
- twins have a trivial behaviour w.r.t. location-domination

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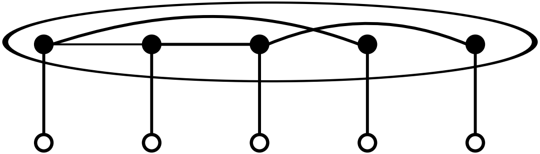
If true, tight: 1. domination-extremal graphs



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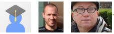
Conjecture (Garijo, González & Márquez, 2014 )

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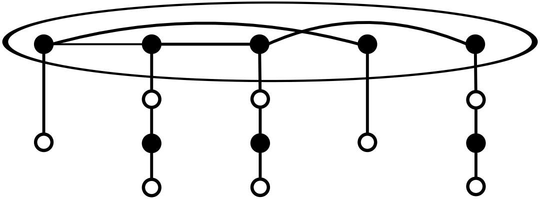
If true, tight: 2. a similar construction



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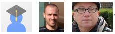
Conjecture (Garijo, González & Márquez, 2014 )

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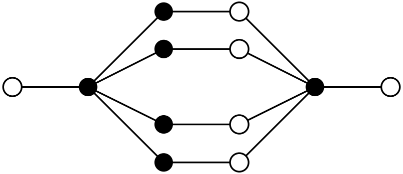
If true, tight: 3. a family with domination number 2



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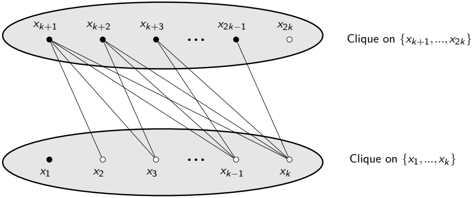
If true, tight: 4. family with dom. number 2: complements of half-graphs



Upper bound: a conjecture - special graph classes

Conjecture (Garijo, González & Márquez, 2014   )

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{n}{2}$.

Theorem (Garijo, González & Márquez, 2014   )

Conjecture true if G has independence number $\geq n/2$. (e.g. bipartite)

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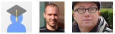
Conjecture (Garijo, González & Márquez, 2014 )

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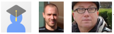
Theorem (Garijo, González & Márquez, 2014 )

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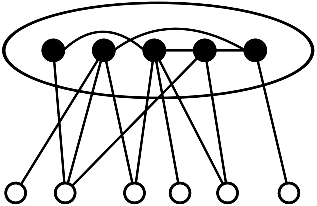
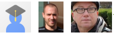
Proof: every vertex cover of a twin-free graph is a locating-dominating set



Conjecture (Garijo, González & Márquez, 2014 )

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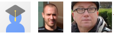
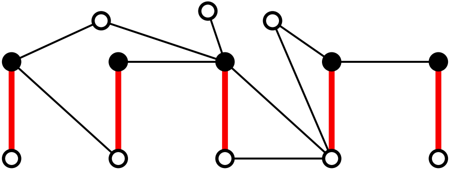
$\alpha'(G)$: matching number of G

Theorem (Garijo, González & Márquez, 2014 )

If G has no 4-cycles, then $LD(G) \leq \alpha'(G) \leq \frac{n}{2}$.

Proof:

- Consider special maximum matching M
- Select one vertex in each edge of M



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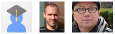
Conjecture (Garijo, González & Márquez, 2014 )

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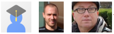
Theorem (Garijo, González & Márquez, 2014 )

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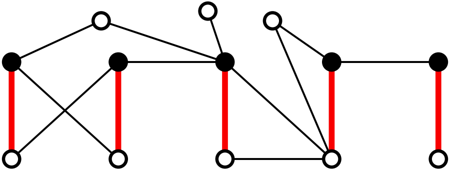
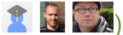


Conjecture (Garijo, González & Márquez, 2014 )

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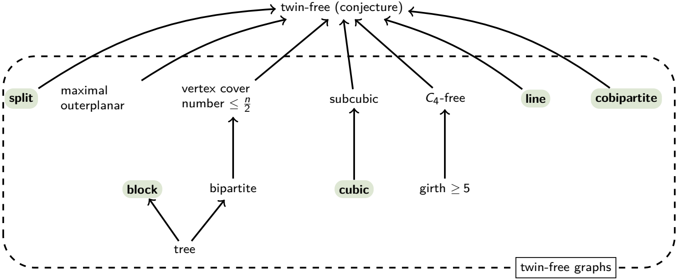
The conjecture is proved for specific graph classes:



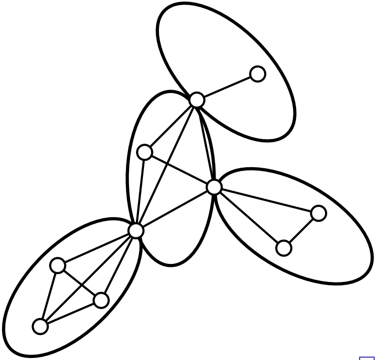
- Garijo, González & Márquez, 2014
- F., Henning, Löwenstein, Sasse, 2016
- F. and Henning, 2016 and 2017
- Chakraborty, F., Henning, Wagler, 2024
- Chakraborty, Hakanen, Lehtilä, 2024+

Block graphs

Theorem (Chakraborty, F., Parreau, Wagler, 2024   )

Any block graph can be partitioned into two LD-sets.

Proof sketch.



□

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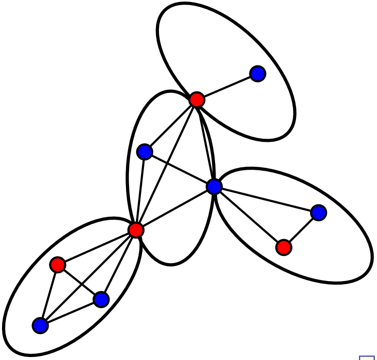
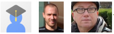
Proof sketch.

- Partition $V(G)$ into two parts **R** and **B**.
- Both **R** and **B** are LD codes of G.
- Either one of $|\mathbf{R}|$ or $|\mathbf{B}| \leq \frac{1}{2}n$.



$\square$

Conjecture (Garijo, González & Márquez, 2014 )

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Is it true that all isolate-free twin-free graphs can be partitioned into two locating-dominating sets?

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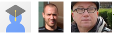
Is it true that all isolate-free twin-free graphs have their LD-number at most the matching number?

A theorem about partitioning

Theorem (Chakraborty, F., Henning, Laihonen, 2025+   )

Every isolate-free (not necessarily twin-free) graph can be partitioned into a dominating set and a locating-dominating set.

Conjecture (Garijo, González & Márquez, 2014 )

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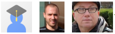
Conjecture (Garijo, González & Márquez, 2014 )

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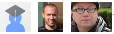
proof of Lemma: consider a smallest dominating set D that maximizes the number of edges inside D. For every $d \in D$, there must be a vertex $f(d)$ only dominated by d (otherwise $D \setminus \{d\}$ is a dominating set). If $f(d) \neq d$, it is a private neighbour of d. If $f(d) = d$, d has no neighbour in D. But since there is no isolated vertex in G, d has a neighbour c in $V(G) \setminus D$, that has 2 neighbours in D. Then, $D \setminus \{d\} \cup \{c\}$ contains more edges than D, a contradiction: so, $f(d) \neq d$. □

Conjecture (Garijo, González & Márquez, 2014 )

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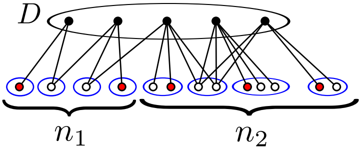
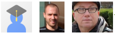


Upper bound: a conjecture - general bound

Conjecture (Garijo, González & Márquez, 2014 )

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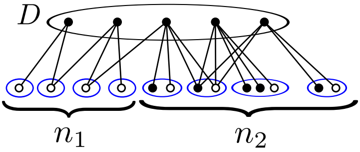
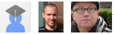


Conjecture (Garijo, González & Márquez, 2014 )

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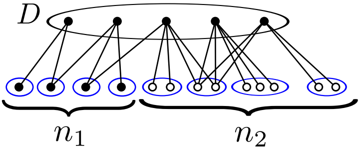
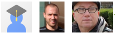


Conjecture (Garijo, González & Márquez, 2014 )

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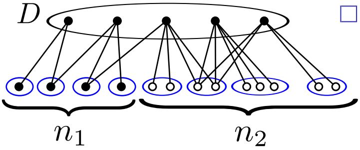

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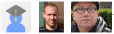
Conjecture (Garijo, González & Márquez, 2014 )

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Theorem (Bousquet, Chuet, Falgas-Ravry, Jacques, Morelle, 2024)

G graph of order n, no isolated vertices, no twins. Then $LD(G) \leq \frac{5}{8}n = 0.625n$.

A similar conjecture for Locating-**total** dominating sets

Theorem (Cockayne, Dawes & Hedetniemi, 1980)

If G is a connected graph on n vertices, then $\gamma_t(G) \leq \frac{2}{3}n$.

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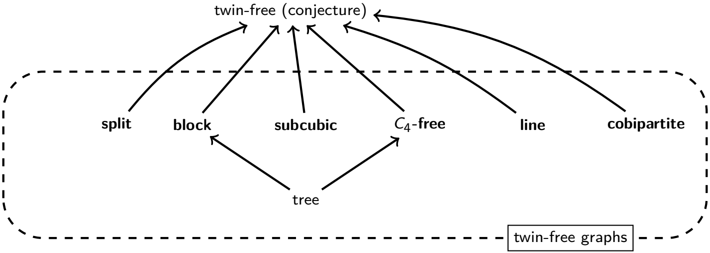


- F. and Henning, 2016 and 2017
- Chakraborty, F., Hakanen, Henning, Wagler, 2024

Open identifying codes in graphs

(a.k.a. open locating-dominating sets)

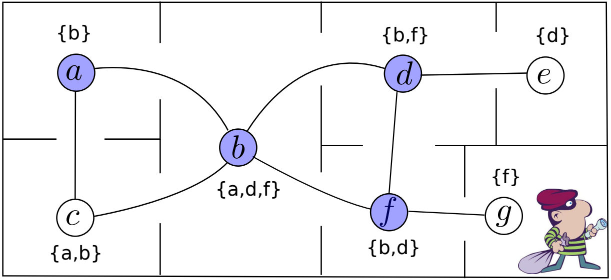
G: undirected graph $\quad N(u)$: set of neighbours of v

Definition - open identifying code (Seo, Slater, 2010  )

Subset D of $V(G)$ such that:

- D is a **total dominating set**: $\forall u \in V(G)$, $N(u) \cap D \neq \emptyset$, and
- D is an **open separating code**: $\forall u \neq v$ of $V(G)$, $N(u) \cap D \neq N(v) \cap D$

Notation. $OID(G)$: open identifying code number of G, minimum size of an open identifying code in G



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Total domination number: $TOTDOM(P_n) \approx \lceil \frac{n}{2} \rceil$

Open id. code number: $OID(P_n) \approx \lceil \frac{2n}{3} \rceil$

Remark

Not all graphs have an open identifying code!

An **isolated vertex** cannot be totally dominated.

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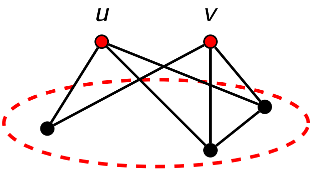
Open twins = pair u, v such that $N(u) = N(v)$.



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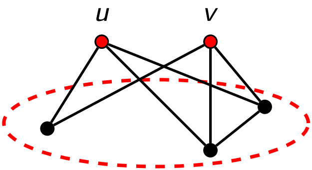


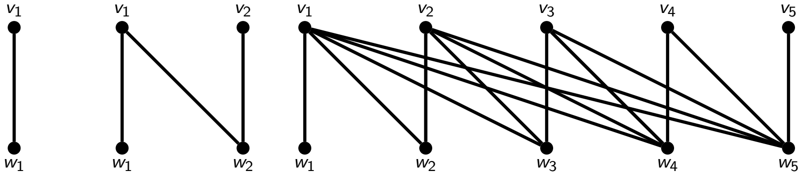
Proposition

A graph is **locatable** if and only if it has no **isolated vertices** and **open twins**.

Upper bound on $OID(G)$?

Definition - Half-graph H_k (Erdős, Hajnal, 1983  )

Bipartite graph on vertex sets $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$, with an edge $\{v_i, w_j\}$ if and only if $i \leq j$.



$H_1 = P_2$ $\quad$ $H_2 = P_4$ $\quad$ H_5

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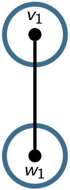

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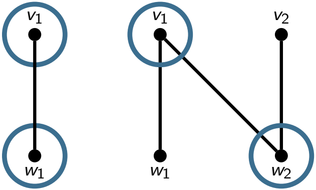

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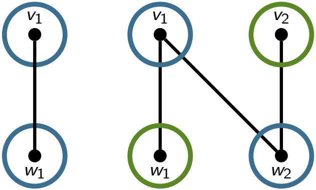

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Some vertices **forced** in any open identifying code because of domination or location

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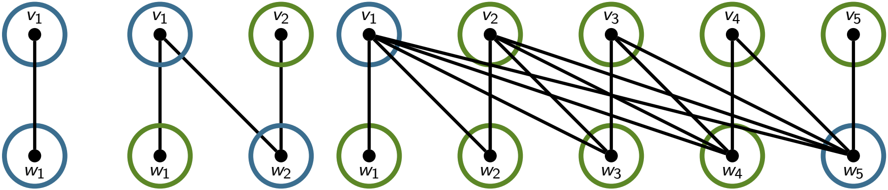

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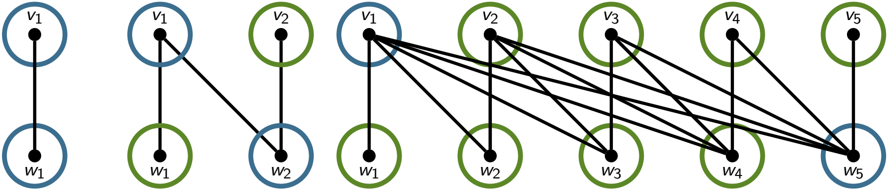

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Some vertices **forced** in any open identifying code because of domination or location

Proposition

For every half-graph H_k of order $n = 2k$, $OID(H_k) = n$.

Theorem (F., Ghareghani, Roshany Tabrizi, Sharifani, 2021   )

Let G be a connected locatable graph of order n.

Then, $OID(G) = n$ if and only if G is a half-graph.

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• By induction, G' is a half-graph. We can conclude that G is a half-graph too, after some case analysis. □

Lower bounds
(neighbourhood complexity)

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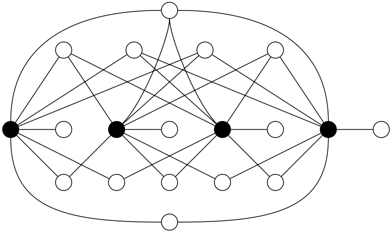
Tight example ($k = 4$):



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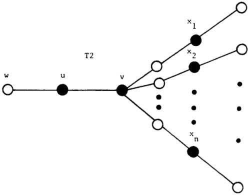

FIG. 2. Tree T2

Proof: Recall: a tree of order n has $n-1$ edges. Consider a LD-set S of size k.

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The $c_2 = n-k-c_1$ others need to have (at least) 2 neighbours in S.

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In total we need $c_1+2(n-k-c_1) = 2n-2k-c_1 \geq 2n-3k$ edges in the tree. So:

$2n-3k \leq n-1$ and so, $n \geq 3k-1$. $\square$

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Theorem (Rall & Slater, 1980's  )

G planar graph, order n, $LD(G)=k$. Then $n \leq 7k-10 \to LD(G) \geq \frac{n+10}{7}$.

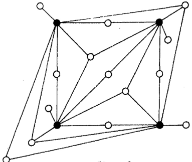
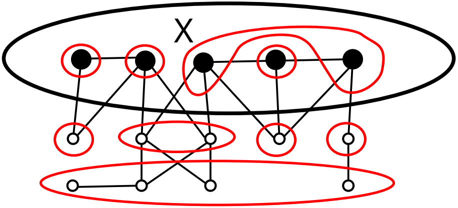
Tight examples: 

Figure 3.

Neighbourhood complexity of a graph G:

maximum number $|\{N(v) \cap X\}|$ of neighbourhoods inside any set X of k vertices, as a function of k



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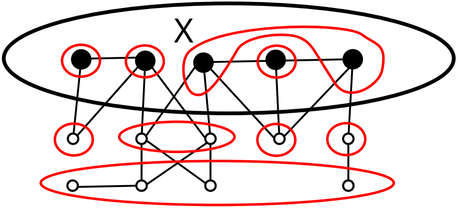

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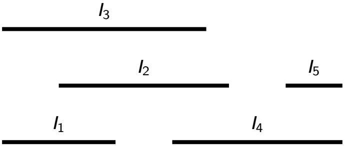
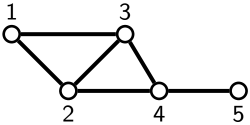
- General graphs : exponential neighbourhood complexity 2^k
- Trees/planar graphs : linear neighbourhood complexity $O(k)$

Definition - Interval graph

Intersection graph of intervals of the real line.





Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017    )

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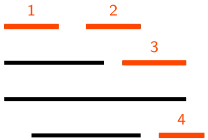

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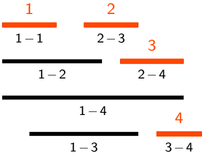

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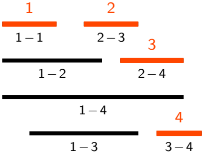

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$$\rightarrow n \leq \sum_{i=1}^{k}(k-i) = \frac{k(k+1)}{2}.$$

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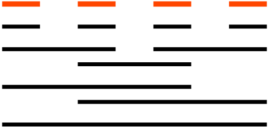
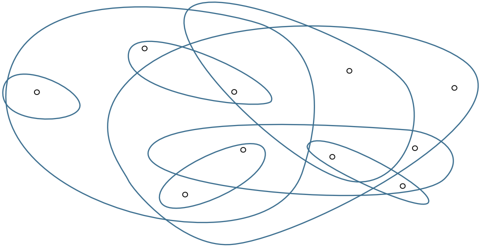



Measure of intersection complexity of sets in a hypergraph $(X, \mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered:

for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.



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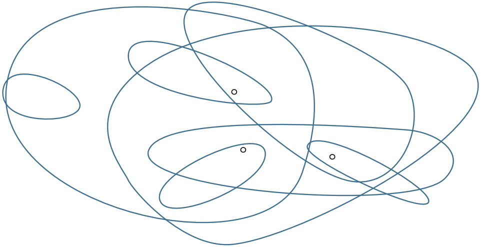

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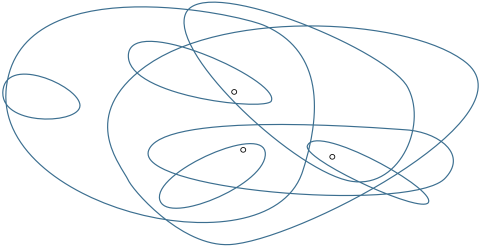

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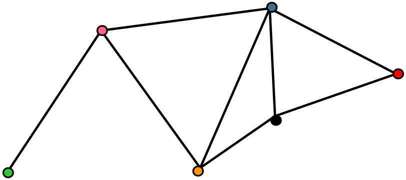
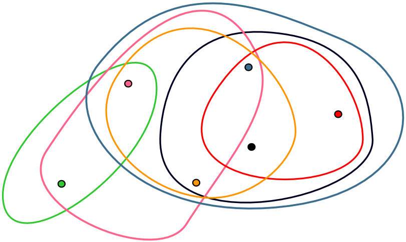
Typically bounded for geometric hypergraphs: 

V-C dimension of a graph: V-C dimension of its open/closed neighbourhood hypergraph





Vapnik-Červonenkis dimension - graphs

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$O(k^2)$: interval, permutation, line...

$O(k)$: cographs, unit interval, bipartite permutation, block...

Sparse/structured graphs

Graph classes of bounded expansion: all shallow minors of its members have bounded average degree → e.g. planar graphs, minor-closed classes, bounded degree...

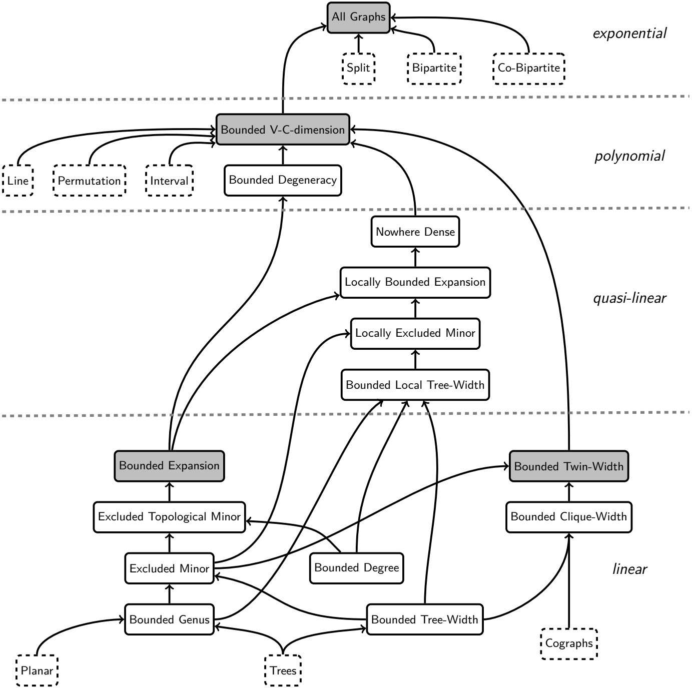
Theorem (Reidl, Sánchez-Villaamil, Stavropoulos, 2019   )

Let $\mathscr{C}$ be a graph class of bounded expansion. Let G in $\mathscr{C}$, order n, and $LD(G)=k$.
Then, $n \leq f(\mathscr{C})k$.

Recently introduced structural measure: twin-width.

Theorem (Bonnet, F., Lehtilä, Parreau, 2024   )

Let G be a graph of twin-width at most d and order n, and $LD(G)=k$.
Then, $n \leq (d+2)2^{d+1}k$.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THANKS FOR YOUR ATTENTION!