Florent Foucaud
Master 2 ICS

Université Clermont Auvergne
2023-2024

Graph algorithms: exercise sheet

(for next lecture: 30/11/2023)

1 Tree-decompositions

Exercise 1 (Cycles).

Let C_n denote the *cycle graph* with n vertices.

1. Represent C_{10} as a partial 2-tree.
2. Deduce from it a tree-decomposition of C_{10} with all bags of size at most 3.
3. Give a different tree-decomposition of C_{10} with all bags of size at most 3 (with a tree of different shape).
4. Prove that when $n \geq 3$, C_n has no tree-decomposition with all bags of size at most 2.
5. Deduce a characterization of the graphs of treewidth 1.

Exercise 2 (3-coloring).

Design a dynamic programming algorithm to find a 3-coloring of a graph (if one exists), given a *nice tree decomposition* of it. The algorithm should run in time $f(k)n^c$, where k is the width of the tree-decomposition and c is a small constant.

2 The dominating set problem

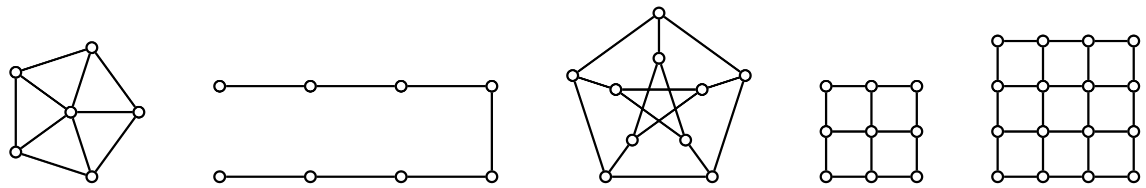
A *dominating set* of a graph G is a subset S of its vertices such that every vertex of G is either in S, or has a neighbour in S. The set of all vertices is trivially a dominating set, so the goal is to find the smallest possible dominating set of a graph.

This problem has natural applications, for example to cover a network by certain types of facilities (radio towers, schools, etc).

> **Dominating Set**
> Input: A graph G, an integer k
> Question: Does G have a dominating set of size at most k?

Exercise 3 (Warmup).

Find a minimum-size dominating set of the following graphs.



Exercise 4 (NP-hardness)**.**

1. Show that DOMINATING SET is in NP.
2. Show that DOMINATING SET is NP-hard, by a reduction from 3-SAT.[1]
3. Carefully modify the graph by adding edges (while keeping the reduction valid) so that the reduction constructs a chordal graph.
   This will show that (unlike INDEPENDENT SET and COLORING) DOMINATING SET is NP-complete, even for chordal graphs.

Exercise 5 (Intervals)**.**

Design a polynomial-time greedy algorithm to find the optimal dominating set of an interval graph (if given the intervals of its representation). Do we need to order the intervals by their starting times or by their ending times?

Exercise 6 (Trees)**.**

Design a dynamic programming algorithm to find the minimum dominating set of a tree (with no weights).[2]

Exercise 7 (Bounded treewidth)**.**

Design a dynamic programming algorithm to find the minimum dominating set of a graph, given a *nice tree decomposition* of it. The algorithm should run in time $f(k)n^c$, where k is the width of the tree-decomposition and c is a small constant.

[1]Hint: you can represent each variable x by a triangle, where one vertex has degree 2 in the graph, and the two other vertices of the triangle represent the literals x and $\overline{x}$. A clause can be represented by a single vertex, connected to the 2 or 3 literals that it contains.

[2]Hint: you can use three types of states for the root of a subtree: in the solution, not in the solution but already dominated, not in the solution and not yet dominated.