GRAPH ALGORITHMS

INDEPENDENT SET is NP-hard
3-COLORING ——
} reductions from 3-SAT.

FOR INTERVALS (→ INTERVAL GRAPHS)

Motivation: genome, scheduling movies/events etc.

Greedy algorithm for IS

1) sort the intervals by increasing <u>end time</u>.
2) SOL = $\emptyset$
3) while there exist unmarked intervals:
   - pick leftmost unmarked interval
   - add it to SOL.
   - mark it and all intersecting intervals

claim 1: the algorithm returns an IS

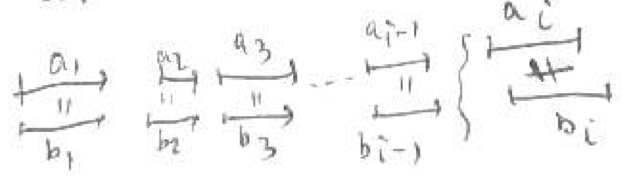
claim 2: the algorithm runs in poly time

claim 3: the algorithm returns a maximum-size IS

Proof: let $A = a_1 a_2 \dots a_k$ be the computed solution.

Assume by contradiction that there exists a larger IS. Among all such sets, we choose one that maximizes its common prefix with A, call it B. $= b_1 \dots b_\ell$ ($\ell > k$)

Let i be the smallest index s.t. $a_i \neq b_i$



By the definition of the algorithm, we have $end(a_i) < end(b_i)$
(otherwise the algorithm would have picked b_i instead of a_i)

Then, $B' = (B \setminus \{b_i\}) \cup \{a_i\}$ is an IS of size $\ell > k$ with a larger common prefix of A than B, contradicting the choice of B ∎

greedy coloring algorithm:

(1) Sort the intervals by increasing start time

(2) order the colors $c_1 \dots c_n$

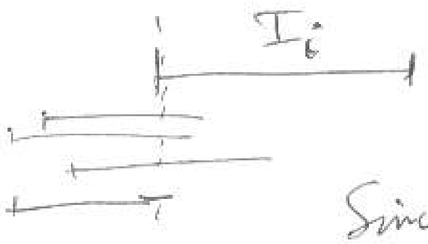
(3) while there remains an uncolored vertex:
- consider the leftmost uncolored vertex
- color it with the smallest available color (not present among the colored intervals that intersect it)

Claim 1: the algorithm returns a valid coloring.

Claim 2: the algorithm runs in polynomial time.

Claim 3: the computed coloring ~~uses~~ exactly $k = \omega(G)$ colors

Proof: consider step i of the algorithm



Consider the set of intervals already colored that intersect I_i.

Since they start before I_i starts, they all intersect the starting point of I_i.

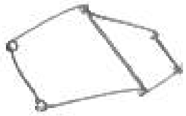
So, they form a clique in G. Hence, there are at most $k - 1 = \omega(G) - 1$ unavailable colors for I_i, thus I_i receives color at most $k = \omega(G)$. $\square$

(Lekkerkerker-Boland, 1962)

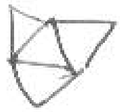
Thm: G is an interval graph if and only if it has no chordless cycle of length ≥ 4 and no asteroidal triple.

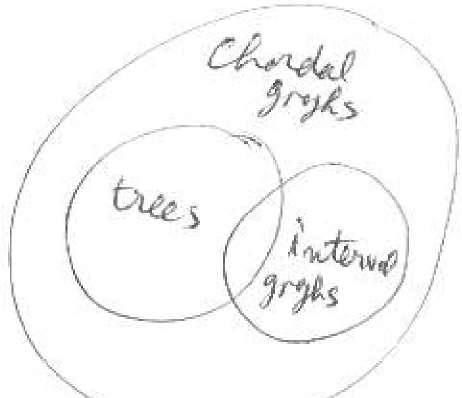
CHORDAL GRAPHS

Def: a graph is *chordal* if every cycle of length ≥ 4 has a *chord*

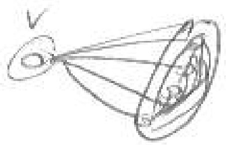


examples: – trees
– complete graphs
– interval graphs

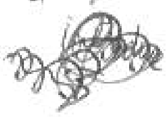




Def: a vertex v is *simplicial* if its neighbors form a clique



give examples



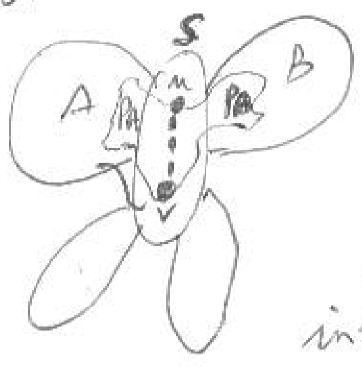
Lemma 1: every *minimal separator* in a chordal graph is a clique.

↑ set of vertices S s.t. $G-S$ has at least two connected components, and no $S' \subset S$ disconnects the graph. ⇝ give examples

"disconnects the graph" =

proof: Consider a minimal separator S and assume by contradiction that there is a non-edge $u \cdots v$ in S.

Let A and B be two connected components of $G-S$.



We claim that u has a neighbor in both A and B: otherwise, $S \setminus \{u\}$ would still separate A, B so S would not be minimal. The same holds for v.

Let P_A / P_B be ~~two~~ shortest paths from u to v only using internal vertices of A / B (they exist by the previous argument and connectivity of A and B).

[not necessarily shortest in G, but in $A \cup \{u, v\} / B \cup \{u, v\}$.]

Both ~~are~~ paths have no chord. There is also no edge between A and B. So, $P_A \cup P_B$ is a cycle of length ≥ 4 with no chord →

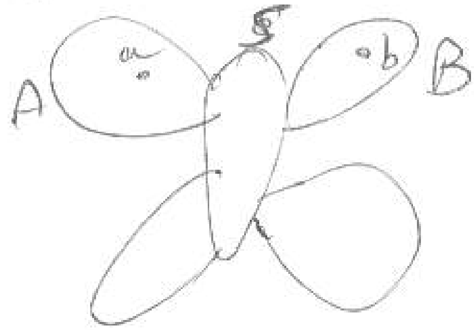
a contradiction since G is chordal. $\square$

<u>Lemma 2</u> (Dirac, 1961): every chordal graph G is either a complete graph, or has two non-adjacent simplicial vertices.

<u>proof</u>: By induction on n.

<u>$n=1$</u>: $G = \cdot \rightarrow$ trivial

Assume it's true for all $n' < n$. If G is complete, there is nothing to prove. If not, let a, b be two non-adjacent vertices and let S be a minimal separator with two components A, B of $G-S$, $a \in A$, $b \in B$.



By lemma 1, S is a clique.

Consider $G_A = G[A \cup S]$
$G_B = G[B \cup S]$

Since $b \notin G_A$, $a \notin G_B$ we can apply induction to G_A, G_B.

- If G_A is a clique, $a = s_A$ is a simplicial vertex of G.
- — G_B — $b = s_B$ —

If G_A is not a clique, it has two non-adj. simplicial vertices.
- Since S is a clique, one of them, s_A, is in A. It is also simplicial in G. The same holds for s_B in B.

Thus, s_A and s_B are two non-adjacent simplicial vertices in G, and the induction is complete. $\square$

Coloring Chordal graphs optimally

1. Compute a simplicial elimination scheme of G:
   pick a simplicial vertex, remove it from G (exists by lemma 2)
   The graph is still chordal: iterate
   we obtain the ordering $v_1, \dots, v_n$
   where v_i is simplicial in $G[v_i \dots v_n]$.

2. For i from $i = n$ to $i = 1$: [reverse order]
   - color v_i with the smallest available color among $c_1, \dots, c_n$.

__Claim 1__: the algorithm runs in polynomial time

__Claim 2__: the algorithm returns a coloring with $k = \omega(G)$ colors

__Proof__: as for interval graphs, when v_i is considered, its colored neighbors form a clique, so there is one available color among $c_1, \dots, c_k$ so it receives a color $\leq c_k$. □

! explain well

__Theorem__: G is chordal $\Leftrightarrow$ G has a simplicial elimination scheme.

__Proof__: ($\Rightarrow$) use lemma 2 inductively.

($\Leftarrow$) Consider a scheme $v_1 \dots v_n$. Let C be a cycle.
$C = c_1 \dots c_k$ ordered as in the scheme. ($k \geq 4$)
c_1 has two neighbors in C. Since it is simplicial in $G[c_1 \dots c_k]$
They are adjacent. This is a chord of C. □