Characterizing the optimum bases of a convex geometry using quasi-closed hypergraphs

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Outline

Outline

1. Definitions
   - Implication Bases
   - Closure System
   - Convex Geometry
2. Optimization of implication base of a convex geometry: general case
3. Acceptant convex geometry

Implication bases

Definition

An **implication** over X is a statement $A \to B$, with $A \subseteq X$, and $B \subseteq X$. An **implication base** is a pair (X, Σ) where Σ is a set of implications on X.

- A is the **premise** of the implication
- B is the **conclusion** of the implication

Example

- $15 \to 2$
- $\Sigma = \{123 \to 4, 15 \to 234, 24 \to 13, 14 \to 3, 5 \to 13\}$

$\Leftrightarrow$

- $(\bar{1} \vee \bar{5} \vee 2)$
- $(\bar{1} \vee \bar{2} \vee \bar{3} \vee 4) \wedge (\bar{1} \vee \bar{5} \vee 2) \wedge (\bar{1} \vee \bar{5} \vee 3) \wedge (\bar{1} \vee \bar{5} \vee 4) \wedge (\bar{2} \vee \bar{4} \vee 1) \wedge (\bar{2} \vee \bar{4} \vee 3) \wedge (\bar{1} \vee \bar{4} \vee 3) \wedge (\bar{5} \vee 1) \wedge (\bar{5} \vee 3)$

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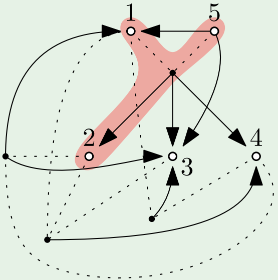
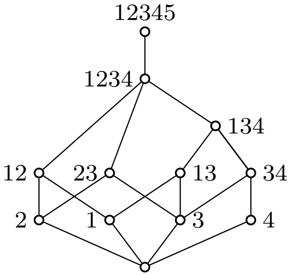


Lattices and implication bases

- (X, Σ) is an implication base of $(X, \mathcal{C})$ if C satisfies Σ if and only if $C \in \mathcal{C}$
  - The set of subset of X satisfied by Σ form a lattice

Example

- $\Sigma = \{15 \rightarrow 234, 24 \rightarrow 13, 5 \rightarrow 13, 123 \rightarrow 4, 14 \rightarrow 3\}$

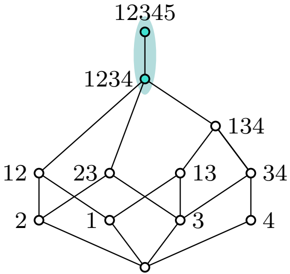


Implication bases and lattices

- We say that $(X, \mathcal{C})$ satisfies Σ if for all $C \in \mathcal{C}$ such that there exist $A \to B \in \Sigma$ with $A \subseteq C$ then $B \subseteq C$

Example

- $\Sigma = \{15 \to 234, 24 \to 13, 5 \to 13, 123 \to 4, 14 \to 3, \}$
- the lattice satisfies $123 \to 4$



Implication bases and lattices

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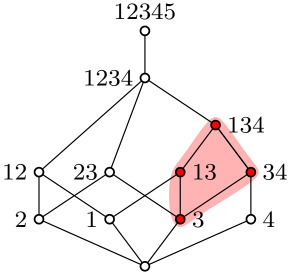
Example

- $\Sigma = \{15 \to 234, 24 \to 13, 5 \to 13, 123 \to 4, 14 \to 3, \}$
- The lattice does not satisfies $3 \to 2$

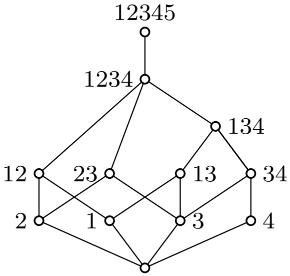


Implication bases and lattices

- A lattice admits several equivalent implication bases

Example

- $\Sigma = \{15 \rightarrow 234, 24 \rightarrow 13,$
  $5 \rightarrow 13, 123 \rightarrow 4, 14 \rightarrow 3\}$
- $\Sigma = \{5 \rightarrow 2, 5 \rightarrow 4,$
  $24 \rightarrow 13, 123 \rightarrow 4, 14 \rightarrow 3\}$
- $\Sigma = \{5 \rightarrow 1234,$
  $24 \rightarrow 13, 123 \rightarrow 4, 14 \rightarrow 3\}$

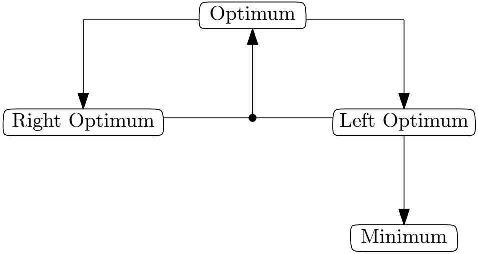


Minimality criterion of an Implication Bases

An implication base of $(X, \mathcal{C})$ is:

- **Minimum** if it minimizes the number of implications
- **Left optimum** if the sum of the size of premises is minimum
- **Right optimum** if the sum of the size of the conclusions is minimum
- **optimum** if it is both left and right optimum

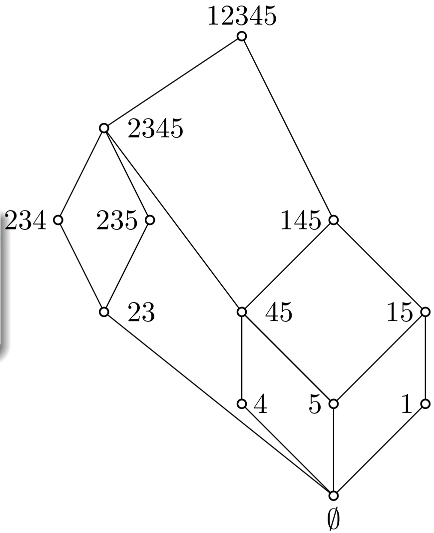
see Wild 2017 for more on implication bases



Left optimization

Example

- $\Sigma = \{2 \rightarrow 3, 3 \rightarrow 2, 123 \rightarrow 4, 124 \rightarrow 35, 14 \rightarrow 5\}$



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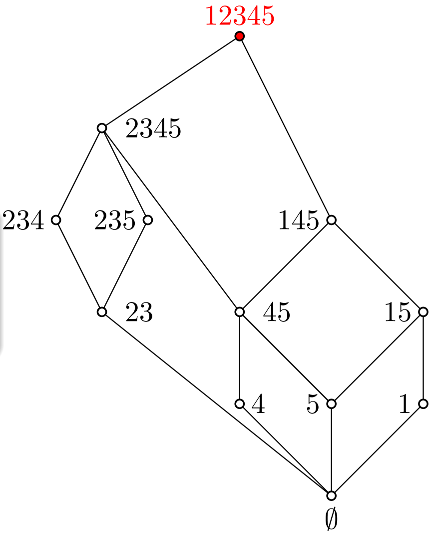

Left optimization

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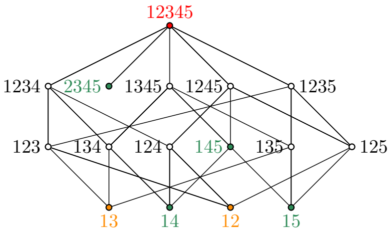
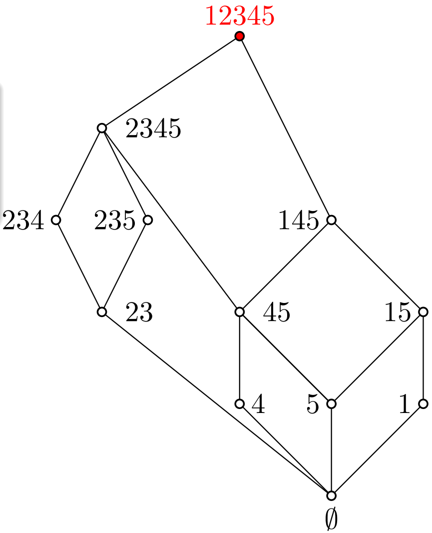
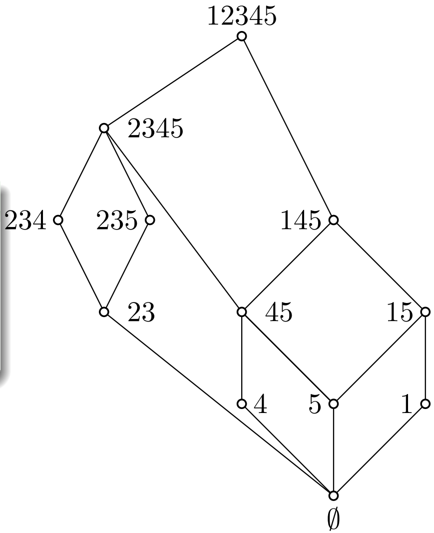




Left optimization

Example

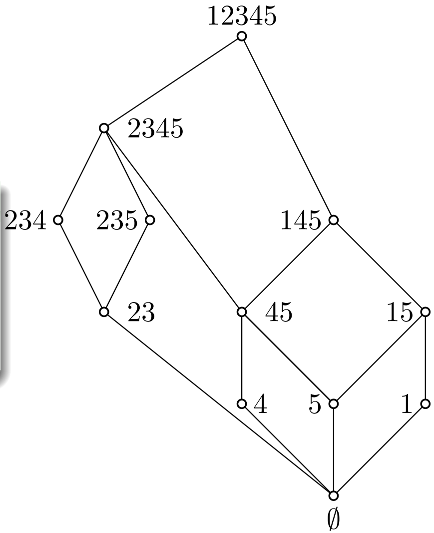
- $\Sigma = \{2 \rightarrow 3, 3 \rightarrow 2,$
  $12 \rightarrow 4, 12 \rightarrow 35, 14 \rightarrow 5\}$
- $\Sigma = \{2 \rightarrow 3, 3 \rightarrow 2,$
  $13 \rightarrow 4, 13 \rightarrow 25, 14 \rightarrow 5\}$



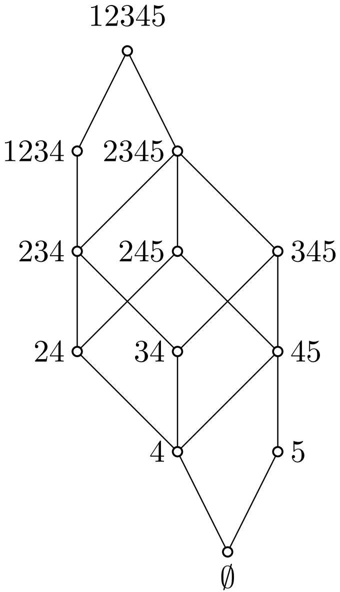
Left optimization

Example

- $\Sigma = \{2 \to 3, 3 \to 2, 12 \to 345, 14 \to 5\}$
- $\Sigma = \{2 \to 3, 3 \to 2, 13 \to 245, 14 \to 5\}$



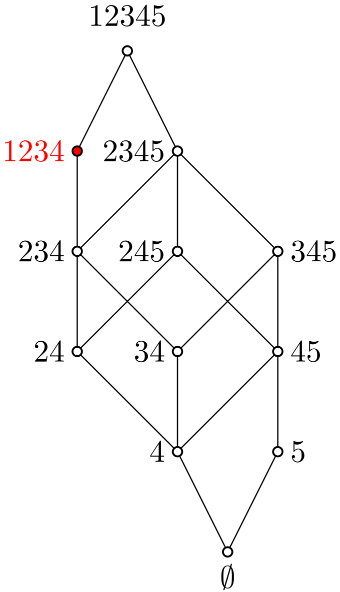
Right optimization



Example

- $\Sigma = \{1 \to 234, 2 \to 4, 3 \to 4, 14 \to 23\}$

Right optimization



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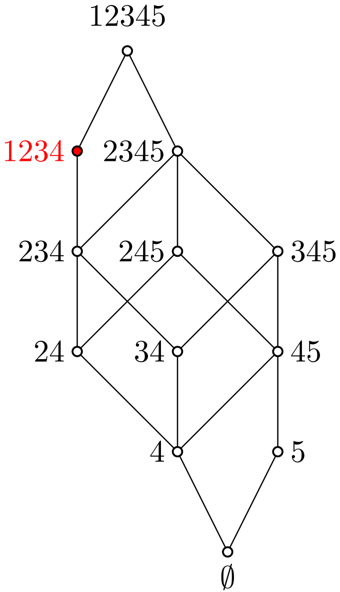

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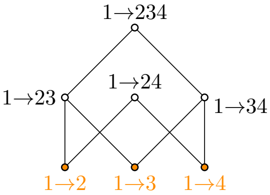
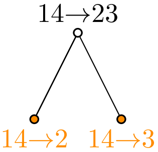
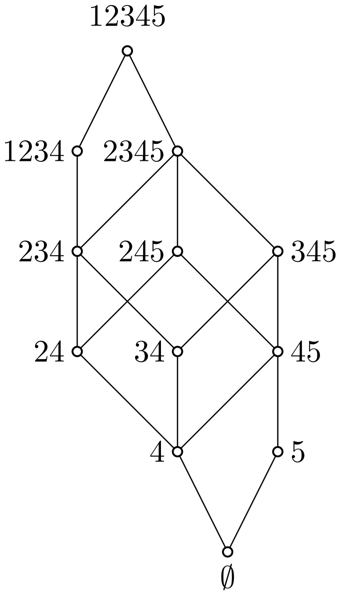




Right optimization



Example

- $\Sigma = \{1 \to 2, 2 \to 4, 3 \to 4, 14 \to 3\}$
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Right optimization

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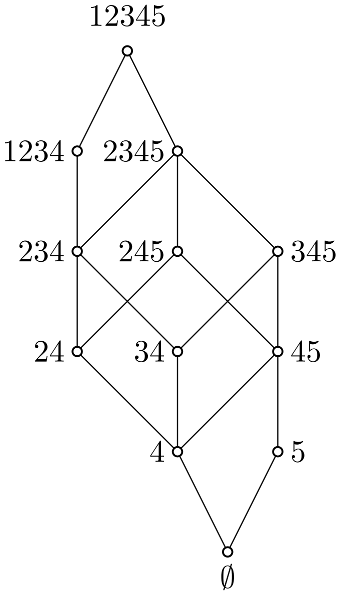

Right optimization

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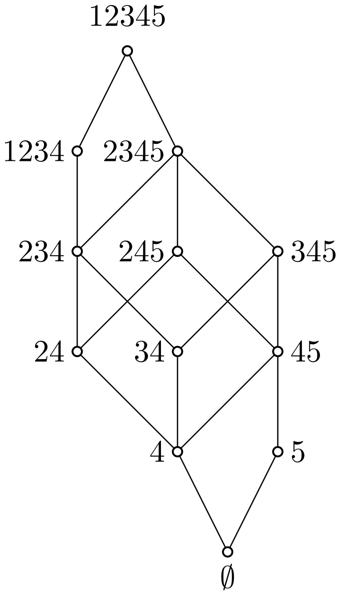
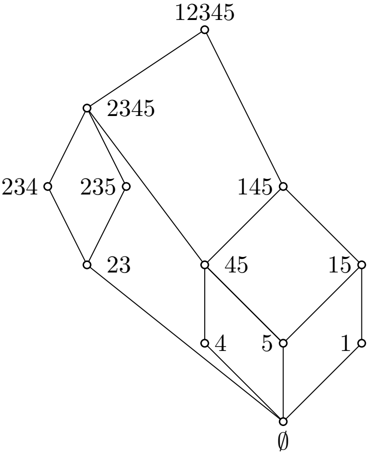
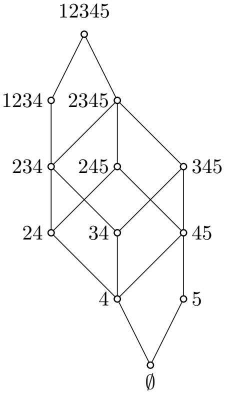


Optimum bases





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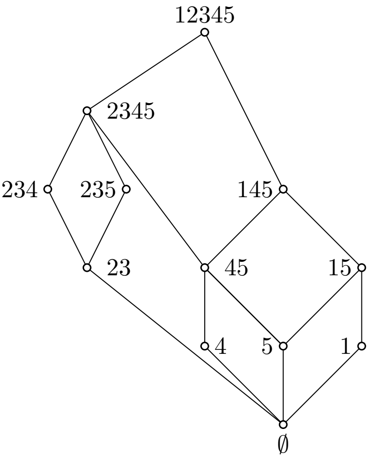

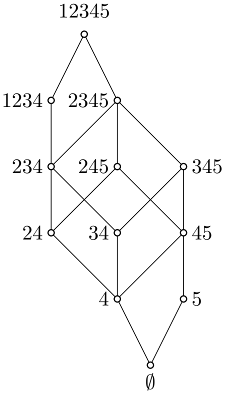

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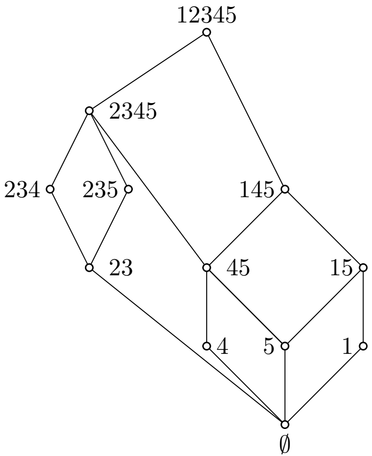
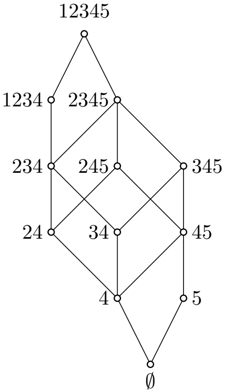
Optimum bases





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Optimization of implication bases

Problem

input: (X, Σ) *an implication base of a closure system* $(X, \mathcal{C})$

output: (X, Σ') *an optimum base of* $(X, \mathcal{C})$

- Maier 1980: Optimization is NP-complete for arbitrary closure system
- Ausiello, D'Atri, and Saccà 1986: Left and right optimization are NP-complete
- Bichoupan 2022 show that in convex geometry the problem is NP-complete
- Convex geometry have some subclasses known to be tractable in polynomial time
  - Hammer and Kogan 1995: Acyclic Convex geometry
  - Nakamura and Kashiwabara 2013: Affine Convex geometry
  - Adaricheva 2017: Double-shelling convex geometry

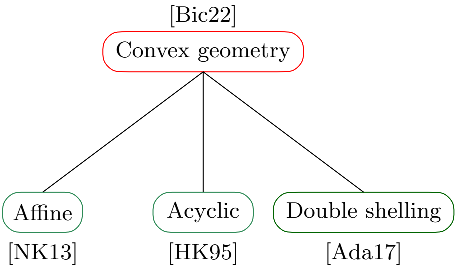
Motivation

Problem

input: (X, Σ) *a minimum implication base of a convex geometry* $(X, \mathcal{C})$

output: (X, Σ') *an optimum base of* $(X, \mathcal{C})$

- When does it remain tractable?



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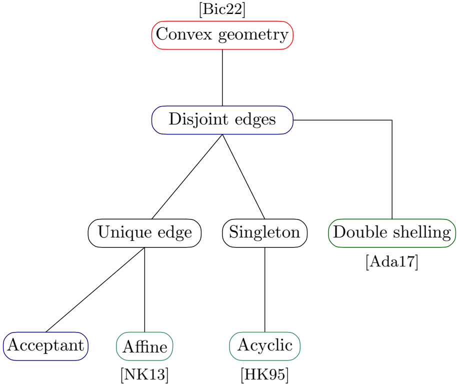


Closure system

Definition

A **closure system** is a pair $(X, \mathcal{C})$ where $\mathcal{C}$ is a collection of subsets of X satisfying:

- $\varnothing, X \in \mathcal{C}$
- $\forall C_1, C_2 \in X, C_1 \cap C_2 \in \mathcal{C}$

Definition

A **closure operator** ϕ on a set X is a map defined on 2^X which is:

- isotone: $A \subseteq B \rightarrow \phi(A) \subseteq \phi(B)$
- extensive: $A \subseteq \phi(A)$
- idempotent: $\phi(\phi(A)) = \phi(A)$

- $\mathcal{C}$ is the set of subset $C \subseteq X$ such that $\phi(C) = C$.

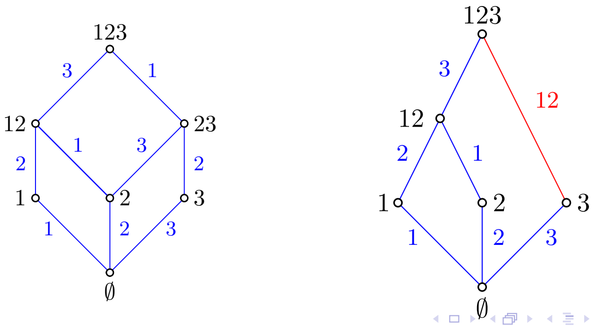
Convex Geometry

Definition

$(X, \mathcal{C})$ is a **convex geometry** if for all $C \in \mathcal{C}$, with $C \neq X$, there exists $x \in X \setminus C$ such that $C \cup x \in \mathcal{C}$

Definition

We denote by $\mathrm{ex}(A)$ for $A \subseteq X$ the **extreme points** of A,
$\mathrm{ex}(A) = \{x : x \in A, x \notin \phi(A \setminus x)\}$



Outline

Optimization of implication base

Definition

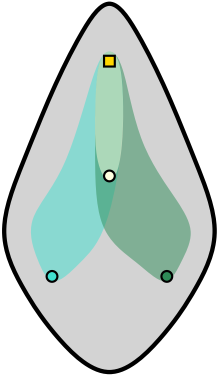
The **equivalence classes** of an implicational base are the sets of implications whose premises share the same closure

Theorem (Mannila and Räihä 1983)

Let $(X,\mathcal{C})$ *be a closure system and let* (X,Σ) *be an implicational base of* $(X,\mathcal{C})$*. Then* (X,Σ) *is optimum if and only if all its equivalence classes are optimum.*

Left optimization: the easy part

- In implication bases, we call a closed set **essential** if it is the closed set of some premise in every equivalent implication base
- In minimum bases each implication is associated to an essential
- In the case of convex geometry, each implication is associated to a different closed set



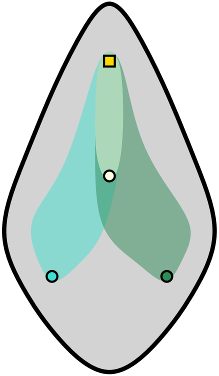
Left optimization: the easy part

Definition

$C \in \mathcal{C}$, a spanning set of C is a subset $A \subseteq X$ such that $\phi(A) = C$.

In convex geometries:

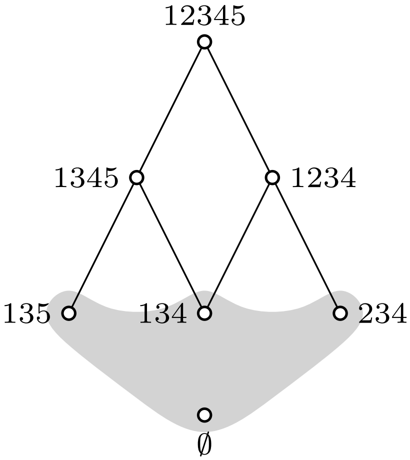
- Every closed set has a unique minimal spanning set
- The extreme points of a closed set is exactly its minimal spanning set



Left optimization: the easy part

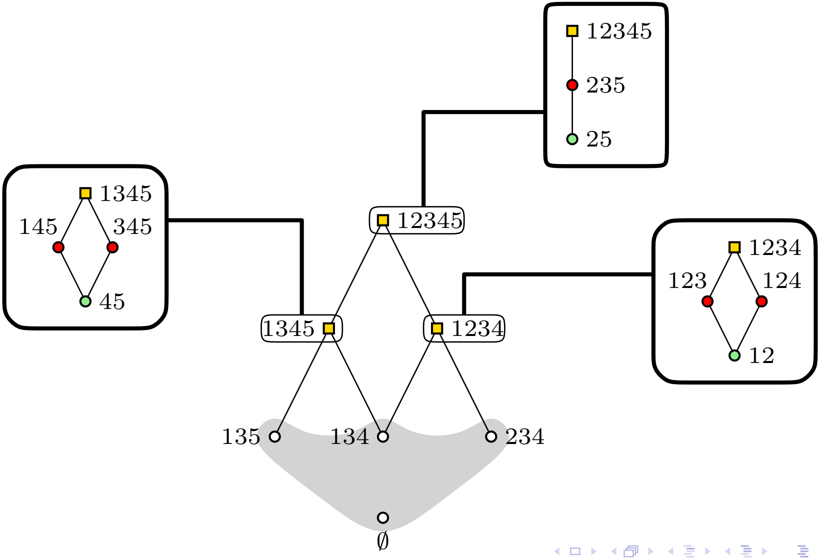
Example

- $\Sigma = \{12 \to 3, 123 \to 4, 45 \to 13, 25 \to 4\}$
- $\mathrm{ex}(1234) = 12$, it is the unique minimal spanning set of 1234, hence $12 \to 4$ can replace $123 \to 4$ in Σ



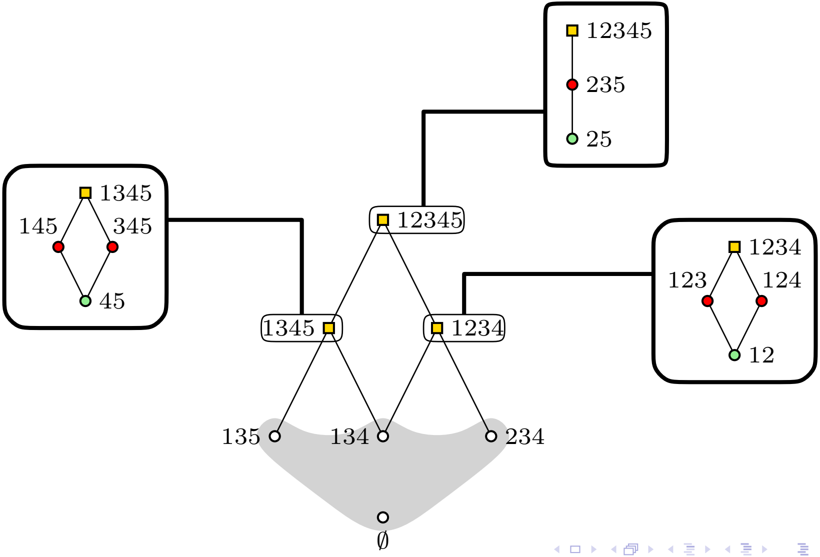
Right optimization: the not so easy part

We call Q **quasi-closed** if $\mathcal{C} \cup Q$ is a closure system



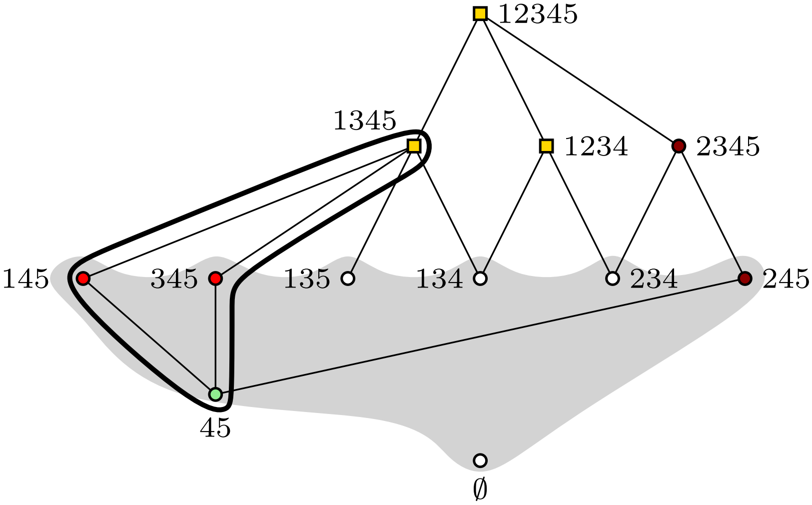
Right optimization: the not so easy part

$$\Sigma = \{12 \rightarrow 34, 45 \rightarrow 13, 25 \rightarrow 4\}$$



Right optimization: the not so easy part

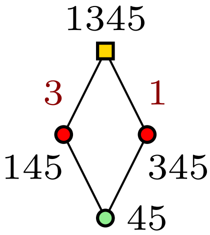
$$\Sigma = \{12 \to 34, 25 \to 4\}$$



Right optimization: the not so easy part

How to correct Σ such that it is an implication base of $\mathcal{C}$ again?

1. How to correct Σ?
   - $ex(C) \rightarrow T$, with $T = \{C \setminus Q : Q \in \max_{\subseteq}\{Q' : Q'$ quasi closed of $C\}\}$



Right optimization: the not so easy part

How to correct Σ such that it is an implication base of $\mathcal{C}$ again?

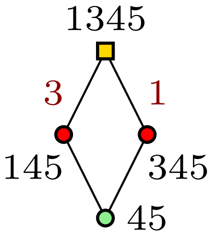
Theorem

Let $(X, \mathcal{C})$ be a convex geometry and Σ a valid collection of implication. If for all essential set $C \in \mathcal{C}$, the associated implication in Σ is of the form $ex(C) \to T$ then Σ is left-optimum

Right optimization: the not so easy part

How to correct Σ such that it is an implication base of $\mathcal{C}$ again?

1. How to correct it with a Right optimum implication?
    - Compute the smallest T possible



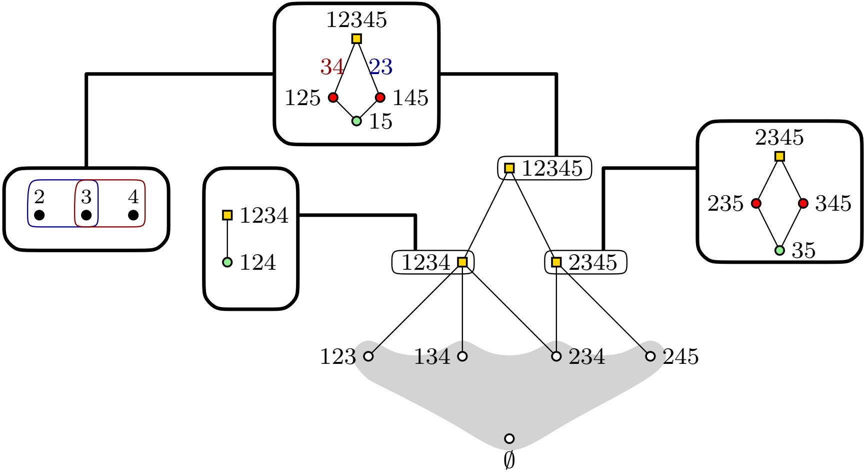
Right optimization: the not so easy part

How to correct Σ such that it is an implication base of $\mathcal{C}$ again?

Theorem

Let $(X,\mathcal{C})$ be a convex geometry and Σ a valid collection of implication. If for all essential set $C \in \mathcal{C}$, the associated implication in Σ is of the form $ex(C) \rightarrow T$ and T is minimum then Σ is optimum

Right optimization: the not so easy part



Right optimization: the not so easy part

Theorem

An implicational base of a convex geometry where each quasi-closed hypergraph has pairwise disjoint edges can be optimized in polynomial time.

Outline

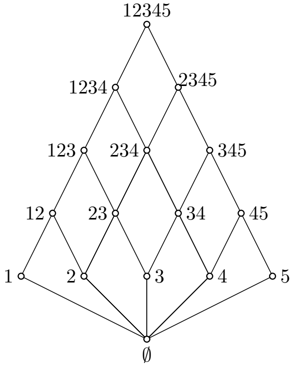
Acceptant convex geometry

Definition

A convex geometry $(X, \mathcal{C})$ is q-acceptant if for all $C \in \mathcal{C}$ we have $|ex(C)| = \min\{q, |C|\}$

Example

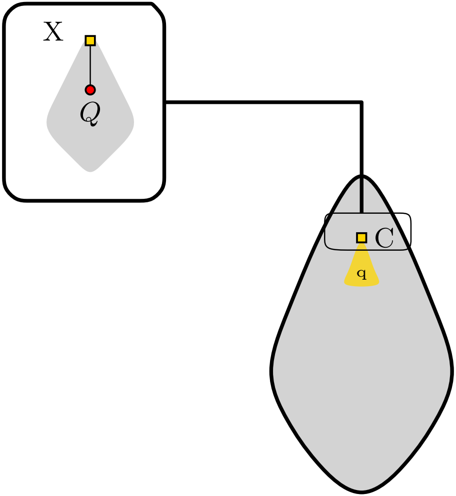
- $ex(12345) = 15$
- $ex(234) = 24$
- $ex(45) = 45$



Acceptant convex geometry

Lemma

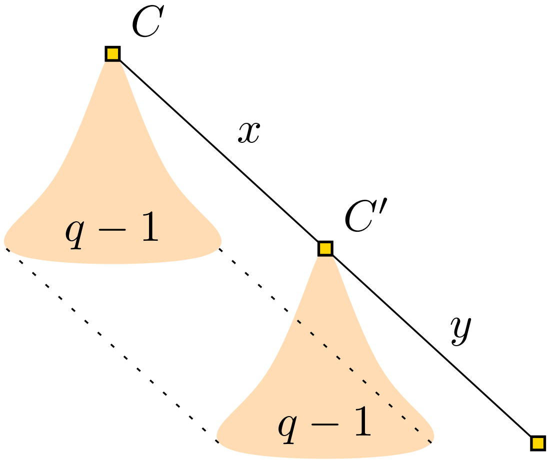
An essential set C has a unique maximal quasi-closed set



Acceptant convex geometry

Lemma

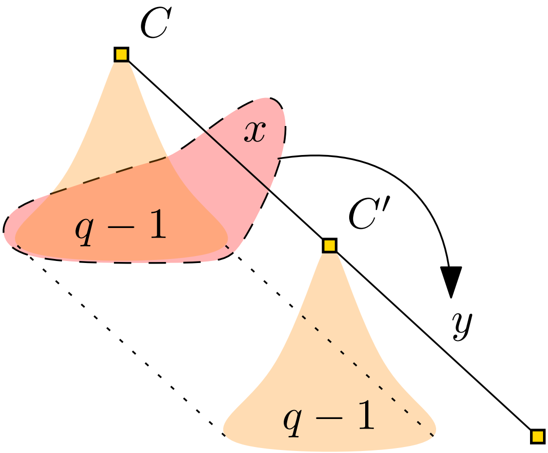
An essential set C has a unique maximal quasi-closed set



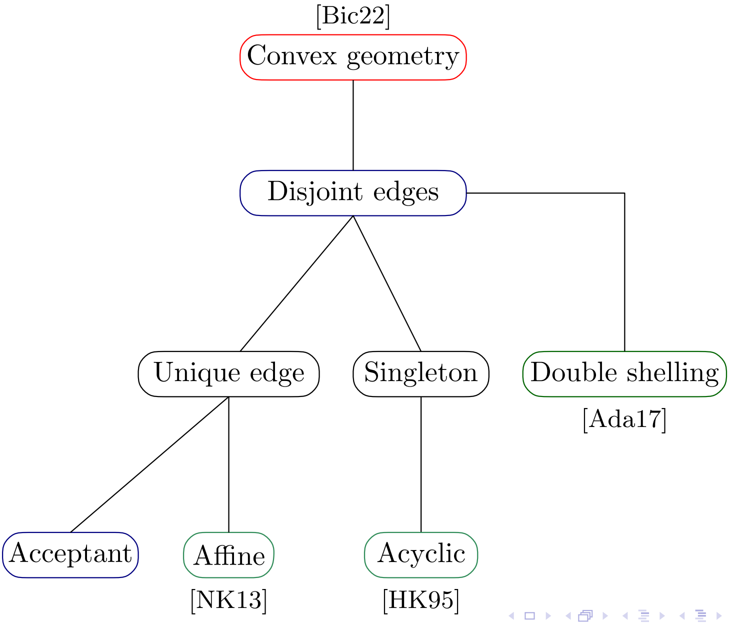
Acceptant convex geometry

Lemma

An essential set C has a unique maximal quasi-closed set



Conclusion


[Bic22]
Convex geometry
Disjoint edges
Unique edge
Singleton
Double shelling
[Ada17]
Acceptant
Affine
[NK13]
Acyclic
[HK95]

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Thank you for your attention

Double-shelling convex geometries

- Let $P = (X, \leq)$ be a poset and let $x, y \in X$
- $[x, y] = \{z : z \in X, x \leq z \leq y\}$ is the **interval** of x, y

Definition

A set $C \subseteq X$ is called a **convex set** of P if for any $x, y \in C$ such that $x < y$, $[x, y] \subseteq C$ holds

Definition

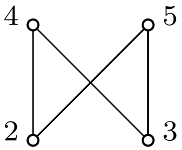
The collection of convex set is called **double-shelling convex geometry**
Its canonical base $\Sigma_c = \{xy \rightarrow [x, y] \setminus \{x, y\} : \exists z \in X, x < z < y \text{ in } P\}$

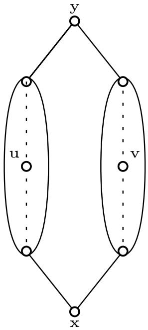
Double-shelling convex geometries

Lemma

Let C be an essential of $\mathcal{C}$ with $\mathrm{ex}(C) = x, y$ and $x < y$ in P Then the edges of $\mathcal{Q}(C)$ are the connected components of $G_p(x, y)$, therefor they are disjoint

$G_p(x, y)$

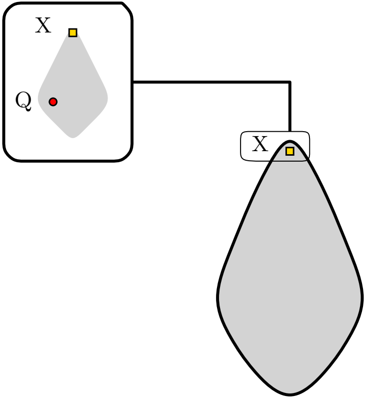




Acyclic convex geometry

Lemma

The maximal quasi-closed sets of an essential set C *are of the form* $C \setminus \{x\}$ *with* $x \in C$



Acyclic convex geometry

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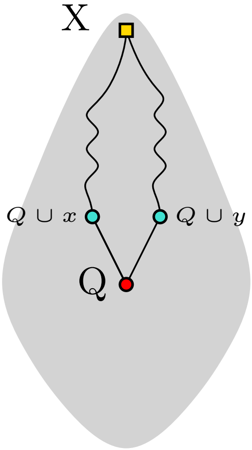


Affine convex geometry

Lemma

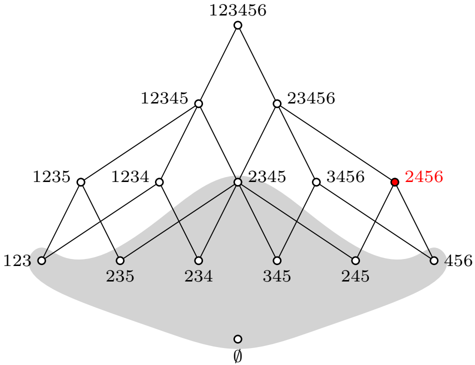
The unique maximal quasi-closed set of any essential set C is $\mathrm{ex}(C)$

Theorem (Nakamura and Kashiwabara 2013)

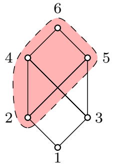
Let $P_1, \ldots, P_m$ be the pseudo-closed sets of $(X, \mathcal{C})$ an affine convex geometry, and for each $1 \leq i \leq m$, let x_i be any element of $\phi(P_i) \setminus P_i$. Then, the pair (X, Σ) with $\Sigma = \{P_1 \rightarrow x_1, \ldots, P_m \rightarrow x_m\}$ is an optimum implicational base of $(X, \mathcal{C})$.

Double-shelling convex geometries

Closure system:



Convex set:



Lemma

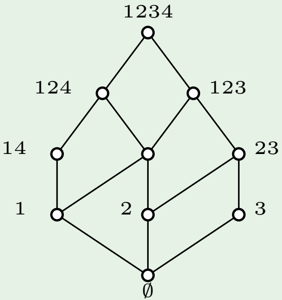
Let C be an essential of $\mathcal{C}$ a double-shelling $\mathcal{Q}(C)$ has disjoint edges for each essential closed set C

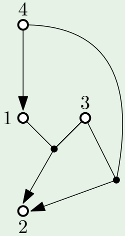
Acyclic convex geometry

Lemma

The maximal quasi-closed sets of an essential set C are of the form $C \setminus \{x\}$ with $x \in C$

Example





Affine convex geometry

Lemma

The unique maximal quasi-closed set of any essential set C is $\mathrm{ex}(C)$

Example