Algorithm q-Acceptant Convex Geometry

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Objectives

1. Characterizing the class of convex geometries associated with q-acceptant choice function.

Introduction

- A choice function over a set A is defined in [1] as a mapping S that associates each subset B of A to a subset $S(B)$ of B.
  - $S(B)$ is the *choosed outcome*, B is a subset of outcomes of A.
- About choice function?
  - Concept of social choice theory which as impact in many field such as game theory, decision strategie or microeconomics
  - There exists different type of choice function satisfying or not a restrained pull of axioms

Definition

- A **closure operator** on a set X is an application $\phi : 2^X \to 2^X$ for all $F \subseteq X$ s.t.:
  - extensivity: $F \subseteq \phi(F)$.
  - monotonicity: $F' \subseteq F \Rightarrow \phi(F') \subseteq \phi(F)$.
  - idempotence: $\phi(\phi(F)) = \phi(F)$.
- A **closed set** F is a set such that $F = \phi(F)$.
- ϕ is **antiexchange** if for each closed set F, and all distinct $x, y \notin F$, if $x \in \phi(F \cup y)$ then $y \notin \phi(F \cup x)$.
- We call $\mathcal{C} = \{\phi(F) \mid F \in 2^X\}$ a **closure system**.
- $\mathcal{C}$ is a **convex geometry** if and only if the associated closure operator ϕ is antiexchange.
- An element x of a closed set F is an **extreme point** if $\phi(F) \setminus x \in \mathcal{C}$.
- In a convex geometry, a closed set M is **irreducible** if there exists a unique x such that $M \cup \{x\} \in \mathcal{C}$.
- We note $\mathcal{C}_x = \{C \mid C \in 2^X, x \in C\}$ and $\mathcal{C}_{\bar{x}} = \{C \mid C \in 2^X, x \notin C\}$
- In the context of convex geometry the extreme point function is our choice function.

Representation of an acceptant

- $\mathcal{C}$ is q-acceptant if for all $F \in \mathcal{C}$ we have $|F| = \min(q, |F|)$

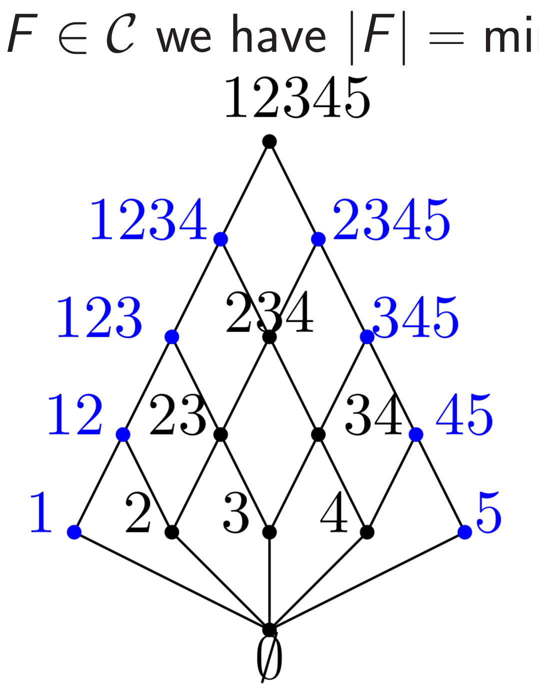

Figure: Example of a 2-acceptant convex geometry with $\mathcal{M}$ in blue

- In a set Σ, called implication bases, we associate the choosed outcome, $ex(F)$, and the outcome, F.
  - $\Sigma = \{15 \to 234, 14 \to 23, 25 \to 34, 13 \to 2, 24 \to 3, 35 \to 4\}$
- An Antichain, noted $\mathcal{A}$, is the set of choosed outcome of size q with their outcome greater than q
  - $\mathcal{A} = \{15,\ 14,\ 25,\ 13,\ 24,\ 35\}$
- The set of irreducible closed set, noted $\mathcal{M}$
  - $\mathcal{M} = \{1234,\ 2345,\ 123,\ 345,\ 12,\ 45,\ 1,\ 5\}$

Problem: Given representation, verify that it represent a q-acceptant convex geometry?

Well known property

- Let $\mathcal{A}$ be a q-acceptant antichain on X. Then $|\mathcal{A}| = \binom{n-1}{q}$ [2]
- In a q-acceptant convex geometry every $F \in 2^X$, $\phi(F)$ has q predecessor.
- Let $|M| \geq q$ and $M \in \mathcal{M}$. Then there exists a unique $x \in X$ such that $M \setminus x \in \mathcal{M}$. [3]

Results - Implication bases

- Let $\Sigma = \{ex(F) \to F \setminus ex(F) | F \in \mathcal{C}, |F| > q\}$ the implication base associated to $(X, \mathcal{C})$. Then, $(X, \mathcal{C})$ is a q-acceptant convex geometry if and only if the following conditions hold:
  1. for each choosed outcome F of Σ, $|F| = q$ and their does not exists a choosed outcome F', $F \neq F'$, such that $\phi(F) = \phi(F')$
  2. the set $\{\phi(F) \mid F$ is a premise of $\Sigma\}$ is exactly the set of closed sets with more than q elements of $(X, \mathcal{C})$
  3. for each subset Y of X of size $q+1$ there exists $F \to \phi(F) \setminus F \in \Sigma$ such that $F \subseteq Y$
- We deduce a polynomial algorithm recognizing the acceptance in 3 part such that for any $F \in 2^X, |F| \geq q$:
  - Verifying that the mapping between $ex(F)$ and $\phi(F)$ is unique.
  - Verifying that there is no closed set $\phi(F)$ with $|ex(F)| > q$
  - Verifying that for all $\phi(F)$ such that $|\phi(F)| = q+1$, we have $|ex(F)| = q$.

Results - Antichain

- Let $(X, \mathcal{C})$ be a non-trivial q-acceptant convex geometry and let $x \in ex(X)$. Then $(X \setminus x, \mathcal{C}_x)$ is $(q-1)$-acceptant and $(X \setminus x, \mathcal{C}_{\bar{x}})$ is q-acceptant.
- Partition of a q-acceptant

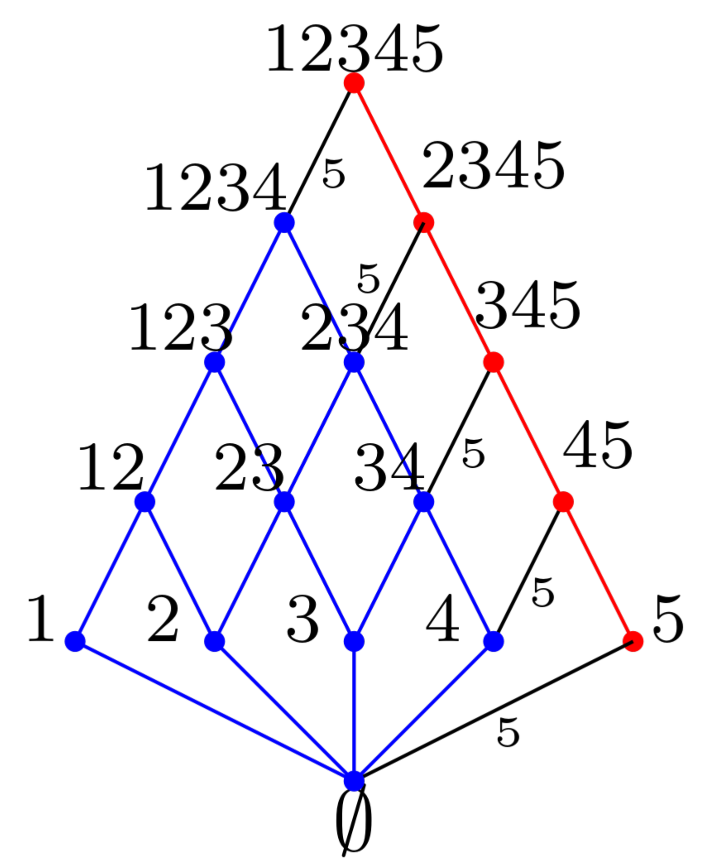

Figure: Example of partition of a 2-acceptant convex geometry on $X = \{1, 2, 3, 4, 5\}$, with in blue the 2-acceptant convex geometry and the 1-acceptant in red.

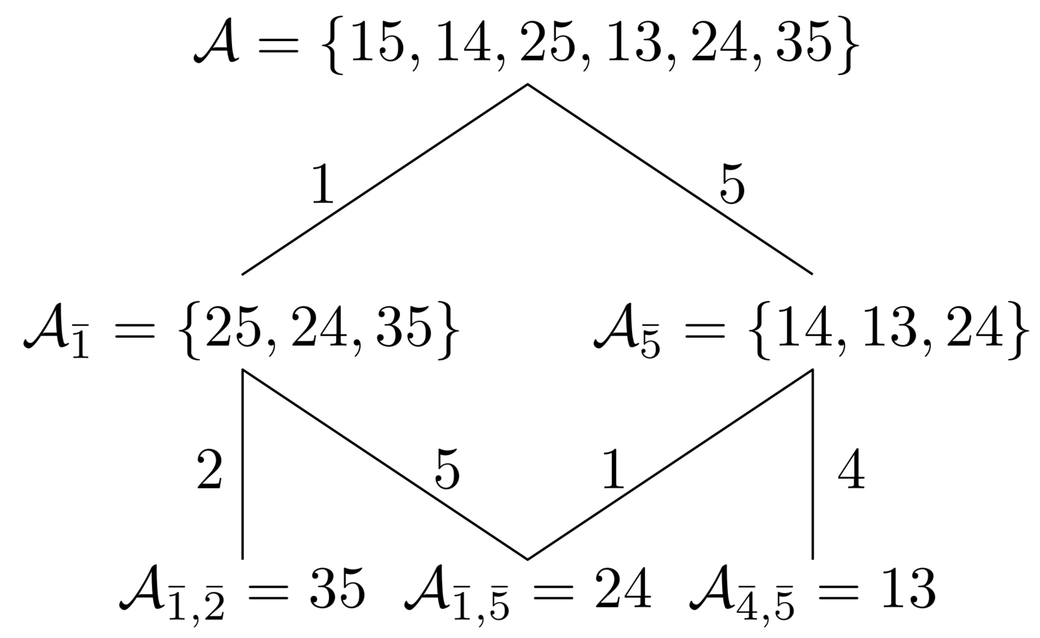

Figure: Recognition of the acceptance of the previous example with the algorithm

- From this property, we propose an algorithm which build the upper part of the lattice from the top.

Coming work

- Is the algorithm proposed for the antichain correct and polynomial?
- Exploiting the structure of the irreducible closed set, we want to leverage what is called *colored poset* to design an algorithm recognizing the acceptance of $\mathcal{M}$.

References

[1] Hervé Moulin.
Choice functions over a finite set: a summary.
Social Choice and Welfare, 2(2):147–160, 1985.

[2] Battal Dogan, Serhat Doğan, and Kemal Yildiz.
On acceptant and substitutable choice rules.
SSRN Electronic Journal, 01 2017.

[3] Serhat Doğan.
Explorations to Refine Aizerman Malishevski's Representation for Path Independent Choice Rules.
PhD thesis, Bilkent Universitesi (Turkey), 2020.