Algorithms in lattices

Thesis committee 2023-2024 - 1st year

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Algorithms in lattices

- Scholarship from MESRI
- Study of algorithm solving computational problem on lattices
- Problem : enumeration of the keys for an implication base

Outline

Outline

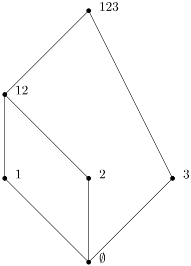
Lattices

Definition

A **lattice** is a partial order in which every pair of elements A and B has a **join** (i.e. least upper bound) and a **meet** (i.e. greatest lower bound)

If $(X, \mathcal{F})$ is a closure system, $(\mathcal{F}, \subseteq)$ is a (closure) lattice

- $X = \{1, 2, 3\}$
- $\mathcal{F} = \{\emptyset, 1, 2, 3, 12, 123\}$



$(\mathcal{F}, \subseteq)$

Closure system

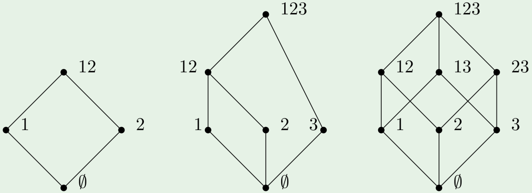
Definition

A **closure system** is a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a collection of subsets of X satisfying :

- $\emptyset, X \in \mathcal{F}$
- $\forall X_1, X_2 \in \mathcal{F},\ X_1 \cap X_2 \in \mathcal{F}$

An element of $\mathcal{F}$ is called a closed set

Example



Closure operator

Definition

A **closure operator** on a set X is a mapping $\phi : 2^X \to 2^X$ which is for every subset S of X :

- extensive: $S \subseteq \phi(S)$
- monotone: for all $S' \subseteq S, \phi(S') \subseteq \phi(S)$
- idempotent: $\phi(S) = \phi(\phi(S))$

Example

- $X = \{1, 2, 3\}, \mathcal{F} = \{\emptyset, 1, 2, 3, 12, 123\}$
- $\phi(S) = \bigcap\{S' : S \subseteq S', S' \in \mathcal{F}\}$
- $\phi(13) = 123$

Implication bases

Definition

An **implication** over X is an expression $L \to R$ where $L, R \subseteq X$

Definition

An **implication base** Σ over X is a collection of implication over X

- The closure of Σ is calculated with a procedure called forward chaining
  - For S a subset of X
  - $S = S_0 \subseteq ... \subseteq S_k = S^\Sigma$
  - $S_i = S_{i-1} \cup \bigcup\{B \mid \exists A \to B \in \Sigma \text{ such that } A \subseteq S_{i-1}\}$
- Different Σ can represent the same closure system

Keys of implication bases

Definition

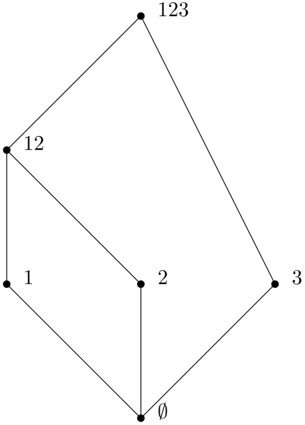
Given an implication base (X, Σ), **a key** of Σ is a subset K of X such that $K^{\Sigma} = X$ and for every $u \in K, (K \setminus u)^{\Sigma} \neq X$.
We call $\mathcal{K}(\Sigma)$ the keys of Σ, i.e, $\mathcal{K}(\Sigma) = min_{\subseteq}\{K \subseteq X : K^{\Sigma} = X\}$

Implication bases

For a closure system $(X, \mathcal{F})$, an equivalent representation is the implication base Σ where implications are $X_i \to \phi(X_i)$ for all $X_i \subseteq X$

Example

- $X = \{1, 2, 3\}, \mathcal{F} = \{\emptyset, 1, 2, 3, 12, 123\}$
- $\Sigma = \{\emptyset \to \emptyset, 1 \to 1, 2 \to 2, 3 \to 3, 12 \to 12, 13 \to 123, 23 \to 123, 123 \to 123\}$
- $\Sigma = \{13 \to 2, 23 \to 1\}$
- $\mathcal{K}(\Sigma) = \{13, 23\}$



Outline

Enumerating keys of an implication base

Key enumeration on an implication base

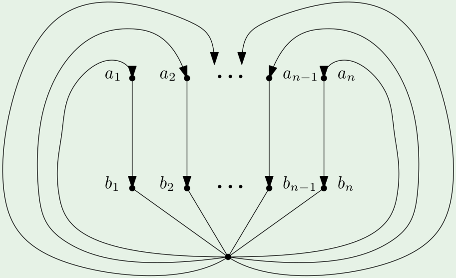
Input. (X, Σ)

Output. $\mathcal{K}(\Sigma)$

- In general, an exponential number of solutions

Example

- $X = \{a_1..a_n\} \cup \{b_1..b_n\}$
- $\Sigma = \{a_i \to b_i, 1 \leq i \leq n\} \cup \{b_1..b_n \to X\}$

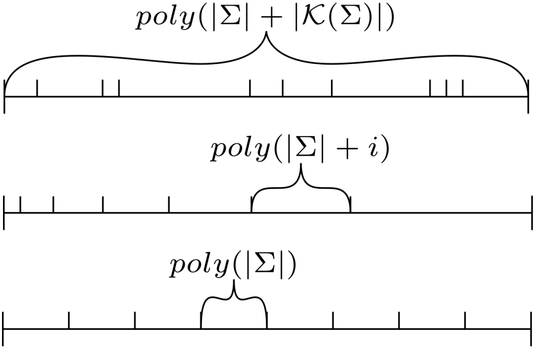


An Enumeration algorithm

- An algorithm that list the answers to an enumeration problem without repetition.
- Output sensitive algorithm

Output sensitive complexity

- output polynomial
- incremental polynomial
- polynomial delay



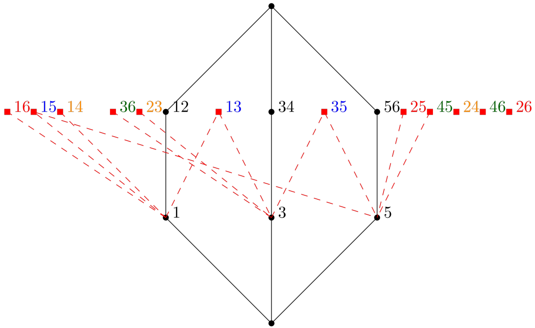
Previous work

- [Lucchesi and Osborn 1978], Polynomial delay and exponential space
  - Database language, supergrah approach
- [Ennaoui and Nourine 2016], Polynomial delay and exponential space
  - Improvment of Lucchesi and Osborn 1978, based on unitary implication
- [Bérczi et al. 2022], Polynomial delay and exponential space
  - Horn logic, similar to Lucchesi and Osborn 1978

Ennaoui and Nourine's work

Approach

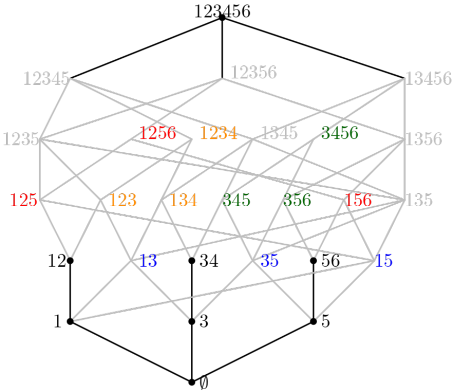
- $\Sigma = \{2 \to 1, 4 \to 3, 6 \to 5, 13 \to 246, 15 \to 246, 35 \to 246\}$
- $\mathcal{K}(\Sigma) = \{13, 15, 35, 23, 25, 24, 26, 14, 45, 46, 16, 36\}$



Ennaoui and Nourine's work

Approach

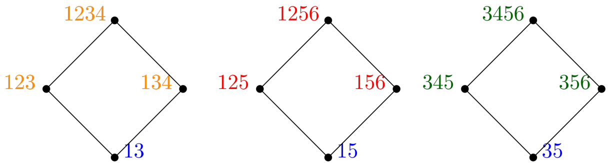
- $\Sigma = \Sigma_u \cup \Sigma_{nu}$
- $\Sigma_u = \{2 \to 1, 4 \to 3, 6 \to 5\}$



Ennaoui and Nourine's work

Enumeration of equivalence classes

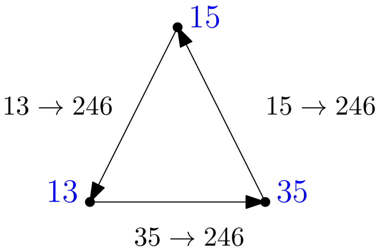
- $\Sigma_u = \{2 \to 1, 4 \to 3, 6 \to 5\}$
- The closure of the keys are divided between equivalence classes
- These classes can be enumerated in time and space polynomial with a reverse search



Ennaoui and Nourine's work

Enumeration of minimal keys

- $\Sigma_{nu} = \{13 \rightarrow 246, 15 \rightarrow 246, 35 \rightarrow 246\}$
- Depth first search traversal of a tree rooted at $\pi^*(X)$ of the super graph
- $O(|X| \times \textit{number of equivalence class})$



Outline

Our goal

- Can we relax a condition over Σ_u such that we enumerate the keys of an implication base with polynomial delay and polynomial space ?
  - Instantiate a reverse search to enumerate the classes from a minimal key
    - determine a parent function
    - determine a child function
  - Enumerate the minimal keys with the remaining implications

Definition

Definition (Wild 2017)

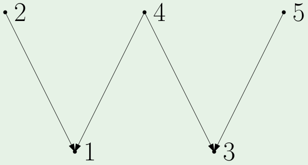
To any family Σ of implication on a set X we can associate the implication-graph $G(\Sigma)$. It has vertex set X and arcs $a \to b$ whenever there is an implication $A \to B \in \Sigma$ with $a \in A$ and $b \in B$, if $G(\Sigma)$ has no directed cycle, Σ is **acyclic**

Definition

In an acyclic implication base, x is an **extreme points** of a set K, if $x \in K$ and $x \notin \phi(K - x)$. We note the set of extreme point of K $ex(K)$

Example

- $\Sigma = \{45 \to 3, 24 \to 1\}$
- ex(345) $= 45$
- ex(1345) $= 145$



A child function

Algorithm 1: $\pi(K)$

Output: A parent key of K

begin

  if K *a minimal key* **then**

    return K

  else

    Let x be the lexicographical smallest attribute in $ex(K)$ such that $(ex(K - x))^{\Sigma} = X$;

    $K = K \setminus \{x\}$;

    while K *is not the closure of a key* **do**

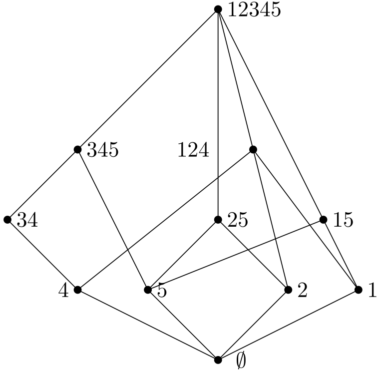
      Let x be the lexicographical smallest attribute in $ex(K)$ such that $(ex(K - x))^{\Sigma} = X$;

      $K = K \setminus \{x\}$;

Some experiments

Example

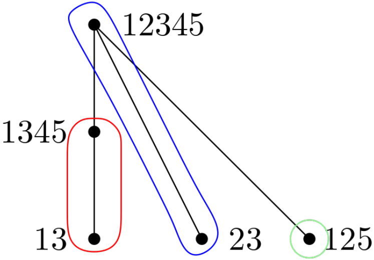
- $\Sigma = \{3 \to 4, 45 \to 3, 24 \to 1, 12 \to 4, 14 \to 2, 13 \to 5\}$
- $\mathcal{K}(\Sigma) = \{13, 23, 125, 145, 245\}$



Some experiments

Example

- $\Sigma = \Sigma_a \cup \Sigma_{na}$
- $\Sigma_a = \{45 \rightarrow 3, 24 \rightarrow 1\}$

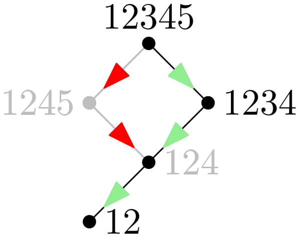


Some experiments

Counter example

- $\Sigma = \{34 \rightarrow 12, 5 \rightarrow 4, 12 \rightarrow 35\}$
- $\Sigma_a = \{34 \rightarrow 12, 5 \rightarrow 4\}$
- $\mathcal{K}(\Sigma) = \{12, 34, 35\}$

Using only the order between the elements is not enough



Conclusion

Key enumeration on an implication base

Input. (X, Σ)

Output. $\mathcal{K}(\Sigma)$

- Generalizing [Ennaoui and Nourine 2016] to reduce the complexity of previous work
  - Enumerate equivalence class with reverse search
    - Elaborate a parent function
    - Elaborate a child function
  - Enumerate minimal keys
    - Are the keys in a strongly connected graph ?

Outline

Formations

- Module SPI (75%):
  - Outils mathématiques pour le doctorant avec Matlab
  - La recherche en sciences pour l'ingénieur au service de l'environnement et des agrosystèmes
  - Metaheuristique pour l'optimisation combinatoire. Évolution et tendances actuelles
- Module OSP (75%):
  - Éthique : conférence plénière et science ouverte (atelier disciplinaire : 17 juin)
  - Zététique et autodéfense intellectuelle
  - Valoriser ses recherches au regard de la science ouverte
  - Concevoir un plan de gestion de données Gérer et partager ses données de la recherche

Participation to conferences

- 41st International Symposium on Theoretical Aspects of Computer Science (STACS 2024) – Clermont-Ferrand
- Workshop on Matroid-Constrained Optimization Problems – Paris

Thanks for your attention

References

Bérczi, Kristóf et al. (2022). “Unique key Horn functions”. In: *Theoretical Computer Science* 922, pp. 170–178.

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