Algorithms in lattices

Thesis committee 2024-2025 - 2nd year

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Algorithms in lattices

- Scholarship from MESRI
- Study of algorithm solving computational problem on lattices
- Problem : Characterization of q-acceptant convex geometries

Outline

Outline

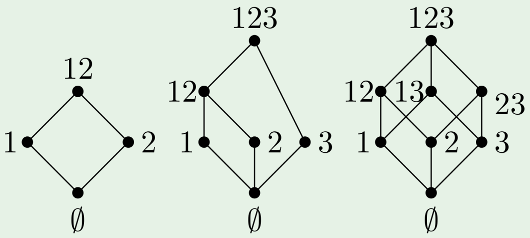
Closure system

Definition

A **closure system** is a pair $(X, \mathcal{C})$ where X is a ground set and $\mathcal{C} \subseteq 2^X$ is closed under intersection and contains X.

- $(\mathcal{C}, \subseteq)$ is a closure **lattice**
- $\mathcal{C}$ induces ϕ a **closure operator** with $\phi(F) = \min_{\subseteq}\{F \mid C \in \mathcal{C}, F \subseteq C\}$

Example

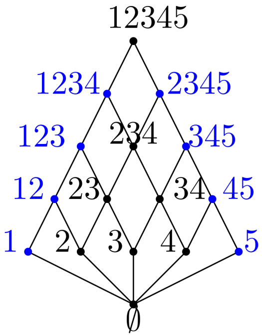


Irreducible closed set

Definition

Closed set $M \in \mathcal{C}$ is meet-irreducible if it is not the intersection of other closed sets

- We note by $\mathcal{M}$ the set of meet-irreducible closed set
- From $\mathcal{M}$ we can compute $\mathcal{C}$



Example

- $X = \{1, 2, 3, 4, 5\}$
- $\mathcal{M} = \{1234,\ 2345,\ 123,\ 345,\ 12,\ 45,\ 1,\ 5\}$

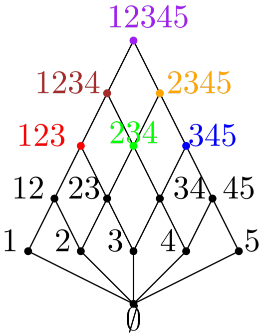
Implication base

Definition

An implication over X is a statement $A \to B$, with $A \subseteq X$, and $B \subseteq X$.
An implication base is a pair (X, Σ) where Σ is a set of implication on X.

Example

- $\Sigma = \{13 \to 2, 24 \to 3, 35 \to 4, 14 \to 23, 25 \to 34, 15 \to 234\}$

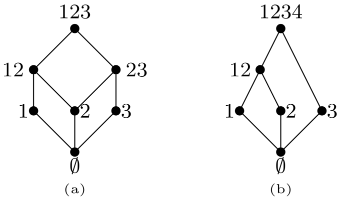


Convex geometry

Definition

$(X, \mathcal{C})$ is a **convex geometry** if for all $C \in \mathcal{C}$, with $C \neq X$, there exists $x \in X \setminus C$ such that $C \cup x \in \mathcal{C}$

- ϕ the closure operator associated to $\mathcal{C}$ is antiexchange
- Convex geometry are a concept rediscovered many times [Monjardet 1985]
- *a* is a convex geometry, *b* is not



Outline

1 Definitions

2 The problem
- Choice function
- Acceptant convex geometry
- previous work

3 Ongoing work

4 Other

Choice function

- Choice function are relevant in many fields such as game theory, decision strategy or microeconomics.

Definition (Moulin 1985)

A choice function over a set A is a mapping S that associates each subset B of A to a subset $S(B)$ of B. $S(B)$ is the *chosen outcome*, B is a subset of outcomes of A.

- One-to-one correspondence between path independent choice function and convex geometries [Koshevoy 1999]
- We interest ourselves in path independent choice functions called *q-acceptant*

Acceptant convex geometry

Definition

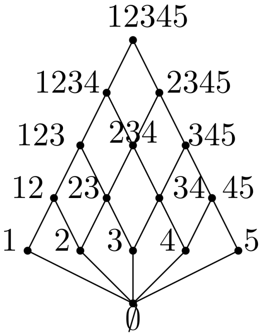
For all $F \subseteq X$, we call $x \in ex(F)$ an **extreme point** of F if $\phi(F) \setminus x \in \mathcal{C}$

Definition

A convex geometry $(X, \mathcal{C})$ is q-acceptant if for all $C \in \mathcal{C}$ we have $|ex(C)| = \min\{q, |C|\}$

Example

- $ex(12345) = 15$
- $ex(234) = 24$
- $ex(45) = 45$



Previous work

- [Alkan 2001], study the structure of the lattice of stable matchings, which is associated to q-acceptant choice functions
- [Dogan, Doğan, and Yildiz 2017], provide some structural lemma on the lattice of q-acceptant choice functions.
- [Chambers and Yenmez 2018], study a super-class of q-acceptant choice functions, lexicographic choice functions

We aim to recognize the acceptance of a convex geometry on their different representation.

Outline

1 Definitions

2 The problem

3 Ongoing work
- Recognition from Σ
- Recognition from $\mathcal{M}$
- Recognition from $\mathcal{A}$

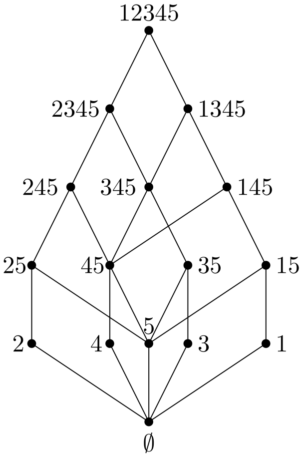
4 Other

Recognition from Σ

Definition

A closed set C is *non-trivial* if $ex(C) \neq C$

- $\Sigma = \{12 \to 345, 23 \to 45, 13 \to 45, 14 \to 5, 24 \to 5\}$
- We verify 2 things:
  - A bijection between the premises of Σ and the non-trivial closed sets
  - The premises of Σ covers $\binom{n}{q+1}$

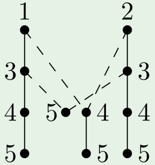


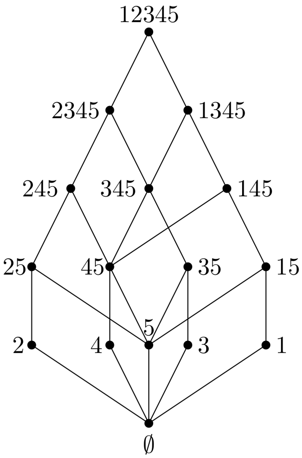
Recognition from $\mathcal{M}$

In a convex geometry every $M \in \mathcal{M}$ has a single successor, $M \cup x$, with $x \in X \setminus M$. We call x a color

Example

- the poset $(\mathcal{M}, \subseteq)$ where each subset is labeled by their color





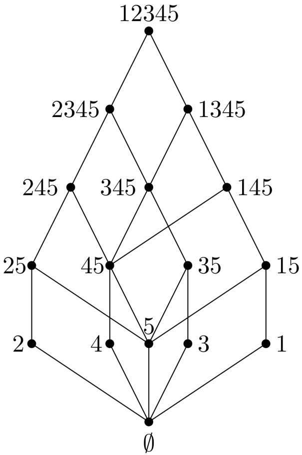
Recognition from $\mathcal{M}$

Dogan's work

Lemma (Doğan 2020)

For all $C \in \mathcal{C}$, $C \neq X$, there exists at most one $M \in \mathcal{M}$ such that $M \prec C$. Moreover, if $C \in \mathcal{M}$, $|C| \geq q$, C cover exactly one M.

We call *q-rigid* a convex geometry which satisfies this lemma

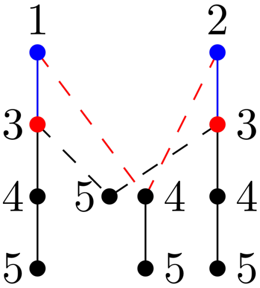


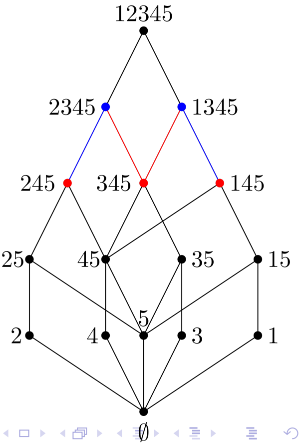
Recognition from $\mathcal{M}$

Example

Proposition

let $(X, \mathcal{C})$ be q-rigid. it is q-acceptant if and only if for all $M_1, M_2 \in \mathrm{Mi}(\mathcal{C})$ with the same color and $|M_1^* \cap M_2^*| \geq q$ then $|ex(M_1^* \cap M_2^*)| \geq q$





Recognition from $\mathcal{A}$

Dogan's work

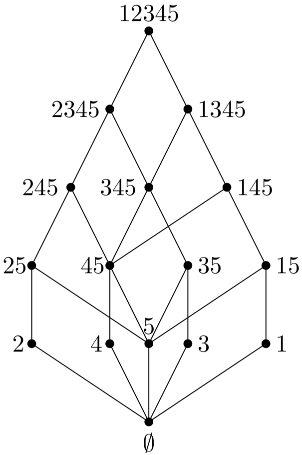
We consider $\mathcal{A} \subset 2^X$ an antichain, we note $|X| = n$

Lemma (Dogan, Doğan, and Yildiz 2017)

Let $\mathcal{A}$ be an antichain on X. If it is acceptant on some q then $|\mathcal{A}| = \binom{n-1}{q}$

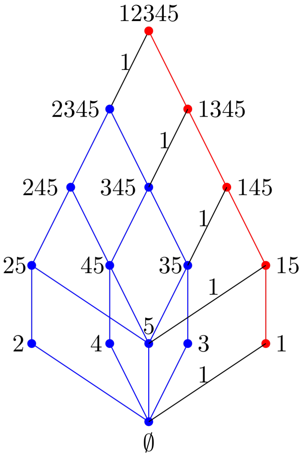
Example

- $\mathcal{A} = \{13, 14, 12, 34, 23, 24\}$



Recognition from $\mathcal{A}$

Relevant property



Lemma

Let $(X, \mathcal{C})$ be a non-trivial q-acceptant convex geometry and let $x \in ex(X)$. Then $(X \setminus x, \mathcal{C}_x)$ is $(q-1)$-acceptant and $(X \setminus x, \mathcal{C}_{\bar{x}})$ is q-acceptant.

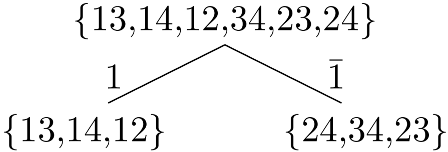
- $\mathcal{C}_x = \{C \setminus x \mid C \in \mathcal{C}, x \in C\}$
- $\mathcal{C}_{\bar{x}} = \{C \in \mathcal{C} \mid x \notin C\}$.

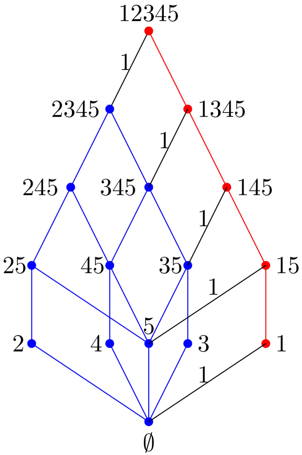
Recognition from $\mathcal{A}$

Relevant property

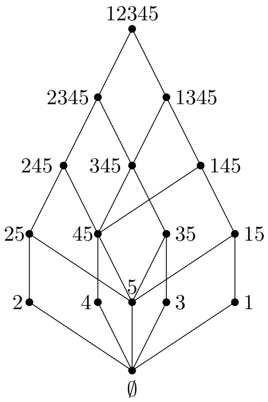
Definition

We call $x \in X$ a *quasi-prime* if $occ(x, \mathcal{A}) = \binom{n-2}{q-1}$. $Q_{\mathcal{A}}$ is the set of quasi prime elements of $\mathcal{A}$.





Recognition from $\mathcal{A}$
Example



- $\mathcal{A} = \{13, 14, 12, 34, 23, 24\}$
- $Q_{\mathcal{A}} = \{1, 2, 3, 4\}$
- $K = 12$

Recognition from $\mathcal{A}$

Example

$$\mathcal{A} = \{13, 14, 12, 34, 23, 24\}$$
$$Q_{\mathcal{A}} = \{1, 2, 3, 4\}$$
$$X = \{1, 2, 3, 4, 5\}$$
$$K = 12$$

Recognition from $\mathcal{A}$

Example

$$\mathcal{A} = \{13, 14, 12, 34, 23, 24\}$$
$$Q_{\mathcal{A}} = \{1, 2, 3, 4\}$$
$$X = \{1, 2, 3, 4, 5\}$$
$$K = 12$$

2 | 1

$$\mathcal{A}_{\bar{2}} = \{13, 14, 34\}$$
$$Q_{\mathcal{A}_{\bar{2}}} = \{1, 3, 4\}$$
$$X = \{1, 3, 4, 5\}$$
$$K = 1$$

$$\mathcal{A}_{\bar{1}} = \{34, 23, 24\}$$
$$Q_{\mathcal{A}_{\bar{1}}} = \{2, 3, 4\}$$
$$\mathrm{X}=\{2,3,4,5\}$$
$$\mathrm{K}=2$$

- We take a look at $\mathcal{A}_{\bar{2}}$
  - We need to complete K
  - We can choose either 3 or 4

Recognition from $\mathcal{A}$

Example

$\mathcal{A} = \{13, 14, 12, 34, 23, 24\}$
$Q_{\mathcal{A}} = \{1, 2, 3, 4\}$
$X = \{1, 2, 3, 4, 5\}$
$K = 12$

2

1

$\mathcal{A}_{\bar{2}} = \{13, 14, 34\}$
$Q_{\mathcal{A}_{\bar{2}}} = \{1, 3, 4\}$
$X = \{1, 3, 4, 5\}$
$K = 13$

$\mathcal{A}_{\bar{1}} = \{34, 23, 24\}$
$Q_{\mathcal{A}_{\bar{1}}} = \{2, 3, 4\}$
X={2,3,4,5}
K=23

3

1

$\mathcal{A}_{\bar{2},\bar{3}} = \{14\}$
$Q_{\mathcal{A}_{\bar{2},\bar{3}}} = \{1, 4\}$
$X = \{1, 4, 5\}$
$K = 1$

$\mathcal{A}_{\bar{1},\bar{2}} = \{34\}$
$Q_{\mathcal{A}_{\bar{1},\bar{2}}} = \{3, 4\}$
X={3,4,5}
$K = 3$

Recognition from $\mathcal{A}$

Example

$\mathcal{A} = \{13, 14, 12, 34, 23, 24\}$
$Q_{\mathcal{A}} = \{1, 2, 3, 4\}$
$X = \{1, 2, 3, 4, 5\}$
$K = 12$

2 1

$\mathcal{A}_{\bar{2}} = \{13, 14, 34\}$
$Q_{\mathcal{A}_{\bar{2}}} = \{1, 3, 4\}$
$X = \{1, 3, 4, 5\}$
$K = 13$

$\mathcal{A}_{\bar{1}} = \{34, 23, 24\}$
$Q_{\mathcal{A}_{\bar{1}}} = \{2, 3, 4\}$
X={2,3,4,5}
K=23

3 1 2 3

$\mathcal{A}_{\bar{2},\bar{3}} = \{14\}$
$Q_{\mathcal{A}_{\bar{2},\bar{3}}} = \{1, 4\}$
$X = \{1, 4, 5\}$
$K = 14$

$\mathcal{A}_{\bar{1},\bar{2}} = \{34\}$
$Q_{\mathcal{A}_{\bar{1},\bar{2}}} = \{3, 4\}$
X={3,4,5}
$K = 34$

$\mathcal{A}_{\bar{1},\bar{3}} = \{24\}$
$Q_{\mathcal{A}_{\bar{1},\bar{3}}} = \{2, 4\}$
$X = \{2, 4, 5\}$
$K = 24$

- Figuring out if the choice matter for the recognition is crucial for this algorithm

Conclusion

On the Recognition of the acceptance of a q-acceptant convex geometry :

- with Σ : Poly-time algorithm bulding the non-trivial part of the lattice
- with $\mathcal{M}$: We found interesting property, we need to prove it is enough to recognize the acceptance and use it to derive an algorithm
- with $\mathcal{A}$: We need to try again to prove that the order does not matter in the partition of $\mathcal{A}$ with the new structural property find out in the study of $\mathcal{M}$

Outline

Formations

- Module SPI (75%) :
  - Outils mathématiques pour le doctorant avec MATLAB
  - La recherche en sciences pour l'ingénieur au service de l'environnement et des agrosystèmes
  - Metaheuristique pour l'optimisation combinatoire. Évolution et tendances actuelles
- Module OSP (100%) :
  - Éthique : conférence plénière, science ouverte et atelier disciplinaire
  - Zététique et autodéfense intellectuelle
  - Valoriser ses recherches au regard de la science ouverte
  - Concevoir un plan de gestion de données Gérer et partager ses données de la recherche
  - Enseigner à l'université

Conférence et cours

- Conférences :
  - 41st International Symposium on Theoretical Aspects of Computer Science (STACS 2024) – Clermont-Ferrand (Mars 2024)
  - Workshop on Matroid-Constrained Optimization Problems – Paris (avril 2024)
- Cours :
  - Représentation en binaire - L1 (cours/TD)
  - Rédaction mathématique et informatiques - L1 (Cours/TD et TP)
  - Système d'information et base de données - L2 (TD et TP)

References

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Thank you for your attention