Characterizing the optimum bases of a convex geometry using quasi-closed hypergraphs

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Outline

Outline

1. Definitions
   - Implication Bases
   - Convex Geometry
2. Optimization of implication base of a convex geometry

Implication bases

Definition

An **implication** over X is a statement $A \to B$, with $A \subseteq X$, and $B \subseteq X$.
An **implication base** is a pair (X, Σ) where Σ is a set of implications on X.

- A is the **premise** of the implication
- B is the **conclusion** of the implication

Example

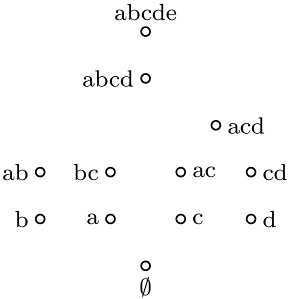
- $\Sigma = \{ae \to bcd, bd \to ac, e \to ac, abc \to d, ad \to c\}$

Closure systems

- $\mathcal{C} = \{C : A \subseteq C \implies B \subseteq C \text{ for all } A \to B \in \Sigma\}$ is called a closure system of Σ
  - $\mathcal{C}$ is associated to ϕ a closure operator
  - for all $C \in \mathcal{C}$ $\phi(C) = C$

Example

$\Sigma = \{ae \to bcd, bd \to ac, e \to ac, abc \to d, ad \to c\}$

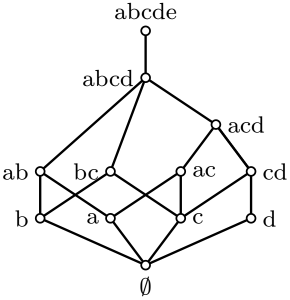


Closure systems

- Σ is an implication base of $\mathcal{C}$ if C satisfies Σ if and only if $C \in \mathcal{C}$
  - The set of subset of X satisfied by Σ form a lattice

Example

$\Sigma = \{ae \rightarrow bcd, bd \rightarrow ac, e \rightarrow ac, abc \rightarrow d, ad \rightarrow c\}$

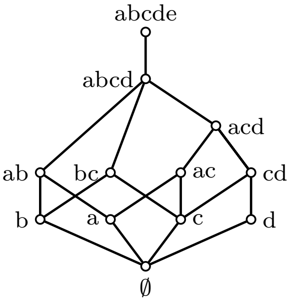


Implication bases and lattices

- A lattice admits several equivalent implication bases

Example

- $\Sigma = \{ae \rightarrow bcd, bd \rightarrow ac, e \rightarrow ac, abc \rightarrow d, ad \rightarrow c\}$
- $\Sigma = \{e \rightarrow b, e \rightarrow d, bd \rightarrow ac, abc \rightarrow d, ad \rightarrow c\}$
- $\Sigma = \{e \rightarrow abcd, bd \rightarrow ac, abc \rightarrow d, ad \rightarrow c\}$

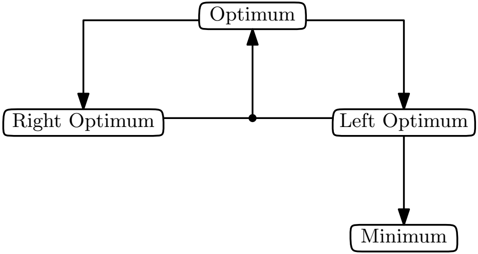


Minimality criterion of an Implication Bases

An implication base of $(X, \mathcal{C})$ is:

- **Minimum** if it minimizes the number of implications
- **Left optimum** if the sum of the size of premises is minimum
- **Right optimum** if the sum of the size of the conclusions is minimum
- **optimum** if it is both left and right optimum

see Wild 2017 for more on implication bases



Optimization of implication bases

Problem

input: (X, Σ) *an implication base of a closure system* $(X, \mathcal{C})$

output: (X, Σ') *an optimum base of* $(X, \mathcal{C})$

- Maier 1980: Optimization is NP-complete for arbitrary closure system
- Ausiello, D'Atri, and Saccà 1986: Left and right optimization are NP-complete
- Bichoupan 2022: In convex geometry the problem is NP-complete
- Convex geometry have some subclasses known to be tractable in polynomial time
    - Hammer and Kogan 1995: Acyclic convex geometry
    - Nakamura and Kashiwabara 2013: Affine convex geometry
    - Adaricheva 2017: Double-shelling convex geometry

Optimization of implication bases

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- Maier 1980: Optimization is NP-complete for arbitrary closure system
- Ausiello, D'Atri, and Saccà 1986: Left and right optimization are NP-complete
- Bichoupan 2022: In convex geometry the problem is NP-complete
- these classes are included in other known tractable classes:
  - Adaricheva 2017:D-geometries contains acyclic convex geometry
  - Adaricheva 2017:Carousel contains affine convex geometry

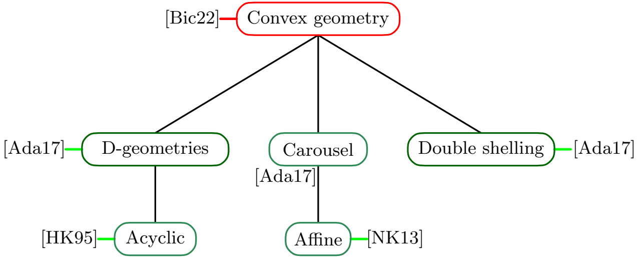
Motivation

Problem

input: (X, Σ) *a minimum implication base of a convex geometry* $(X, \mathcal{C})$

output: (X, Σ') *an optimum base of* $(X, \mathcal{C})$

- When does it remain tractable?



Motivation

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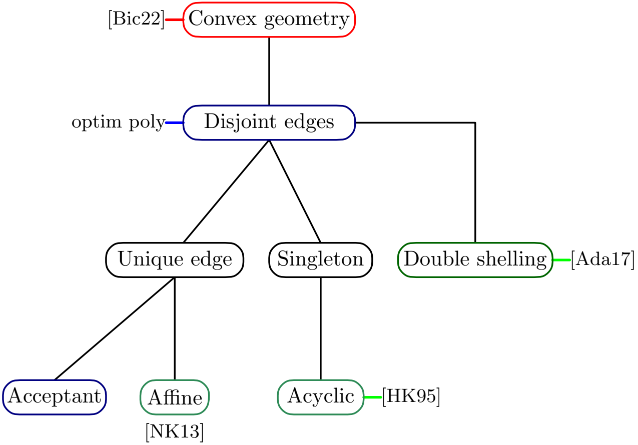


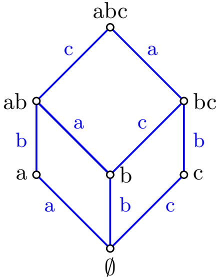
Convex Geometry

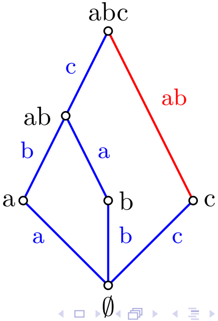
Definition

$(X, \mathcal{C})$ is a **convex geometry** if for all $C \in \mathcal{C}$, with $C \neq X$, there exists $x \in X \setminus C$ such that $C \cup x \in \mathcal{C}$

Definition

We denote by $\mathrm{ex}(A)$ for $A \subseteq X$ the **extreme points** of A,
$\mathrm{ex}(A) = \{x : x \in A, x \notin \phi(A \setminus x)\}$





Outline

Optimization of implication base

Definition

The **equivalence classes** of an implicational base are the sets of implications whose premises share the same closure

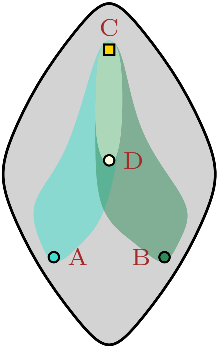
Theorem (Mannila and Räihä 1983)

Let $(X, \mathcal{C})$ *be a closure system and let* (X, Σ) *be an implicational base of* $(X, \mathcal{C})$*. Then* (X, Σ) *is optimum if and only if all its equivalence classes are optimum.*

Left optimization: the easy part

- In implication bases, we call a closed set **essential** if it is the closed set of some premise in every equivalent implication base

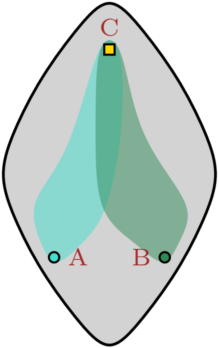
$(P(X), \subseteq)$



Left optimization: the easy part

- In implication bases, we call a closed set **essential** if it is the closed set of some premise in every equivalent implication base
- In minimum bases each implication is associated to an essential

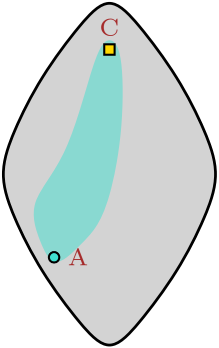
$(P(X), \subseteq)$



Left optimization: the easy part

- In implication bases, we call a closed set **essential** if it is the closed set of some premise in every equivalent implication base
- In minimum bases each implication is associated to an essential
- In the case of convex geometry, each implication is associated to a different closed set

$(P(X), \subseteq)$



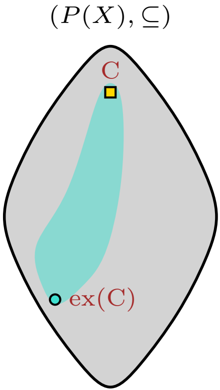
Left optimization: the easy part

Definition

$C \in \mathcal{C}$, a spanning set of C is a subset $A \subseteq X$ such that $\phi(A) = C$.

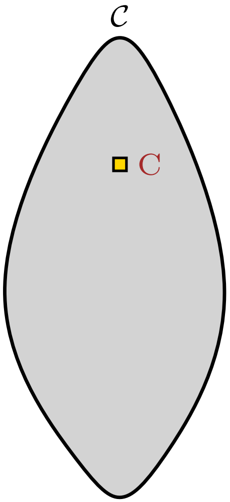
In convex geometries:

- Every closed set has a unique minimal spanning set
- The extreme points of a closed set is exactly its minimal spanning set



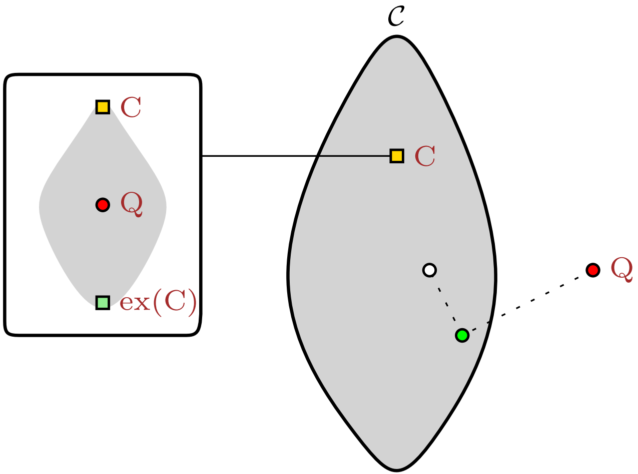
Right optimization: the not so easy part

Consider an essential C



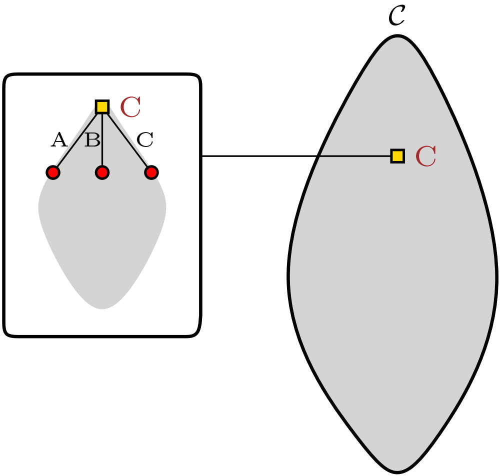
Right optimization: the not so easy part

Q is a **quasi-closed set** of C if $Q \subset C$ and $\mathcal{C} \cup Q$ is a closure system



Right optimization: the not so easy part

$\mathcal{Q}(C)$ is the quasi closed hypergraph with the element of C as vertex and $\{C \setminus Q : Q$ is an inclusion-wise maximal quasi-closed set of $C\}$ as edges



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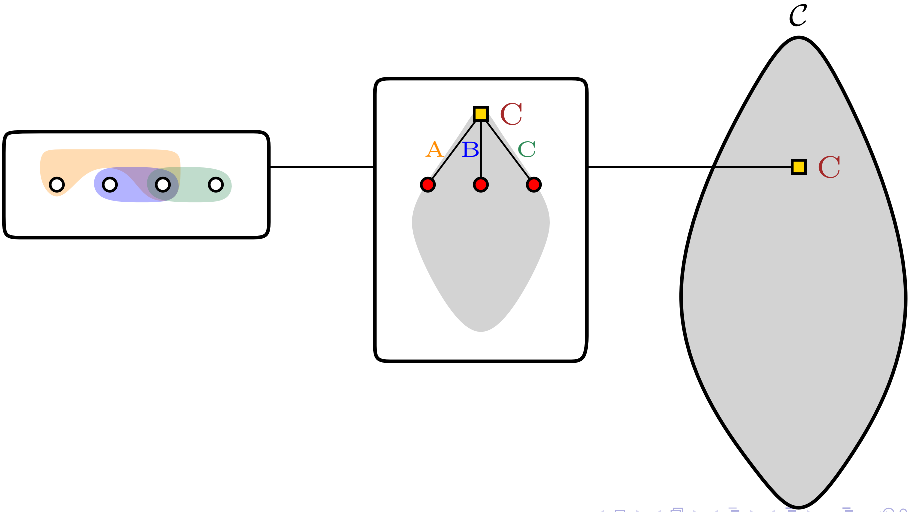
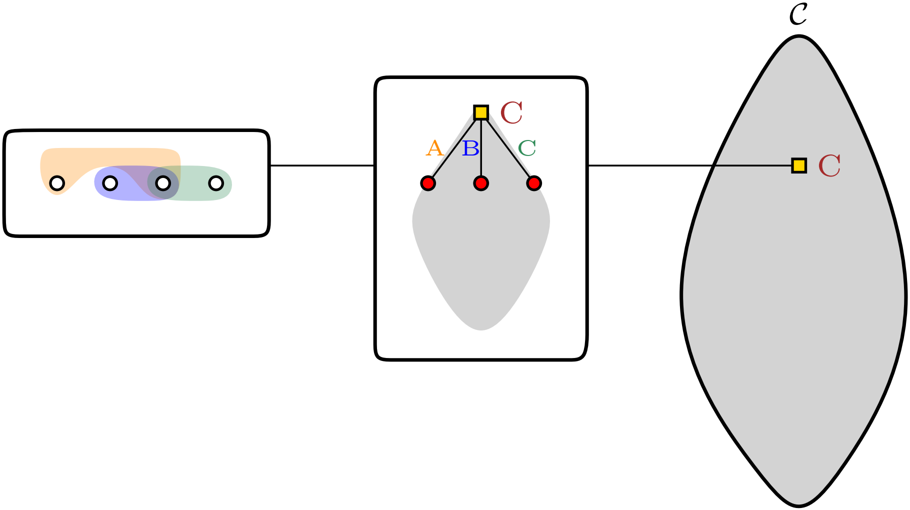


Right optimization: the not so easy part

The implication of C is $\mathrm{ex}(C) \rightarrow T$ with T a hitting set of $\mathcal{Q}(C)$.



Right optimization: the not so easy part

Theorem

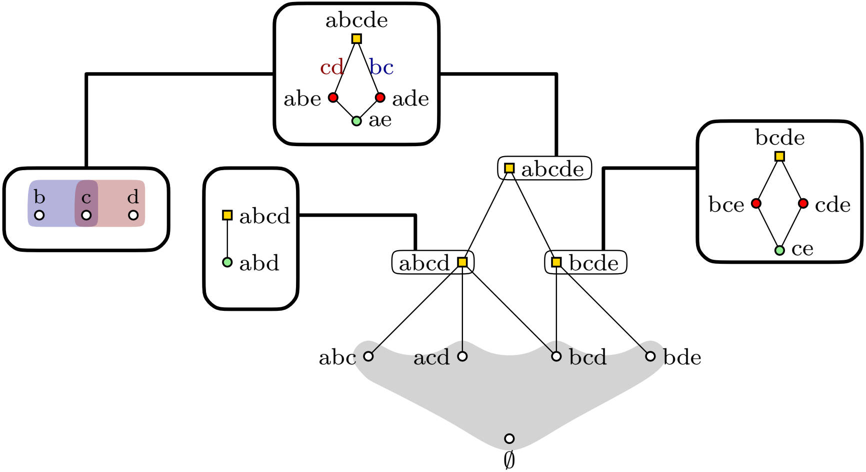
Let $(X, \mathcal{C})$ be a convex geometry and Σ a valid collection of implication. If for all essential set $C \in \mathcal{C}$, the associated implication in Σ is of the form $ex(C) \to T$ then Σ is left-optimum

Right optimization: the not so easy part

Theorem

Let $(X, \mathcal{C})$ be a convex geometry and Σ a valid collection of implication. If for all essential set $C \in \mathcal{C}$, the associated implication in Σ is of the form $ex(C) \to T$ and T is minimum then Σ is optimum

Right optimization: the not so easy part

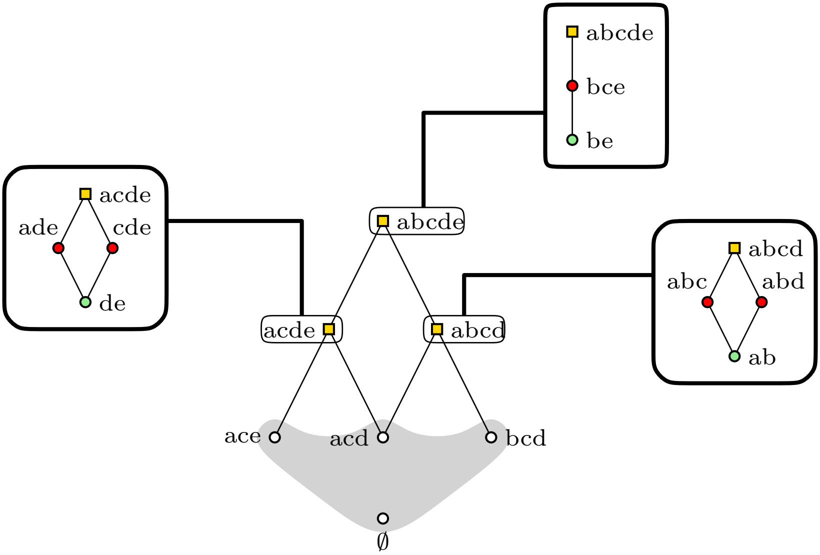


Right optimization: the not so easy part

Theorem

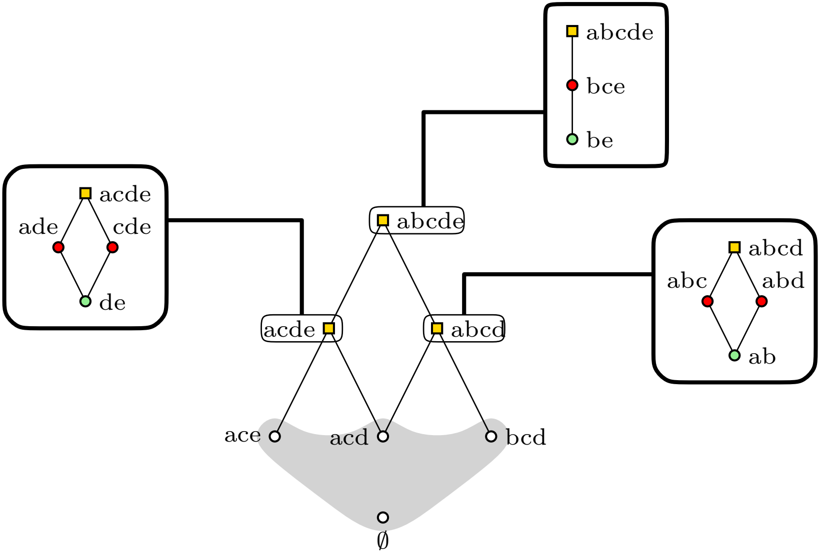
An implicational base of a convex geometry where each quasi-closed hypergraph has pairwise disjoint edges can be optimized in polynomial time.

Right optimization: the not so easy part

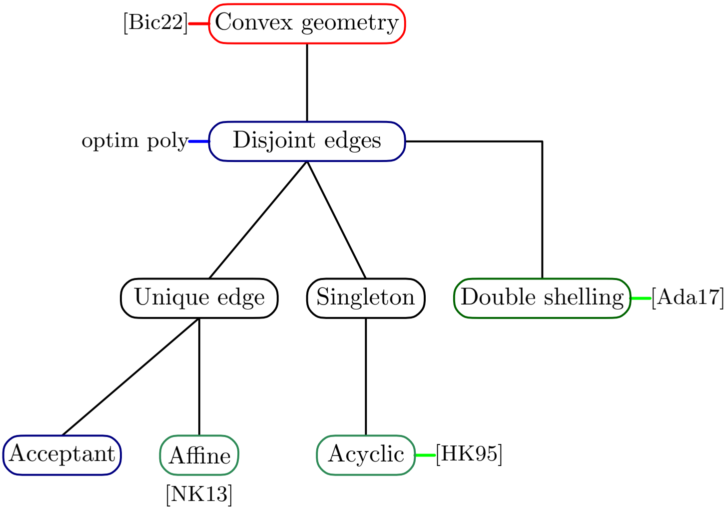


Right optimization: the not so easy part

$$\Sigma = \{ab \to cd, de \to ac, be \to d\}$$

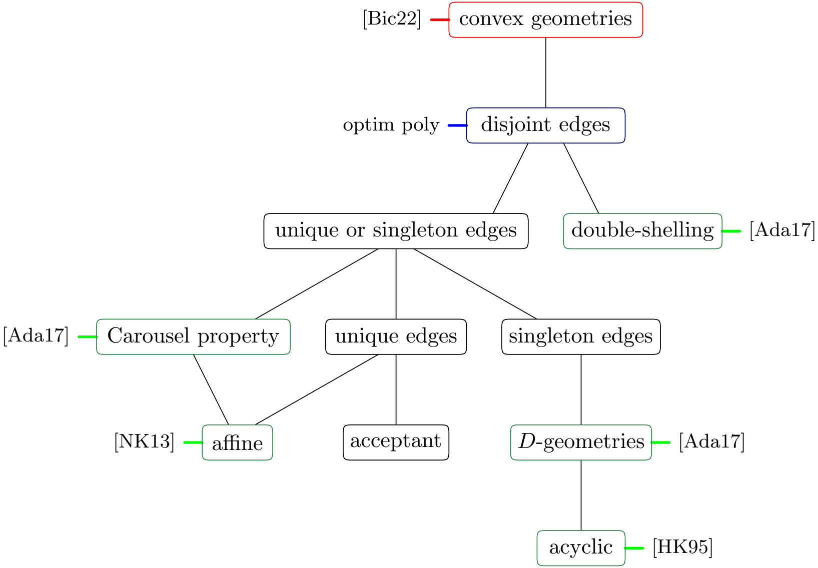


Conclusion



[Bic22]
Convex geometry
optim poly
Disjoint edges
Unique edge
Singleton
Double shelling
[Ada17]
Acceptant
Affine
[NK13]
Acyclic
[HK95]

Conclusion



[Bic22]
convex geometries
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unique or singleton edges
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[Ada17]
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Carousel property
unique edges
singleton edges
[NK13]
affine
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D-geometries
[Ada17]
acyclic
[HK95]

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Thank you for your attention

(Long version on Arxiv !)