Recent results and techniques on the Erdős-Pósa property

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LIRMM (University of Montpellier) and MIMUW (University of Warsaw)

January 2017, LaBRI, Bordeaux.

The Erdős–Pósa Theorem

Theorem (Erdős and Pósa, 1965)

max number of pairwise vertex-disjoint cycles in $G = k$

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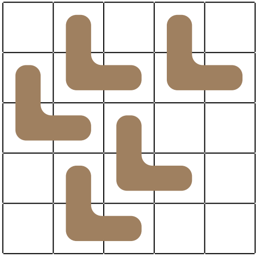

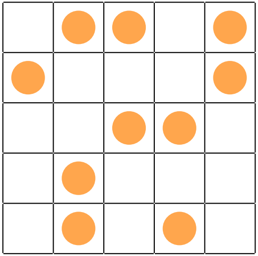

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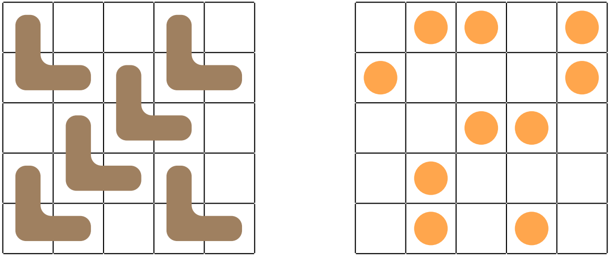

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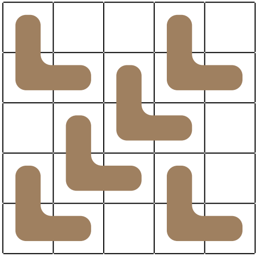

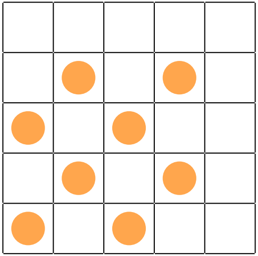

The Erdős–Pósa property

Definition

$\mathcal{H}$ has the EP property if there is a function f s.t.

max number of pairwise vertex-disjoint subgraphs of G that are in $\mathcal{H} = k$

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min number of vertices meeting every such subgraph $\leq f(k)$.

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Two directions of research:

1. prove that your favorite class has the EP property;
2. find the smallest possible f.

Outline of the talk

The Erdős–Pósa Theorem from girth

G: graph with max number of pairwise vertex-disjoint cycles k
Goal: find a small set of $O(k \log k)$ vertices meeting all cycles.

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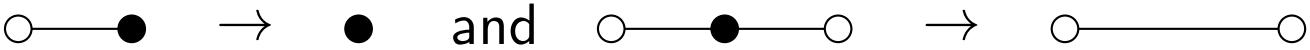
Reductions: 

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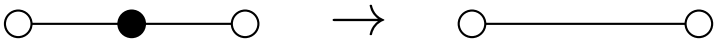
Reductions:  and 

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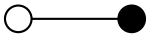
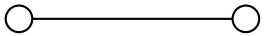
Reductions:  $\rightarrow$  and  $\rightarrow$ 

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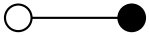
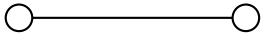
Reductions:  $\to$  and  $\to$ 

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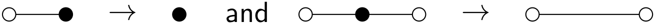
Reductions: 

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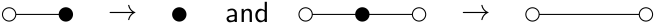

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For every forest F, there is a constant c s.t.:
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Theorem (Chatzidimitriou, R., Sau, Thilikos, 2015)

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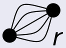

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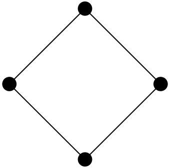
H is a *minor* of *G* if it can be obtained by deleting vertices or edges and contracting edges.



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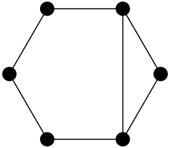

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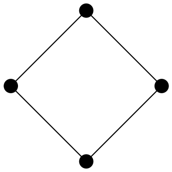

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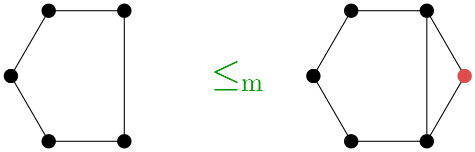

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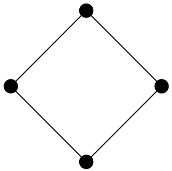

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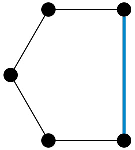

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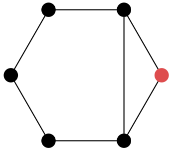

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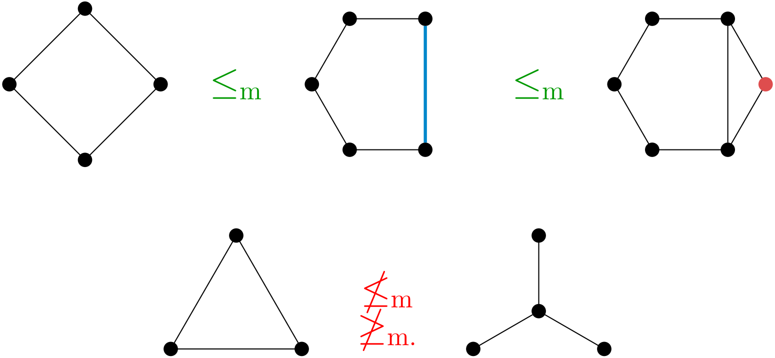

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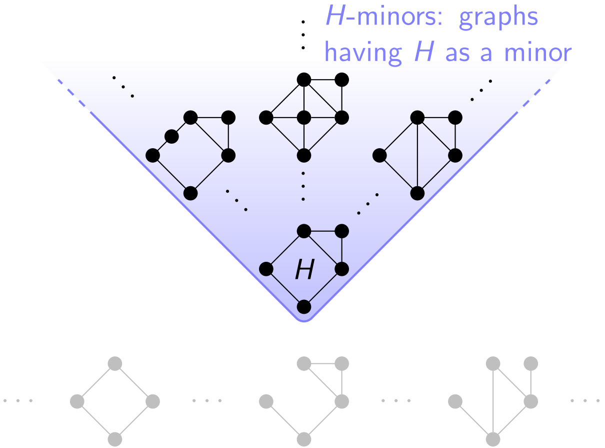

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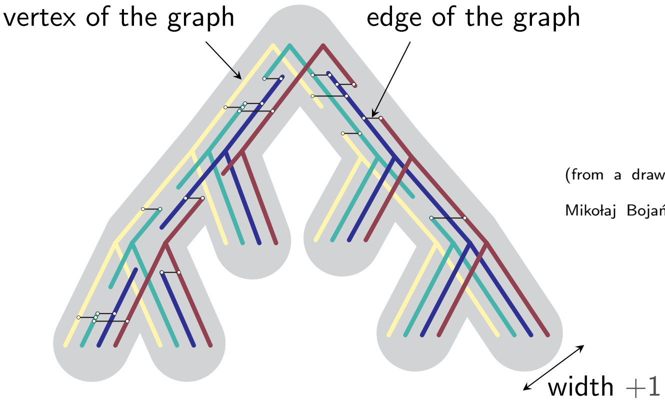

(from a drawing by Mikołaj Bojańczyk)

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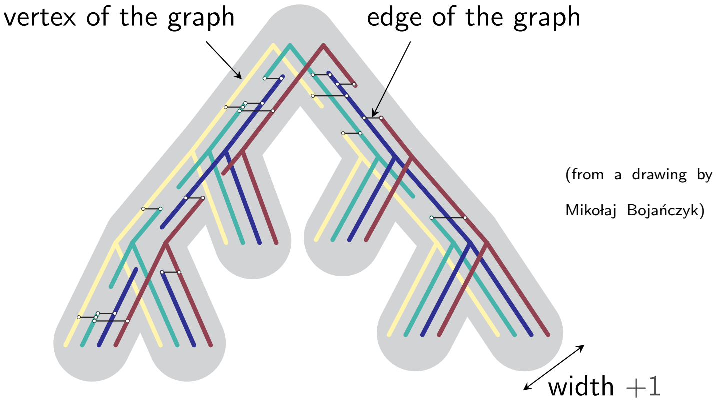

- treewidth: min with of such a decomposition;
- $\forall H$ planaire, $\mathbf{tw}(G) > f(|H|) \Rightarrow H$ is a minor of G (GM.V).

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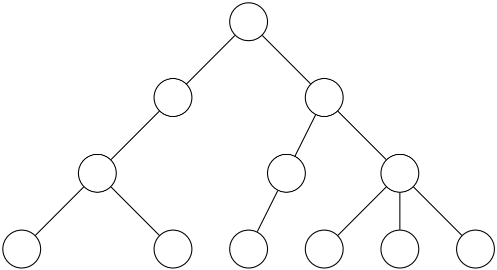

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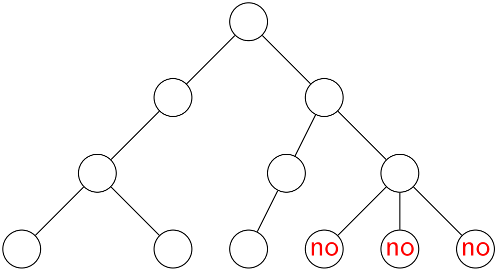

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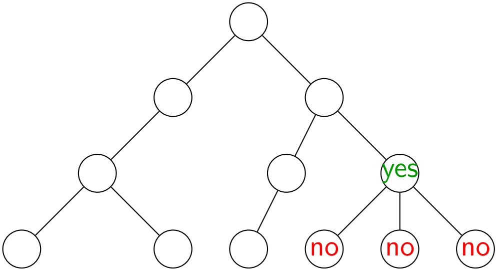

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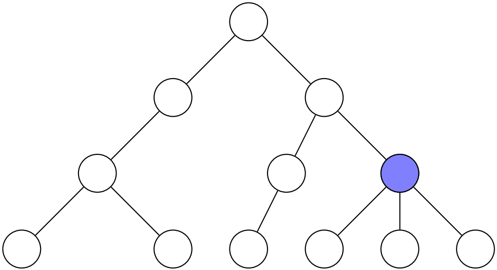

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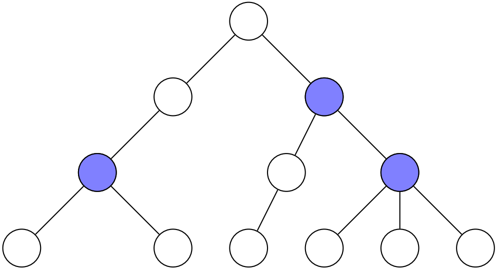

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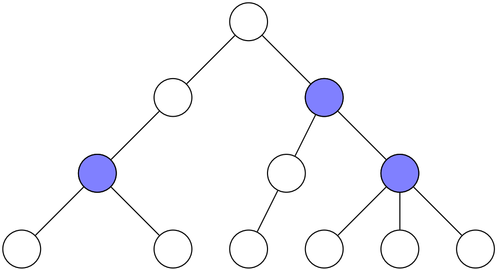

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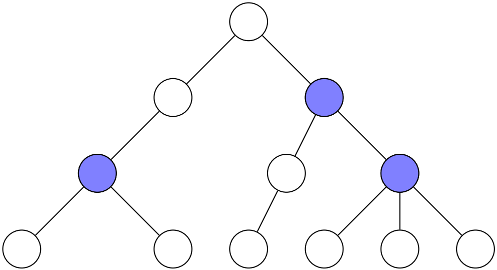

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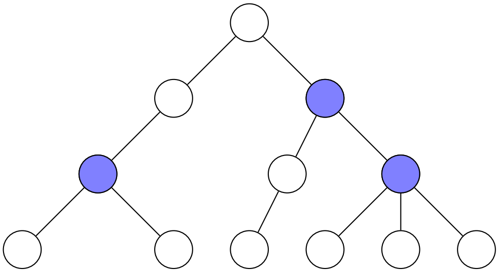

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Outline of the talk

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Theorem (Erdős and Pósa, 1965)

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$\rightarrow$ polynomial kernel parameterized by k and Δ.

Other applications

Theorem (Chatzidimitriou, R., Sau, Thilikos, 2015)

There is a $O(\log \mathrm{OPT})$ approximation of

- *the max number of pairwise vertex-disjoint θ_r-minors;*
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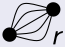

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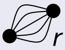

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Also: applications of EP in bioinformatics.

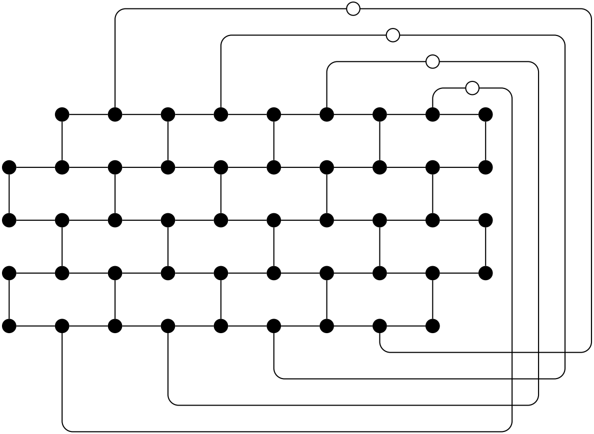
Outline of the talk

1. Introduction
2. Proof techniques
3. Algorithmic applications
4. Variants
5. Open problems

Restricting the class of host graphs

Theorem (Dejter and Neumann-Lara, 1987)

Odd cycles do not have the Erdős–Pósa property.



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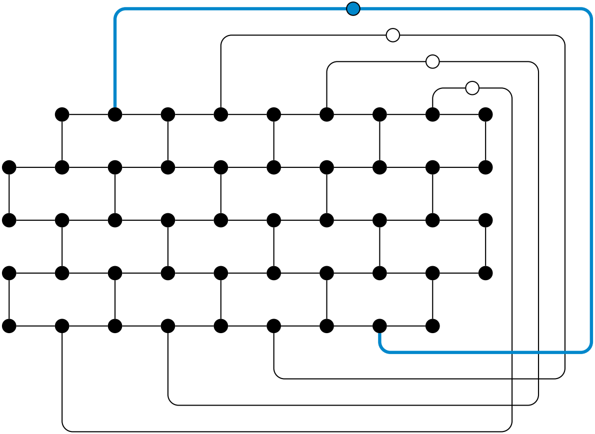

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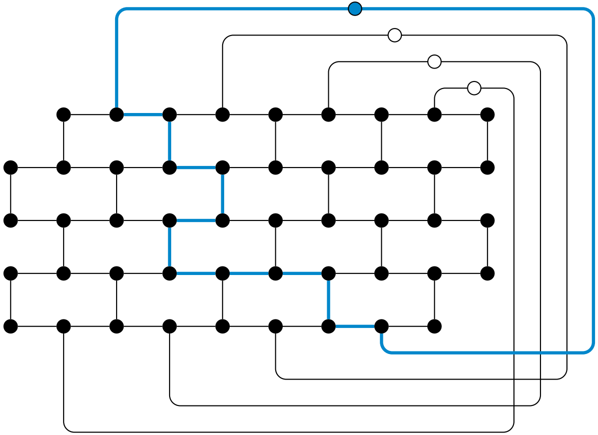

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Theorem (Ma, Yu, and Wang, JoCO 2013)

For every planar G*:*
max number of pairwise vertex-disjoint cycles $G = k$
$\Rightarrow$ *min number of vertices meeting every cycle* $\leq 3k$ *(in general:* $\Omega(k \log k)$*).*

(Jone's conjecture: $2k$)

The edge-version

Theorem (Erdős and Pósa, 1962)

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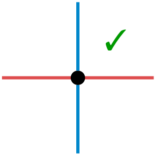
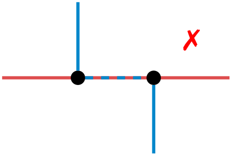
$\Downarrow$

min number of **edges** *meeting every cycle* $\leq c \cdot k \log k$.





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Theorem (Haxell 1999)

max number of pairwise **edge**-*disjoint triangles in* $G = k$
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Theorem (Giannopoulou, Kwon, R., Thilikos, WG 2016)

For every planar subcubic H, there is a poly function f s.t.:
max number of pairwise edge-disjoint H-immersions in $G = k$
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Already seen: odd cycles in planar graphs.

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Theorem (Thomassen, JGT 1988)

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Theorem (Fiorini and Herinckx, JGT 2014)

For every $t \in \mathbb{N}^*$,
max number of pairwise vertex-disjoint cycles of length $\geq t$ in $G = k$
$\Rightarrow$ min nb. of vert. meeting every such cycle $\leq (6t + 4 + o_t(1))k \log k$.

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Other results (mostly for cycles) combining all variants mentioned so far.

What about directed graphs?

Theorem (Reed, Robertson, Seymour, and Thomas, Combinatorica 1996)

There is a function f s.t, for every directed graph G,
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Theorem (Amiri, Kawarabayashi, Kreutzer, and Wollan, 2016)

For every strongly-connected H that is $\overrightarrow{\text{minor}}$ of the cylindrical grid, there is a function f s.t, for every directed graph G,
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Recap

1. EP property: min-max relation between invariants of packing and covering;
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Many variants: restricted host class, edge-version, length constraints, prescribed vertices, digraphs, etc..

Outline of the talk

Some conjectures

Conjecture (Tuza's conjecture, 1984)

For every G:
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Conjecture (?)

For every planar H, there is a constant c s.t. for every G,
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(best known: $k \operatorname{polylog} k$, lower bound: $> \left(\frac{1}{2} + o(1)\right) k \log k$).

Some open problems

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Do induced cycles of length ≥ 4 have the EP property?

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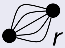

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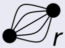

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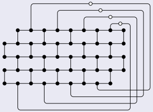

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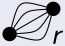

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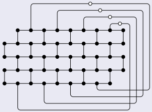

Thank you!