Obstructions to bounded cutwidth

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Joint work with:
- Archontia Giannopoulou (TU Berlin);
- Michał Pilipczuk (University of Warsaw);
- Dimitrios M. Thilikos (LIRMM–CNRS); and
- Marcin Wrochna (University of Warsaw).
Cutwidth

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- the **width** of a layout $\sigma$:

$$\max_{i=1,2,\ldots,n-1} \#(\text{edges with one endpoint} \leq i \text{ and second} > i).$$
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- The *cutwidth* of $G$:
  $$\text{ctw}(G) = \min \{\text{width}(\sigma) : \sigma \text{ is a layout of } G\}.$$
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Class considered of this talk: graphs of cutwidth $\leq k$. 

Immersions

$\leq_{\text{imm}}$

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\begin{align*}
\{ & \text{vertices} \mapsto \text{distinct vertices} \\
& \text{edges} \mapsto \text{edge-disjoint paths} \}
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“Monotonicity” of cutwidth:

\[H \leq_{\text{imm}} G \Rightarrow \text{ctw}(H) \leq \text{ctw}(G).\]
**Immersion**

\[ \leq_{\text{imm}} \]

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- edges $\mapsto$ edge-disjoint paths

"Monotonicity" of cutwidth:

\[ H \leq_{\text{imm}} G \Rightarrow \text{ctw}(H) \leq \text{ctw}(G). \]

\{G, ctw(G) \leq k\} is immersion-closed.
$K_5$ and $K_{3,3}$ are the *minor-obstructions* of planar graphs.
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\textit{Obstruction} for a class = \textit{minimal} element of its complementary (depends on the class and on the order)

The set of obstructions may be infinite!
Obstructions for bounded cutwidth

What about obstructions of \( \{ G, \text{ctw}(G) \leq k \} \)?

**Theorem (Robertson and Seymour, consequence of GM.XXII)**

For every \( k \in \mathbb{N} \), \( \{ G, \text{ctw}(G) \leq k \} \), has finitely many obstructions.
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\[ \text{ctw} > k \]

\[ \text{ctw} \leq k \]

How many?
Bounding the size of the obstructions

$s_k := \text{max size of an obstruction for cutwidth } \leq k$

(immersion-min. graph with \text{ctw} > k)

Results of Lagergren (1998):

$G_{\text{minor-obstruction}}$ for pathwidth \( \leq k \Rightarrow |G| = 2 \cdot O(k^4) \);

$G_{\text{minor-obstruction}}$ for treewidth \( \leq k \Rightarrow |G| = 2^2 \cdot O(k^5) \).
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Govindan and Ramachandramurthi, 2001

\[
\frac{1}{2} (3^{k-5} - 1) \leq s_k
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\[ \frac{1}{2} (3^{k-5} - 1) \leq s_k = 2^{O(k^3 \log k)} \]

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- $G$ minor-obstruction for pathwidth $\leq k$ $\Rightarrow |G| = 2^{O(k^4)}$;

- $G$ minor-obstruction for treewidth $\leq k$ $\Rightarrow |G| = 2^{2^{O(k^5)}}$. 
How to show that obstructions are small?

General idea

If an obstruction is too large, some part of it is redundant.
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If an obstruction is too large, some part of it is redundant.

We define an equivalence relation on bounded subgraphs:

\[
\begin{array}{ccc}
\sim & \iff & \forall G, \text{ctw}(G) = \text{ctw}(\overline{G})
\end{array}
\]
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If an obstruction is too large, some part of it is redundant.

We define an equivalence relation on boundaried subgraphs:

\[ A \sim B \iff \forall C, \text{ctw}(C) = \text{ctw}(C') \]

Let \( G \) be an obstruction of \( \{ G, \text{ctw}(G) \leq k \} \):
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If an obstruction is too large, some part of it is redundant.

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\[ \sim \iff \forall G, \text{ctw}(G) = \text{ctw}(\text{equivalent graph}) \]

Let \( G \) be an obstruction of \( \{G, \text{ctw}(G) \leq k\} \):

- replace a subgraph with an equivalent one that is smaller;
  
  (this does not change the cutwidth)
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If an obstruction is too large, some part of it is redundant.

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\[ \sim \iff \forall \begin{array}{c} \vline \\ \vline \end{array}, \text{ctw} \begin{array}{c} \begin{array}{c} \vline \\ \vline \end{array} \end{array} = \text{ctw} \begin{array}{c} \begin{array}{c} \vline \\ \vline \end{array} \end{array} \]

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- replace a subgraph with an equivalent one that is smaller;
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- we prove that the obtained graph is an immersion of \( G \);
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General idea

If an obstruction is too large, some part of it is redundant.

We define an equivalence relation on boundaried subgraphs:

\[ \sim \iff \forall G, \ ctw(G) = ctw \left( \begin{array}{c} G \\ \end{array} \right) \]

Let \( G \) be an obstruction of \( \{G, \ ctw(G) \leq k\} \):

- replace a subgraph with an equivalent one that is smaller;
  (this does not change the cutwidth)
- we prove that the obtained graph is an immersion of \( G \);
- contradicts the minimality of \( G \)!
Bounding the number of equivalence classes

Key Lemma
If $ctw(G) \leq k$ and $|E(A, B)| \leq \ell$, then there exists an optimum-width layout of $G$ with $O(k\ell)$ blocks.

Relevant information about $B$:
$O(k\ell)$ numbers up to $k$.

→ finite number of equivalence classes for fixed $k$, $\ell$.
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An ordering \( v_1 \ldots, v_n \) of is \textit{linked} if, for every \( i < j \) and every \( t \),

\[\text{either there are } t \text{ edge-disj. paths from } v_1, \ldots, v_i \text{ to } v_j, \ldots, v_n; \]

or

\[\exists k, i \leq k < j \text{ s.t. } |E(\{v_1, \ldots, v_k\}, \{v_k+1, \ldots, v_n\})| < t.\]
An ordering $v_1, \ldots, v_n$ is said to be **linked** if, for every $i < j$ and every $t$,

- either there are $t$ edge-disjoint paths from $v_1, \ldots, v_i$ to $v_j, \ldots, v_n$;
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\[ v_i \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad v_{k+1} \quad \cdots \quad v_n \]

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\[ t \]

\textbf{Lemma}

Every graph has a \textit{linked} ordering of optimal width.
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**Lemma**

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Proof: non-linked orderings can be *improved* without increasing the width.
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\[ v_i \quad v_k \quad v_j \]

\[ t \]

Lemma

\textit{Every graph has a linked ordering of optimal width.}

Proof: non-linked orderings can be \textit{improved} without increasing the width. Similar notions: linked path decompositions, linked tree decompositions.
Lemma

If $w$ is a word of length $N$ over $[r]$, there is a $p \in [r]$ s.t.:

- **some subword $u$ contains numbers $\geq p$**;
- **$u$ contains $p$ at least $N$ times**.
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Lemma

If $w$ is a word of length $N^r$ over $[r]$, there is a $p \in [r]$ s.t.:

- some subword $u$ contains numbers $\geq p$;
- $u$ contains $p$ at least $N$ times.

If $|G| > N^r$, some contiguous subsequence of $v_1, \ldots, v_n$ has cuts $\geq p$ and $\geq N$ cuts of size $p$. 
Bounding the size of obstructions

Goal: show that obstructions for $\text{ctw} \leq k$ are small.

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- consider a linked optimal ordering of $G$:

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- assign a type to every prefix ("equivalence class for $\text{ctw}$")
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![Graph diagram]

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![Diagram of linked ordering]

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![Diagram showing a linked optimal ordering]

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  \begin{align*}
    &0 & 1 & 2 & 3 & 4 & 5 & 6 \\
    &\text{blue} & \text{orange} & \text{brown} & \text{red} & \text{green} \\
  \end{align*}

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- shrink using edge-disjoint paths:

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  - recall: there are finitely many different types
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  - this immersion of $G$ has cutwidth $k + 1$: contradiction.
Computing cutwidth

**Problem:** deciding given \((G, k)\) if \(\text{ctw}(G) \leq k\).

- non-uniform, non-constructive FPT (by the finiteness of obstructions);
Computing cutwidth

**Problem:** deciding given \((G, k)\) if \(\text{ctw}(G) \leq k\).
- non-uniform, non-constructive FPT (by the finiteness of obstructions);
- constructive FPT algorithm with running time \(2^{O(k^2)} \cdot n\).
  (Thilikos, Bodlaender, and Serna)
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**Theorem (Giannopoulou, Pilipczuk, R., Thilikos, Wrochna, 2016)**

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**Ingredients:**
- equivalence classes of subgraphs w.r.t. cutwidth;
- DP on graphs of bounded cutwidth;
- “edge-removal” lemma.
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Slightly slower… but much easier!
Extension to tree-like parameters

Tree-cut decomposition:

Associated parameter: $\text{tcw}$
(tree-cut width)

Small $\text{tcw}$ implies:

- *small* bags;
- *thin* edges;
- small number of *thick* neighbors.
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Small \textbf{tcw} implies:
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\text{treewidth and minors} \sim \text{tree-cut width and immersions}
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treewidth and minors \sim tree-cut width and immersions

Hope for similar results in this context (work in progress).
Our contribution:

- a single exponential upper-bound on the size of the obstructions for $\text{ctw} \leq k$;
- a simpler FPT algorithm for cutwidth.
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Thank you!