Polynomial expansion and sublinear separators

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Join work with Louis Esperet (G-SCOP, Grenoble).
Theorem (Plotkin, Rao, and Smith, SODA 1994)

*polynomial expansion* $\Rightarrow$ *sublinear separators*

Theorem (Dvořák and Norin, SIDMA 2016)

*sublinear separators* $\Rightarrow$ *polynomial expansion*.
Minors:

(Bounded-depth) minors

\[ \text{Minors:} \]

\[ \text{radius} (G) \leq r: \quad G \text{ has a vertex at distance } \leq r \text{ of the others.} \]

Ex: a 0-minor is a subgraph.
Minors:

- disjoint;
(Bounded-depth) minors

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r-minors, r-shallow minors, minors of depth $\leq r$:

- disjoint;
- connected;
- adj. if the corresponding vertices are so;
- radius $\leq r$.

$\text{radius}(G) \leq r$: $G$ has a vertex at distance $\leq r$ of the others.
(Bounded-depth) minors

$r$-minors, $r$-shallow minors, minors of depth $\leq r$:

- disjoint;
- connected;
- adj. if the corresponding vertices are so;
- radius $\leq r$.

$\text{radius}(G) \leq r$: $G$ has a vertex at distance $\leq r$ of the others.

Ex: a 0-minor is a subgraph.
- **r-minor of** $G$: obtained by contracting disjoint subgraphs of radius $\leq r$ in a subgraph of $G$;
- **r-minor of** $G$: obtained by contracting disjoint subgraphs of radius $\leq r$ in a subgraph of $G$;
- $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|H|}, \ H \subseteq G \right\}$
- $r$-minor of $G$: obtained by contracting disjoint subgraphs of radius $\leq r$ in a subgraph of $G$;
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- $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|H|}, \ H \text{ is a } 0\text{-minor of } G \right\}$
- $\nabla_r(G) = \max \left\{ \frac{|E(H)|}{|H|}, \ H \text{ is a } r\text{-minor of } G \right\}$
  (density of $r$-minors of $G$ as a function of $r$)
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  (density of $r$-minors of $G$ as a function of $r$)

$\nabla_r(G)$

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \cdots & |G| \\
\end{array}
\]

$\nabla_r(G)$

- $C$ has **bounded expansion** if $\exists f$ s.t.
  $\forall r \in \mathbb{N}, \ \forall G \in C, \nabla_r(G) \leq f(r)$
- **$r$-minor of $G$**: obtained by contracting disjoint subgraphs of radius $\leq r$ in a subgraph of $G$;

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  (density of $r$-minors of $G$ as a function of $r$)

- $\mathcal{C}$ has **bounded expansion** if $\exists f$ s.t.

  $$\forall r \in \mathbb{N}, \forall G \in \mathcal{C}, \nabla_r(G) \leq f(r)$$

- $\mathcal{C}$ has **polynomial expansion** if $f$ is polynomial.
Separators

- separator \((A, B)\): 

\[ A \cup B \]

\[ A \cap B \]

\[ A \setminus B \]

\[ B \setminus A \]

\[ \leq 2^{3|G|} \]

- sublinear separator: separator of size \(O(n^{\beta})\) with \(0 \leq \beta < 1\).

Ex: planar graphs have balanced separators of size \(O(n^{1/2})\).
Separators

- separator \((A, B)\): 
  \[ A \cap B \]

- order of \((A, B)\): \(|A \cap B|\)

Ex: planar graphs have balanced separators of size \(O(n^{1/2})\).

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Polynomial expansion and sublinear separators
Separators

- **separator** $(A, B)$: $A \cap B$
- **order of** $(A, B)$: $|A \cap B|$
- **balanced separator** $(A, B)$: $|A \setminus B|, |B \setminus A| \leq \frac{2}{3} |G|$

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Polynomial expansion and sublinear separators
Separators

- separator \((A, B)\):

\[ A \quad B \]

- order of \((A, B)\): \(|A \cap B|\)

- balanced separator \((A, B)\):

\[ A \\ B \]

\[ |A \setminus B|, |B \setminus A| \leq \frac{2}{3} |G| \]

- sublinear separator: separator of size \(O(n^\beta)\) with \(0 \leq \beta < 1\).
Separators

- separator \((A, B)\):

- order of \((A, B)\): \(|A \cap B|

- balanced separator \((A, B)\):

\[
|A \setminus B|, |B \setminus A| \leq \frac{2}{3} |G|
\]

- sublinear separator: separator of size \(O(n^\beta)\) with \(0 \leq \beta < 1\).

Ex: planar graphs have balanced separators of size \(O(n^{1/2})\).
Why do we care about (small) separators?

- they give structural information;
- they are connected to several parameters (e.g. treewidth);
- they have algorithmic applications (divide and conquer).

Lemma (Robertson and Seymour, 1986)

Any graph $G$ has a balanced separator of order at most $tw(G) + 1$.

Theorem (Dvořák and Norin, 2014)

Let $\beta \in [0, 1)$. For all $H \subseteq G$, $H$ has a balanced separator of order $\leq c |H| \beta$.

\[
\forall H \subseteq G, tw(H) \leq 105 c |H| \beta.
\]
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- they give structural information;
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**Lemma (Robertson and Seymour, 1986)**

*Any graph $G$ has a balanced separator of order at most $\textrm{tw}(G) + 1$.*
Why do we care about (small) separators?

- they give structural information;
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  \[ \forall H \subseteq G, \ H \ has \ a \ balanced \ separator \ of \ order \ \leq c|H|^\beta \]
  \[ \Downarrow \]
  \[ \forall H \subseteq G, \ \text{tw}(H) \leq 105c|H|^\beta. \]
Let \( C \) be a subgraph-closed class and \( \delta \in (0, 1] \).
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**Theorem (Plotkin, Rao, and Smith, SODA 1994)**

$C$ has expansion $O\left(r^{1/\delta}\right)$

$\Downarrow$

graphs of $C$ have balanced separators of size $O\left(n^{1-\delta}\right)$.
Let $\mathcal{C}$ be a subgraph-closed class and $\delta \in (0, 1]$.

**Theorem (Plotkin, Rao, and Smith, SODA 1994)**

$\mathcal{C}$ has expansion $O\left(r^{1/\delta}\right)$

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**Theorem (Dvořák and Norin, SIDMA 2016)**

graphs of $\mathcal{C}$ have balanced separators of size $O\left(n^{1-\delta}\right)$

$\Downarrow$

$\mathcal{C}$ has expansion $O\left(r^{5/\delta^2}\right)$. 
Let \( C \) be a subgraph-closed class and \( \delta \in (0, 1] \).

**Theorem (Plotkin, Rao, and Smith, SODA 1994)**

\[ C \text{ has expansion } O\left(\frac{r^{1/\delta}}{\delta}\right) \]

\[ \Downarrow \]

\[ \text{graphs of } C \text{ have balanced separators of size } O\left(n^{1-\delta}\right). \]

**Theorem (Dvořák and Norin, SIDMA 2016)**

\[ \text{graphs of } C \text{ have balanced separators of size } O\left(n^{1-\delta}\right) \]

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sublinear balanced separators \( \iff \) polynomial expansion
Let $\mathcal{C}$ be a subgraph-closed class and $\delta \in (0, 1]$.

**Theorem (Plotkin, Rao, and Smith, SODA 1994)**

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**Conjecture (Dvořák and Norin, SIDMA 2016)**

graphs of $\mathcal{C}$ have balanced separators of size $O\left(n^{1-\delta}\right)$

$\Downarrow$

$\mathcal{C}$ has expansion $O\left(r^{c/\delta}\right)$. ?

sublinear balanced separators $\iff$ polynomial expansion
Let $C$ be a subgraph-closed class and $\delta \in (0, 1]$.

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- $C$ has expansion $O\left(r^{1/\delta}\right)$
- $\implies$
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**Theorem (Esperet and R., 2017)**

- Graphs of $C$ have balanced separators of size $O\left(n^{1-\delta}\right)$
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Sublinear balanced separators $\iff$ polynomial expansion
A few words about expanders

Expander: small subsets of $V$ have linearly many neighbors at least.

$|N(S)| \geq \alpha |S|$ whenever $|S| \leq n/2$
A few words about expanders

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A few words about expanders

Expander: small subsets of $V$ have linearly many neighbors at least.

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Expander: \(|N(S)| \geq \alpha |S|\) whenever \(|S| \leq n/2\).

**Lemma**

\(G\) is an \(\alpha\)-expander \(\Rightarrow\) \(\text{tw}(G) \geq \frac{\alpha}{3(\alpha+1)} \cdot |G| - 1\).
Expander: $|N(S)| \geq \alpha |S|$ whenever $|S| \leq n/2$.

**Lemma**

If $G$ is an $\alpha$-expander, then $\text{tw}(G) \geq \frac{\alpha}{3(\alpha+1)} \cdot |G| - 1$.

$(A, B)$ balanced separator of order $\text{tw} + 1$
Expanders have linear treewidth

Expander: \(|N(S)| \geq \alpha |S|\) whenever \(|S| \leq n/2\).

Lemma

\(G\) is an \(\alpha\)-expander \(\Rightarrow tw(G) \geq \frac{\alpha}{3(\alpha + 1)} \cdot |G| - 1\).

\((A, B)\) balanced separator of order \(tw + 1\)

\begin{align*}
|A \setminus B| & \leq 2n/3 \\
|B \setminus A| & \leq 2n/3
\end{align*}
Expanders have linear treewidth

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$(A, B)$ balanced separator of order $\text{tw} + 1$

\[
\left\{
\begin{array}{l}
|A \setminus B| \leq 2n/3 \\
|A \cap B| \leq \text{tw} + 1 \\
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\end{array}
\right.
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\begin{align*}
|A \setminus B| &\leq n/2 \\
|A \cap B| &\leq \text{tw} + 1 \\
|A \cap B| &\geq \alpha |A \setminus B| \\
|B \setminus A| &\leq 2n/3 \\
|A \setminus B| + |A \cap B| &\geq n/3 \\
\frac{|A \cap B|}{\alpha} + |A \cap B| &\geq n/3
\end{align*}
\]
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Lemma

\(G\) is an \(\alpha\)-expander \(\Rightarrow \text{tw}(G) \geq \frac{\alpha}{3(\alpha+1)} \cdot |G| - 1\).

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\end{align*}
\]

\[
\begin{align*}
|A \setminus B| + |A \cap B| &\geq n/3 \\
\frac{|A \cap B|}{\alpha} + |A \cap B| &\geq n/3 \\
|A \cap B| &\geq \frac{\alpha}{\alpha+1} \cdot \frac{n}{3}
\end{align*}
\]
Expanders have linear treewidth

Expander: \(|N(S)| \geq \alpha|S|\) whenever \(|S| \leq n/2\).

**Lemma**

\(G\) is an \(\alpha\)-expander \(\Rightarrow \text{tw}(G) \geq \frac{\alpha}{3(\alpha+1)} \cdot |G| - 1\).

\((A, B)\) balanced separator of order \(\text{tw} + 1\)

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\end{align*}
\]

\[
\begin{align*}
\frac{|A \cap B|}{\alpha} + |A \cap B| &\geq n/3 \\
|A \cap B| &\geq \frac{\alpha}{\alpha+1} \cdot n/3 \\
\text{tw} + 1 &\geq \frac{\alpha}{\alpha+1} \cdot n/3
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Expanders have linear treewidth

Expander: $|N(S)| \geq \alpha |S|$ whenever $|S| \leq n/2$.

**Lemma**

$G$ is an $\alpha$-expander $\Rightarrow tw(G) \geq \frac{\alpha}{3(\alpha+1)} \cdot |G| - 1$.

$(A, B)$ balanced separator of order $tw + 1$

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\begin{align*}
|A \setminus B| &\leq n/2 \\
|A \cap B| &\leq tw + 1 \\
|A \cap B| &\geq \alpha |A \setminus B| \\
|B \setminus A| &\leq 2n/3 \\
|A \setminus B| + |A \cap B| &\geq n/3 \\
\frac{|A \cap B|}{\alpha} + |A \cap B| &\geq n/3 \\
|A \cap B| &\geq \frac{\alpha}{\alpha+1} \cdot \frac{n}{3} \\
tw + 1 &\geq \frac{\alpha}{\alpha+1} \cdot \frac{n}{3}
\end{align*}
\]

Expanders have linear treewidth!
Theorem (Shapira and Sudakov, Combinatorica 2015)

Any graph $G$ contains a subgraph $H$ s.t.:

1. $d(H) \geq 0.9 \cdot d(G)$;
2. $H$ is an $\frac{1}{\text{polylog}|H|}$-expander.
Finding subgraphs of linear treewidth

Theorem (Shapira and Sudakov, Combinatorica 2015)

Any graph $G$ contains a subgraph $H$ s.t.:

1. $d(H) \geq 0.9 \cdot d(G)$;
2. $H$ is an $\frac{1}{\text{polylog}|H|}$-expander.

Corollary

Any graph $G$ contains a subgraph $H$ with

1. $d(H) \geq 0.9 \cdot d(G)$;
2. $\text{tw}(H) = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right)$. 

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Sketch of the proof

Current goal

\( C \) has balanced sep. of order \( O(n^{1-\delta}) \) \( \Rightarrow \) \( C \) has expansion \( O(r^{c/\delta}) \)
Sketch of the proof

Current goal

\( C \) has balanced sep. of order \( O(n^{1-\delta}) \) \( \Rightarrow \) \( C \) has expansion \( O(r^c/\delta) \)

\( G \in C \)
Sketch of the proof

Current goal

\[ C \text{ has balanced sep. of order } O(n^{1-\delta}) \Rightarrow C \text{ has expansion } O\left(\frac{r^c}{\delta}\right) \]

\[ G \in C \]

\[ F \text{ } r\text{-minor of } G \]
Sketch of the proof

Current goal

\[ d(F) = O\left(\frac{r^c}{\delta}\right) \]

\[ G \in C \]

\[ F \text{ } r\text{-minor of } G \]
Sketch of the proof

Current goal

\[ d(F) = O \left( r^{c/\delta} \right) \]

\( G \in \mathcal{C} \)

\( F \) \( r \)-minor of \( G \)

Lemma (expander lemma)

\textit{Every }\( F \text{ has a sgr. } H \text{ with } d \geq 0.9d(F) \text{ and } tw = \Omega \left( \frac{|H|}{\text{polylog } |H|} \right). \)
Sketch of the proof

Current goal

\[ d(F) = O\left(r^{c/\delta}\right) \]

\[ G \in C \]

\( F \) \( r \)-minor of \( G \)

\( H \) sgr. of \( F \) with \( \text{tw}(H) = \Omega\left(\frac{|H|}{\text{polylog}|H|}\right) \) and \( d(H) \geq 0.9 \cdot d(F) \)

Lemma (expander lemma)

Every \( F \) has a sgr. \( H \) with \( d \geq 0.9d(F) \) and \( \text{tw} = \Omega\left(\frac{|H|}{\text{polylog}|H|}\right) \).
Sketch of the proof

Current goal

\[ d(H) = O \left( r^{c/\delta} \right) \]

\( G \in \mathcal{C} \)

\( F \ r\text{-minor of } G \)

\( H \text{ sgr. of } F \text{ with } \text{tw}(H) = \Omega \left( \frac{|H|}{\text{polylog } |H|} \right) \) and \( d(H) \geq 0.9 \cdot d(F) \)
Sketch of the proof

Current goal

\[ d(H) = O\left(\frac{r^c}{\delta}\right) \]

\[ G \in \mathcal{C} \]

\[ F \text{ is an } r\text{-minor of } G \]

\[ H \text{ is a separator of } F \text{ with } \text{tw}(H) = \Omega\left(\frac{|H|}{\text{polylog}|H|}\right) \text{ and } d(H) \geq 0.9 \cdot d(F) \]

Theorem (Chekuri and Chuzhoy, SODA 2015)

Every \( G \) with \( \text{tw} = k \) has a separator with \( \Delta \leq 3 \) and \( \text{tw} \geq \frac{k}{\text{polylog}k} \).
Sketch of the proof

Current goal

\[ d(H) = O \left( r^{c/\delta} \right) \]

\( G \in \mathcal{C} \)

\( F \) \( r \)-minor of \( G \)

\( H \) sgr. of \( F \) with \( \text{tw}(H) = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \) and \( d(H) \geq 0.9 \cdot d(F) \)

\( H' \) sgr. of \( H \) with \( \Delta \leq 3 \) and \( \text{tw}(H') = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \)

Theorem (Chekuri and Chuzhoy, SODA 2015)

Every \( G \) with \( \text{tw} = k \) has a sgr. with \( \Delta \leq 3 \) and \( \text{tw} \geq \frac{k}{\text{polylog} k} \).
Sketch of the proof

Current goal

\[ d(H) = O \left( r^{c/\delta} \right) \]

1. \( G \in \mathcal{C} \)
2. \( F \) is an \( r \)-minor of \( G \)
3. \( H \) is a separator of \( F \) with \( \text{tw}(H) = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \) and \( d(H) \geq 0.9 \cdot d(F) \)
4. \( H' \) is a separator of \( H \) with \( \Delta \leq 3 \) and \( \text{tw}(H') = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \)
5. \( G' \) is a separator of \( G \) corresponding to \( H' \): \( \text{tw}(G') = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \)
Sketch of the proof

Current goal

\[ d(H) = O\left(\frac{r^c}{\delta}\right) \]

\[ G \in \mathcal{C} \]

\[ F \text{ r-minor of } G \]

\[ H \text{ sgr. of } F \text{ with } \text{tw}(H) = \Omega\left(\frac{|H|}{\text{polylog}|H|}\right) \text{ and } d(H) \geq 0.9 \cdot d(F) \]

\[ H' \text{ sgr. of } H \text{ with } \Delta \leq 3 \text{ and } \text{tw}(H') = \Omega\left(\frac{|H|}{\text{polylog}|H|}\right) \]

\[ G' \text{ sgr. of } G \text{ corresponding to } H': \text{tw}(G') = \Omega\left(\frac{|H|}{\text{polylog}|H|}\right) \]

Theorem (Dvořák and Norin, 2014)

*Balanced separators of order* \( O(n^{1-\delta}) \Rightarrow \text{tw} = O(n^{1-\delta}) \)
Current goal

\[ d(H) = O\left( r^{c/\delta} \right) \]

\( G \in C \)

\( F \) is an r-minor of \( G \)

\( H \) is a separator of \( F \) with \( \text{tw}(H) = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \) and \( d(H) \geq 0.9 \cdot d(F) \)

\( H' \) is a separator of \( H \) with \( \Delta \leq 3 \) and \( \text{tw}(H') = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \)

\( G' \) is a separator of \( G \) corresponding to \( H' \): \( \text{tw}(G') = \Omega \left( \frac{|H|}{\text{polylog}|H|} \right) \)

\( \alpha \frac{|H|}{\text{polylog}|H|} \leq \text{tw}(G') \leq \beta |G'|^{1-\delta} \)

Theorem (Dvořák and Norin, 2014)

\textit{Balanced separators of order } \( O(n^{1-\delta}) \Rightarrow \text{tw} = O(n^{1-\delta}) \)
Sketch of the proof

**Current goal**

\[ d(H) = O \left( r^{c/\delta} \right) \]

\( G \in \mathcal{C} \)

\( F \ r\text{-minor of } G \)

\( H \text{ sgr. of } F \) with \( \text{tw}(H) = \Omega \left( \frac{|H|}{\text{polylog} |H|} \right) \) and \( d(H) \geq 0.9 \cdot d(F) \)

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Sketch of the proof

Current goal

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Sketch of the proof

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Let $C$ be a subgraph-closed class and $\delta \in (0, 1]$. 
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**Theorem (Plotkin, Rao, and Smith, SODA 1994)**

$C$ has expansion $O\left(\frac{r}{\delta}\right)$

$\Downarrow$

graphs of $C$ have balanced separators of size $O\left(n^{1-\delta}\right)$.

**Theorem (Esperet and R., 2017)**

graphs of $C$ have balanced separators of size $O\left(n^{1-\delta}\right)$

$\Downarrow$

$C$ has expansion $O\left(r^{\frac{c}{\delta}}\right)$. 

Open: find the infimum $c$ s.t. the above theorem holds.

Know: $\frac{1}{2} \leq c \leq 1$. 

Thank you!
Let $C$ be a subgraph-closed class and $\delta \in (0, 1]$.

**Theorem (Plotkin, Rao, and Smith, SODA 1994)**

$C$ has expansion $O\left(\frac{r^{1/4\delta - 1}}{\delta - 1}\right)$

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- graphs of $\mathcal{C}$ have balanced separators of size $O \left( n^{1-\delta} \right)$
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Open: find the infimum $c$ s.t. the above theorem holds.
Let $C$ be a subgraph-closed class and $\delta \in (0, 1]$.

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Jean-Florent Raymond
Let $C$ be a subgraph-closed class and $\delta \in (0, 1]$.

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