Tight Erdős-Pósa bounds for minors

Jean-Florent Raymond
(TU Berlin)

Joint work with Wouter Cames van Batenburg, Tony Huynh, and Gwenaël Joret (Université Libre de Bruxelles).
Packing and covering in bipartite graphs

Max. number of disjoint edges?

pack $K_2 = 3$

Min. number of vertices to cover all edges?

cover $K_2 = 3$

cover = pack (Kőnig’s Theorem, 1931)
Packing and covering in bipartite graphs

Max. number of disjoint edges?

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cover = pack (Kőnig’s Theorem, 1931)
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\[ \text{cover} = \text{pack} \]

(König’s Theorem, 1931)
Packing and covering cycles

Max. number of disjoint cycles: pack cycles = 4

Min. number of vertices to cover all cycles: cover cycles = 8

pack ⩽ cover ⩽ c

(Erdős-Pósa Theorem, 1965)
Max. number of disjoint cycles?
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pack \leq cover \leq c

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Erdős-Pósa Theorem, 1965

Jean-Florent Raymond
Packing and covering cycles

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\text{pack}_{\text{cycles}} = 4
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\[
\text{pack} \leq \text{cover} \leq c \cdot \text{pack} \log \text{pack}
\]

(Erdős-Pósa Theorem, 1965)
The Erdős-Pósa Theorem

Theorem (Erdős and Pósa, Can. J. Math. 1965)

Every graph has one of the following:

- $k$ vertex-disjoint cycles;
- a feedback vertex set of size $O(k \log k)$.
### The Erdős-Pósa Theorem

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large packing vs. small cover
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Min-max theorem (like König’s and Menger’s theorems, etc.).
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large packing vs. small cover

Min-max theorem (like König’s and Menger’s theorems, etc.).

Our goal: generalize from cycles to minor-models.
**Definition**

An *H*-model in $G$ is a set $\{S_u\}_{u \in V(H)}$ of disjoint subsets of $V(G)$ s.t.

- the $G[S_u]$’s are connected;
- edge $uv$ in $H \Rightarrow$ edge between $S_u$ and $S_v$ in $G$. 

---

**Diagram:**

The figure on the left shows a graph $G$ with a highlighted subset that represents an $H$-model. The figure on the right shows the graph $H$ that is a minor of $G$. The symbol $\cong$ indicates that $H$ is a minor of $G$. 

---

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---

![Diagram of $G$ and $H$ with an $H$-model highlighted]
**Minor models**

**Definition**

An *H-model* in $G$ is a set $\{S_u\}_{u \in V(H)}$ of disjoint subsets of $V(G)$ s.t.

- the $G[S_u]$’s are connected;
- edge $uv$ in $H$ $\Rightarrow$ edge between $S_u$ and $S_v$ in $G$.

$G$ has a *$H$-model* $\iff H$ is a minor of $G$
**Definition**

$H$ has the **Erdős-Pósa property** if there is a function $f$ s.t., for every graph $G$ and $k \in \mathbb{N}$,

- $G$ has $k$ vertex-disjoint $H$-models; or
- there is $X \subseteq V(G)$ s.t. $G - X$ is $H$-minor free and $|X| \leq f(k)$.

**Theorem (Robertson and Seymour, JCTB 1986)**

$H$ has the Erdős-Pósa property iff $H$ is planar.
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*H* has the **Erdős-Pósa property** if there is a function *f* s.t., for every graph *G* and *k* ∈ ℤ⁺,

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The Erdős-Pósa property of minor models

Definition

H has the Erdős-Pósa property if there is a function f s.t., for every graph G and \( k \in \mathbb{N} \),

- G has \( k \) vertex-disjoint H-models; or
- there is \( X \subseteq V(G) \) s.t. \( G - X \) is H-minor free and \( |X| \leq f(k) \).

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Theorem (Robertson and Seymour, JCTB 1986)

H has the Erdős-Pósa property iff H is planar.

With which gap?
### A non-exhaustive history of Erdős-Pósa gaps

| Graph $H$ | EP gap $O(k \log k)$ | Reference |  |
|-----------|----------------------|------------|  |
| $K_3$     |                      | Erdős and Pósa | (Can. J. Math.’65) |

Best possible:
- $H$ not planar: no Erdős-Pósa property;
- $H$ has a cycle: no $o(k \log k)$ gap.
### A non-exhaustive history of Erdős-Pósa gaps

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- $H$ not planar $\Rightarrow$ no Erdős-Pósa property;
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**Best possible:**

- $H$ not planar $\Rightarrow$ no Erdős-Pósa property;
- $H$ has a cycle $\Rightarrow$ no $o(k \log k)$ gap.
The key lemma

Our main theorem follows from the statement:

“every graph has a small $H$-model or a large useless part”
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“every graph has a small $H$-model or a large useless part”

**Lemma (Cames van Batenburg, Huynh, Joret, R., 2018+)**

For every graph $G$ and every planar graph $H$,

- $G$ has an $H$-model of size $O(\log |G|)$;  
  
  or

- $G = A \cap B$  
  $|B| \geq \text{large}(|A \cap B|)$
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For every graph $G$ and every planar graph $H$,

- $G$ has an $H$-model of size $O(\log |G|)$;

  or

- $G = \begin{array}{c}
    A \\
    \cap \\
    B
\end{array}$  \quad $G[B]$ is $H$-minor free \\
  $|B| \geq \text{large}(|A \cap B|)$

The constant hidden in the “$O$” notation depends on:

- the graph $H$;

- the definition of “large”.

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Tight Erdős-Pósa bounds for minors
Proof sketch for $H = K_3$

Goal: “$G$ has a small $H$-model or a large useless part”
**Proof Sketch for** $H = K_3$

Goal: “$G$ has a *small* $H$-model or a large *useless* part”

Maximum collection of disjoint paths of length $\ell$: (covering $G$, for simplicity)

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PROOF SKETCH FOR $H = K_3$

Goal: “$G$ has a small $H$-model or a large useless part”

Maximum collection of disjoint paths of length $\ell$:
(covering $G$, for simplicity)

- either every path sees $\geq 3$ other paths
Proof sketch for $H = K_3$

Goal: “$G$ has a small $H$-model or a large useless part”

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  cycle of length $O(\ell \cdot \log |G|)$
Proof sketch for \( H = K_3 \)

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- $|B| \geq \text{large}(|A \cap B|)$

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PROOF SKETCH FOR $H = K_3$

Goal: “$G$ has a small $H$-model or a large useless part”

Maximum collection of disjoint paths of length $\ell$:
(covering $G$, for simplicity)

- $B$ is $K_3$-minor free
- $|B| \geq$ large($|A \cap B|$)

- either every path sees $\geq 3$ other paths:
  cycle of length $O(\ell \cdot \log |G|)$
- or one path sees $\leq 2$ other paths:
  cycle of length $\leq 2\ell$ or large useless part.
How to generalize?

Crucial property: we can conclude when two paths are connected with many edges.
How to generalize?

Crucial property: we can conclude when two paths are connected with many edges.

Possible extension to $H = K_4$:

Pack cycles of bounded size first, then paths.
How to generalize?

Crucial property: we can conclude when two paths are connected with many edges.

Possible extension to $H = K_4$:

Pack cycles of bounded size first, then paths.

$\leadsto$ gap $O(k \log k)$ when $H$ is a wheel

(Aboulker, Fiorini, Huynh, Joret, R. and Sau, 2018)
Orchards

An a-orchard in G consists in collections

• $P_1; \ldots; P_a$ of vertex-disjoint (horizontal) paths; and

• $T_1; \ldots; T_b$ of vertex-disjoint (vertical) trees,

s.t. for every $i \in \{1, \ldots, a\}$, $j \in \{1, \ldots, b\}$:

• $P_i \nsubseteq T_j = \emptyset$ and connected;

and

• each leaf of $T_j$ lies on some horizontal path.
Orchards

An $a \times b$-orchard in $G$ consists in collections

- $P_1, \ldots, P_a$ of vertex-disjoint (horizontal) paths; and
- $T_1, \ldots, T_b$ of vertex-disjoint (vertical) trees,

s.t. for every $i \in [a], j \in [b]$:

- $P_i \cap T_j \neq \emptyset$ and connected; and
- each leaf of $T_j$ lies on some horizontal path.
Decomposition into orchards

\[ G \]
Decomposition into orchards

$G$ max. collection of disjoint $m \times \omega(m)$-orchards in $G$
Decomposition into orchards

- Max. collection of disjoint $m \times \omega(m)$-orchards in $G$
- Max. collection of disjoint $(m - 1) \times \omega(m - 1)$-orchards in what remains
Decomposition into orchards

\[ G \]

- max. collection of disjoint \( m \times \omega(m) \)-orchards in \( G \)
- max. collection of disjoint \((m - 1) \times \omega(m - 1)\)-orchards in what remains
- \( \vdots \)
- max. collection of disjoint \( 1 \times \omega(1) \)-orchards in what remains
Decomposition into orchards

$G$ → max. collection of disjoint $m \times \omega(m)$-orchards in $G$

$\vdots$

max. collection of disjoint $(m - 1) \times \omega(m - 1)$-orchards in what remains

$\vdots$

max. collection of disjoint $1 \times \omega(1)$-orchards in what remains

leftovers
Decomposition into orchards

\[ \text{max. collection of disjoint } m \times \omega(m)\text{-orchards in } G \]

\[ \text{max. collection of disjoint } (m - 1) \times \omega(m - 1)\text{-orchards in what remains} \]

\[ \vdots \]

\[ \text{max. collection of disjoint } 1 \times \omega(1)\text{-orchards in what remains} \]

\[ \text{leftovers} \]

- many edges between two orchards \( \Rightarrow \) small model or \textit{better} decomposition
Decomposition into orchards

\begin{itemize}
    \item many edges between two orchards \implies small model or \textit{better} decomposition
    \item only few edges between two orchards \implies small separation
\end{itemize}
Consequences
**Consequence 1/4: Algorithms**

\[ \text{pack}_H(G) \] max. number of disjoint \( H \)-models in \( G \)

\[ \text{cover}_H(G) \] min. size of a cover of \( H \)-models in \( G \)
**Consequence 1/4: Algorithms**

$\text{pack}_H(G)$ max. number of disjoint $H$-models in $G$

$\text{cover}_H(G)$ min. size of a cover of $H$-models in $G$

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<th>Problem</th>
<th>Exact</th>
<th>Approximate</th>
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| $\text{pack}_{K_3}$ | CYCLE PACKING | NPC | • polytime $O(\log \text{OPT})$-approx.  
• $O(\log(n)^{1/2-\epsilon})$-approx. is quasi-NP-hard |
| $\text{cover}_{K_3}$ | FVS | NPC | • polytime 2-approx. |
**Consequence 1/4: Algorithms**

\[
\text{pack}_H(G) \quad \text{max. number of disjoint } H\text{-models in } G
\]

\[
\text{cover}_H(G) \quad \text{min. size of a cover of } H\text{-models in } G
\]

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| pack_{K_3} | CYCLE PACKING | NPC | • polytime \(O(\log \text{OPT})\)-approx.  
• \(O(\log(n)^{1/2-\epsilon})\)-approx. is quasi-NP-hard |
| cover_{K_3} | FVS | NPC | • polytime 2-approx. |

**Theorem (from our results)**

*For every planar graph \(H\), there is a polytime \(O(\log(\text{OPT}))\)-approximation algorithm for \(\text{pack}_H\).*
**Consequence 1/4: Algorithms**

The function $\text{pack}_H(G)$ represents the maximum number of disjoint $H$-models in $G$, while $\text{cover}_H(G)$ represents the minimum size of a cover of $H$-models in $G$.

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**Theorem (from our results)**

For every planar graph $H$, there is a polytime $O(\log(\text{OPT}))$-approximation algorithm for $\text{pack}_H$.

(idem for $\text{cover}_H$, but $O(1)$-approximations are already known)
**Theorem (Stiebitz, JGT 1996)**

*Every graph of large minimum degree has a partition into many subgraphs of large minimum degree.*
Consequence 2/4: Large treewidth graph decomposition

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Same for treewidth?
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If $G$ has treewidth at least

- $\text{poly}(r) \cdot k \text{polylog}(k + 1)$ (Chekury and Chuzhoy, 2013)

then it has $k$ disjoint subgraphs of treewidth at least $r$. 
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Theorem (Thomassen, JGT 1988)

For every \( m \in \mathbb{N}_{\geq 1} \) there is a function \( f \) s.t., for every \( k \in \mathbb{N} \) and every graph \( G \),

- \( G \) contains \( k \) vertex-disjoint cycles of length \( 0 \mod m \),
- or there is a subset \( X \) of at most \( f(k) \) vertices s.t. \( G - X \) has no such cycle.

From Thomassen's proof:
\[
f(k) = 2^{2O(k)}
\]

Chekuri and Chuzhoy (2013):
\[
f(k) = k \text{polylog} k
\]

From our result:
\[
f(k) = O(k \log k) \text{ (tight)}
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Consequence 3/4: Packing cycles with modularity constraints

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- from our result: $f(k) = O(k \log k)$ (tight)
EP65: every gap for $K_3$ is an $\Omega(k \log k)$.
Consequence 4/4: Erdős-Pósa in minor-closed classes

EP65: every gap for $K_3$ is an $\Omega(k \log k)$.

Theorem (Bienstock and Dean, JCTB 1992)

$k \mapsto 54k$ is a gap for $K_3$ in planar graphs.
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For every planar graph $H$ and every proper minor-closed class $\mathcal{G}$, there is a $O(k)$ gap for $H$ *in $\mathcal{G}$.*

The previous theorem also follows from our results.
Open problems
**The right gap**

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In our proof, the hidden constant (which depends on \( H \)) is:

- not known to be computable;
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There is a function $f(k, \ell) = O(k \log k + k\ell)$ such that $C_\ell$ has gap $f(\cdot, \ell)$, for every $\ell \geq 3$. 
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Same behavior?
## Variants

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Other variants: directed, induced, weighted, labelled, etc.
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Thank you for your attention!