# Enumerating minimal dominating sets in triangle-free graphs

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19/10/2018
Dagstuhl Seminar on Algorithmic Enumeration

Joint work with **Marthe Bonamy** (LaBRI, Bordeaux), **Oscar Defrain** (LIMOS, Clermont-Ferrand), and **Marc Heinrich** (LIRIS, Lyon).

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**Simple observation**

$D \subseteq V(G)$ minimal dominating set iff

- $D$ dominates $G$, and
- every $v \in D$ has a private neighbor ($v$ is irredundant).

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Running time polynomial in (input size + output size)

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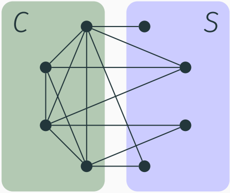
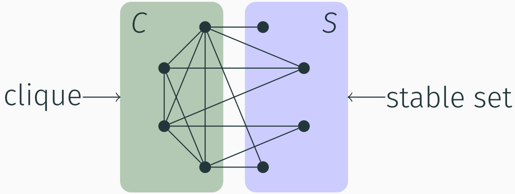
**Theorem (Bonamy, Defrain, Heinrich, R., 2018+)**

*There is an output-polynomial algorithm enumerating minimal dominating sets in triangle-free graphs.*

# Minimal dominating sets in split graphs

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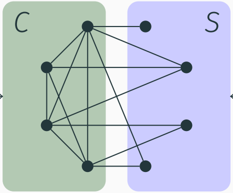
clique
stable set

clique ⟶ 

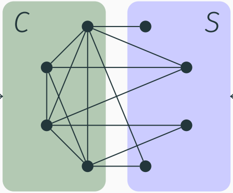
⟵ stable set (maximal)

$D$ minimal DS in $G$

$\Updownarrow$

$D \cap C$ is irredundant (every vertex has a private neighbor)

and ...

clique⟶  ⟵stable set (maximal)



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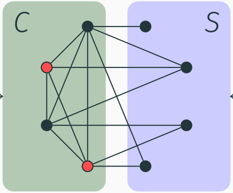
clique⟶  ⟵stable set (maximal)



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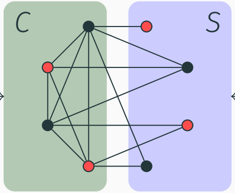
clique⟶  ⟵stable set (maximal)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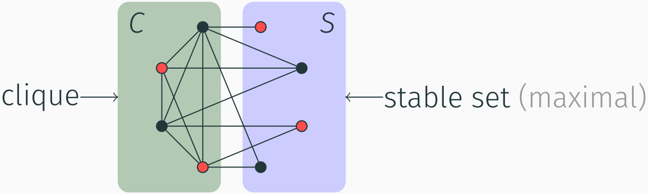


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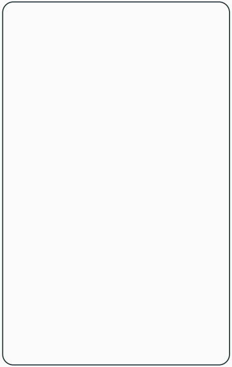
**Enumeration:** complete in $S$ every irredundant $X \subseteq C$
(linear delay, Kanté et al. 2014)

**Goal:** enumeration of minimal DS in triangle-free graphs



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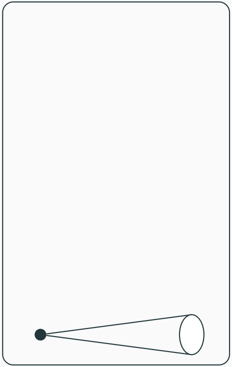

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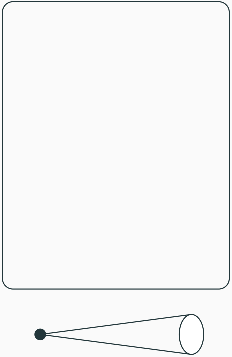

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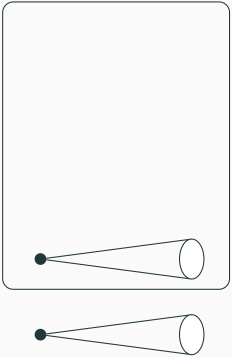

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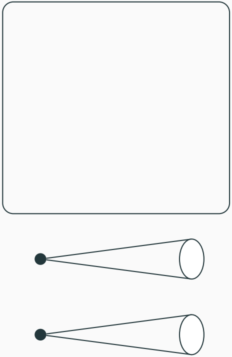

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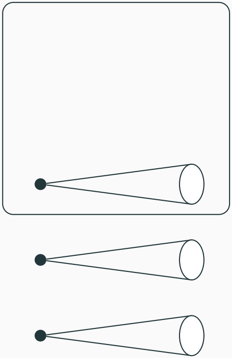

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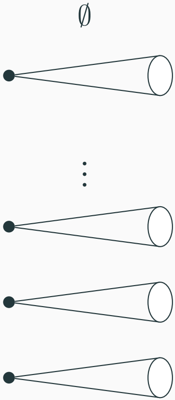

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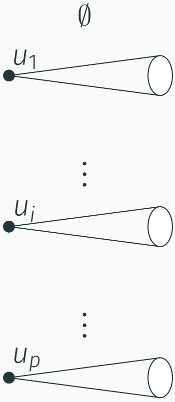

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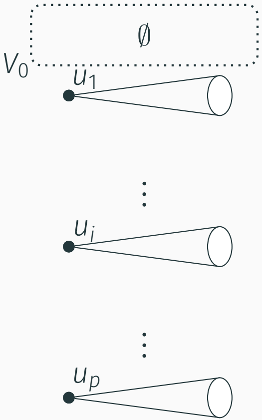

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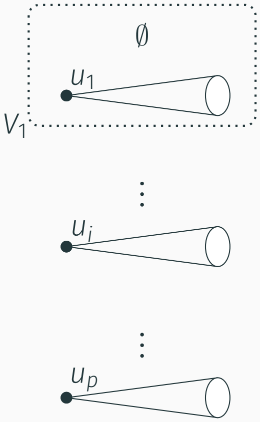

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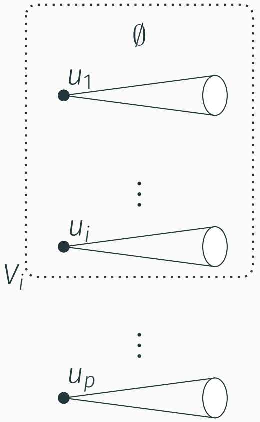

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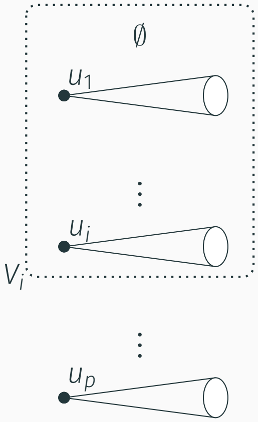


**Plan:**

1. enumerate the minimal DS of $V_i$
2. use them to enumerate those of $V_{i+1}$

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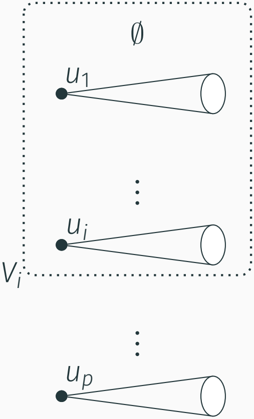


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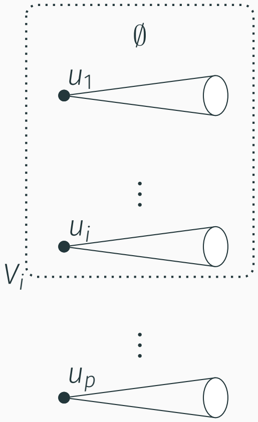


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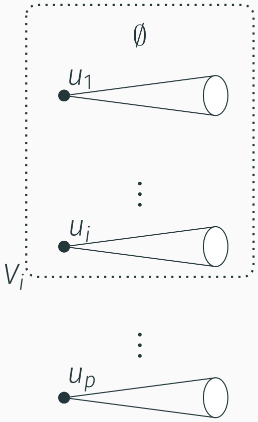


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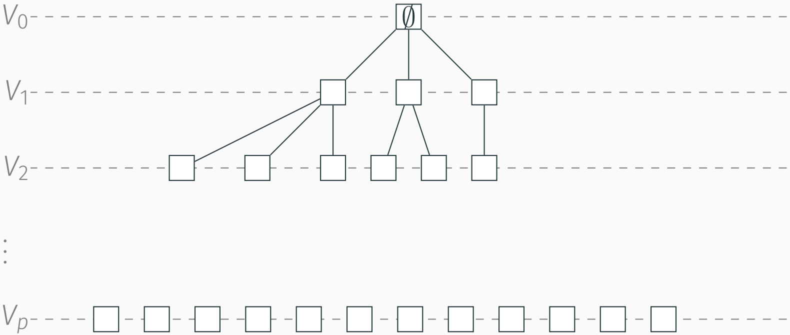
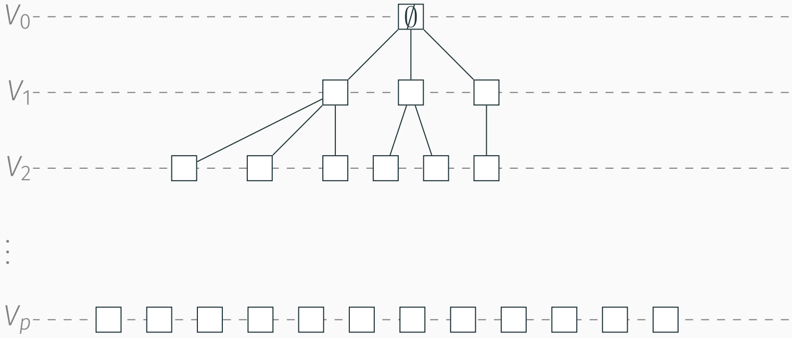
1. enumerate the minimal DS of $V_i$
   with vertices of $G$
2. use them to enumerate
   those of $V_{i+1}$
   extend each minimal DS of $V_i$ to minimal DS of $V_{i+1}$

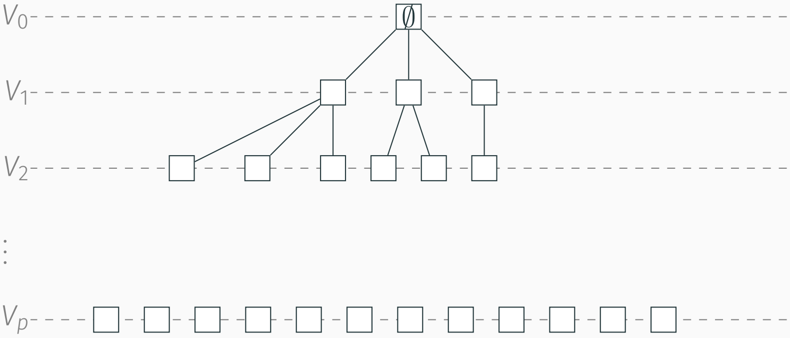
# Growing partial minimal dominating sets






## Important properties:

- no cycle

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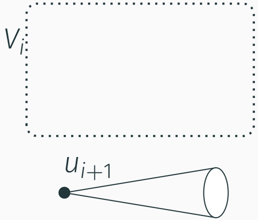

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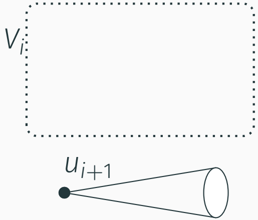


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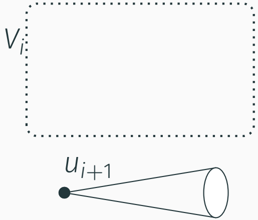


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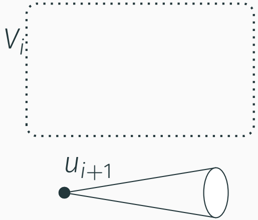


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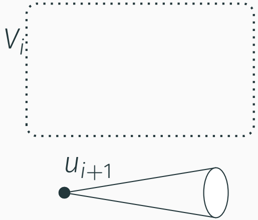


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**Lemma:** $|$candidate extensions of $D| \leqslant |$minimal DS of $G|$
(so we can try them all even if only few *work*)

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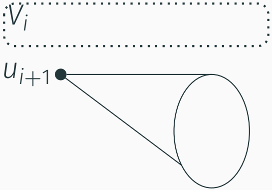
Candidate ext.: minimal $X$ s.t. $D \cup X$ dominates $V_{i+1}$



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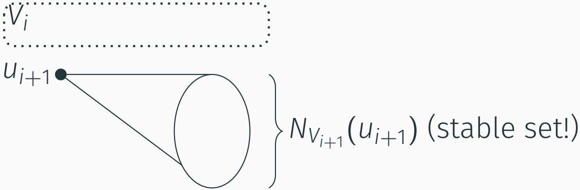

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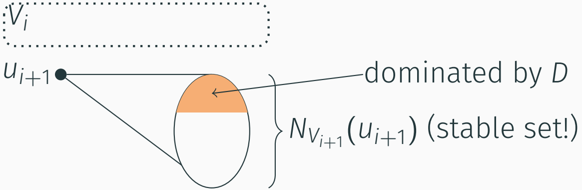

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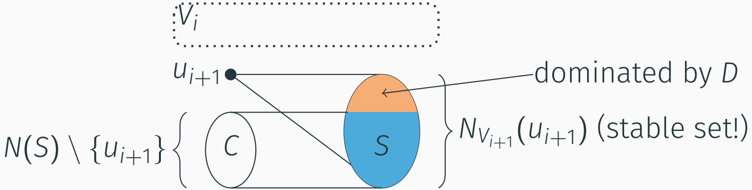

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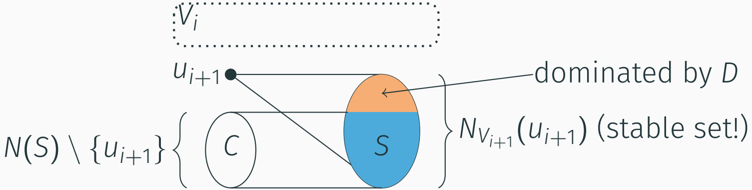


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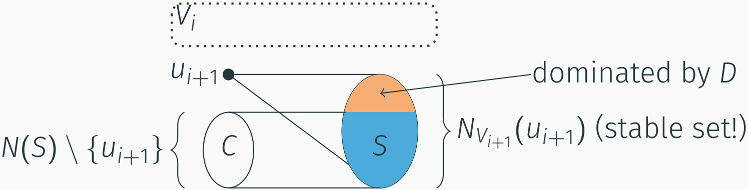


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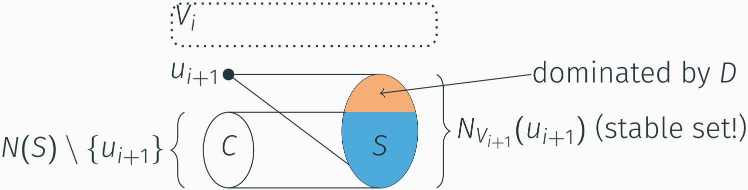


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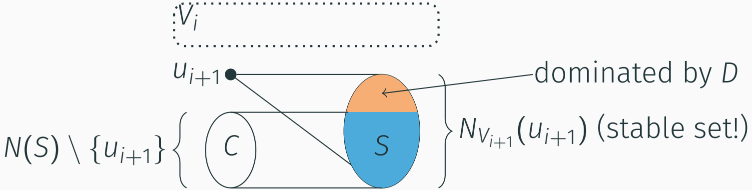


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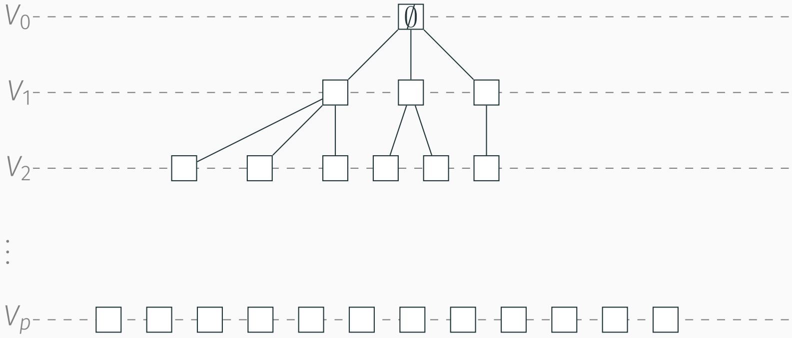
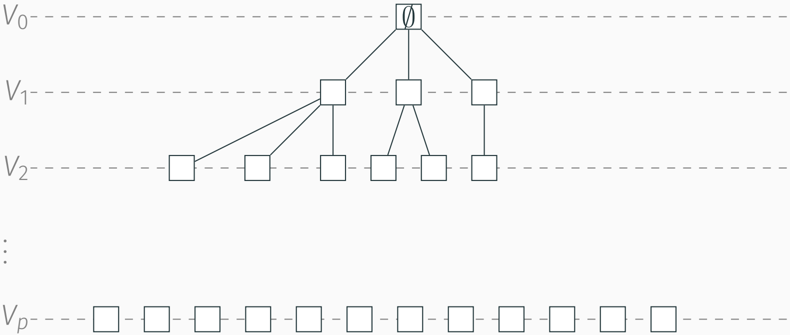
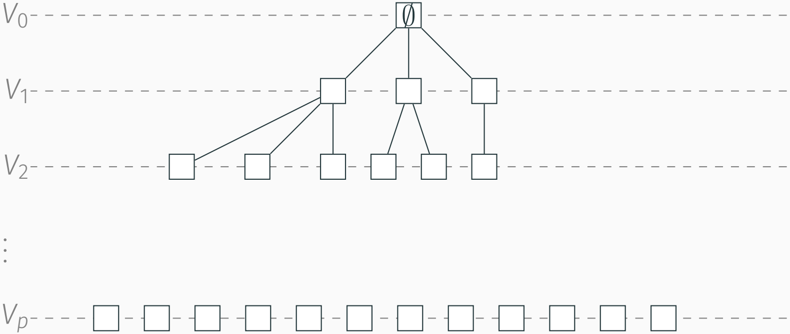
- or $u_{i+1}$ not dom. by $D$: they should dom. $u_{i+1}$ as well
  $\rightarrow$ irredundant $\{t\} \cup Q$ s.t. $\begin{cases} t \in N(u_{i+1}) \\ Q \subseteq C \text{ minimal DS of } \mathrm{Split}(C, S) \end{cases}$

# The algorithm






For each $D$ minimal DS of $V_i$:

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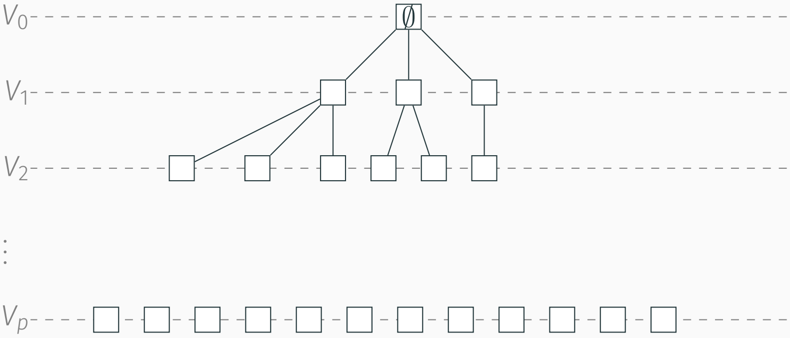


For each $D$ minimal DS of $V_i$:

- compute all candidate extensions;
- only keep the $X \cup D$'s that are minimal and children of $D$.

**Theorem (Bonamy, Defrain, Heinrich, R., 2018+)**

*The set $\mathcal{D}(G)$ of minimal dominating sets of any triangle-free graph $G$ can be enumerated in time $\mathrm{poly}(|G|) \cdot |\mathcal{D}(G)|^2$ and polynomial space.*

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