Enumerating minimal dominating sets in triangle-free graphs

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(TU Berlin)

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Joint work with **Marthe Bonamy** (LaBRI, Bordeaux), **Oscar Defrain** (LIMOS, Clermont-Ferrand), and **Marc Heinrich** (LIRIS, Lyon).

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Problem: possibly many objects!

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Two perspectives about complexity

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Output-sensitive: in terms of input+output size

- [Fredman and Khachiyan, 1996]:
  $O\left(s^{\log s}\right)$-time for Minimal Dominating Set
  $s = \text{poly}\,|G| + |\mathcal{D}(G)|$

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"Fast" output-sensitive algorithms

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“FAST” OUTPUT-SENSITIVE ALGORITHMS

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Note: output space is not counted.

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Simple observation

$D \subseteq V(G)$ minimal dominating set iff

- D dominates G, and
- every $v \in D$ has a private neighbor (v is irredundant).

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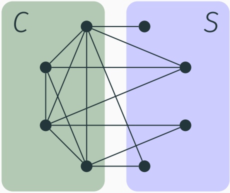
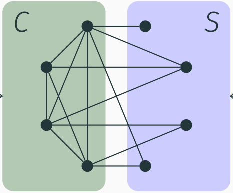
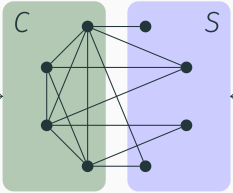
Theorem (Bonamy, Defrain, Heinrich, R., 2018+)
There is an output-polynomial algorithm enumerating minimal dominating sets in triangle-free graphs.

Minimal dominating sets in split graphs



Minimal dominating sets in split graphs

clique⟶  ⟵stable set clique⟶  ⟵stable set (maximal)

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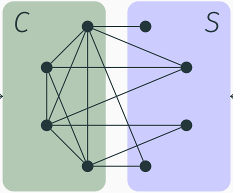
clique⟶  ⟵stable set (maximal)

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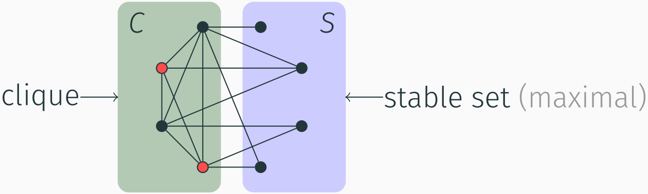

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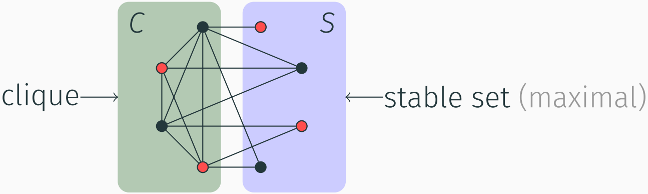

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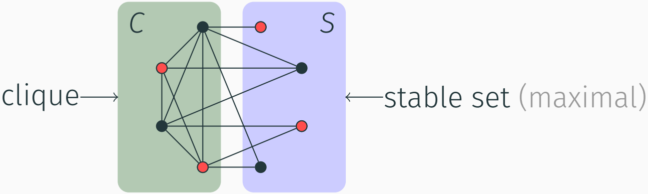

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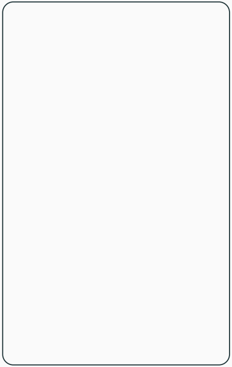
Enumeration: complete in S every irredundant $X \subseteq C$
$\rightsquigarrow$ linear delay [Kanté, Limouzy, Mary, and Nourine, 2014]

Goal: enumeration of minimal DS in triangle-free graphs



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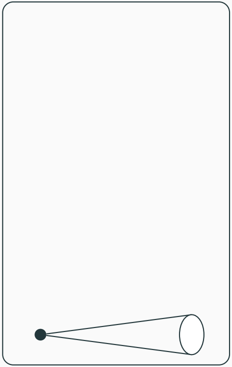

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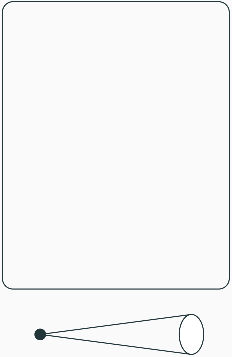

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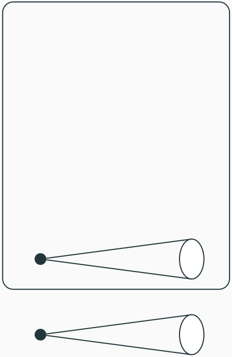

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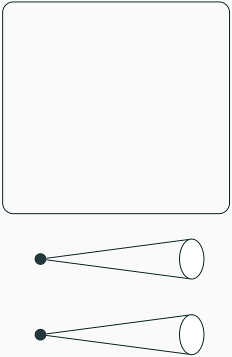

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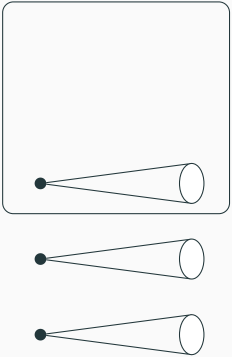

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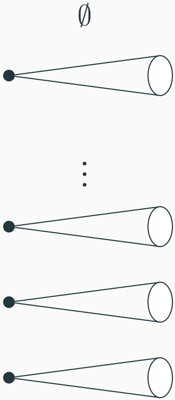

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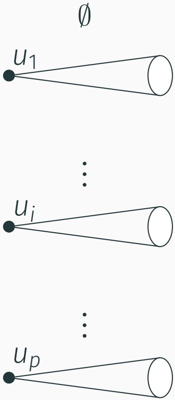

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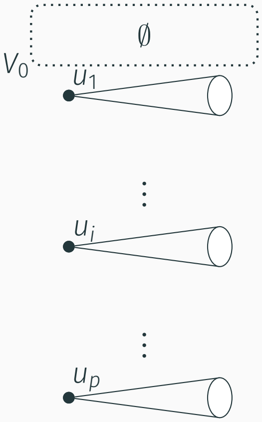

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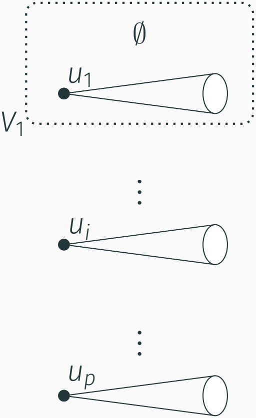

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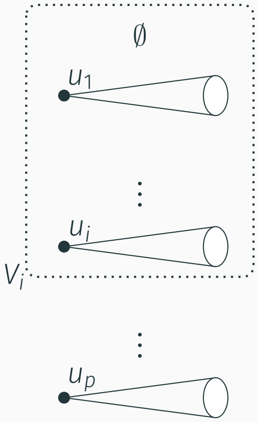

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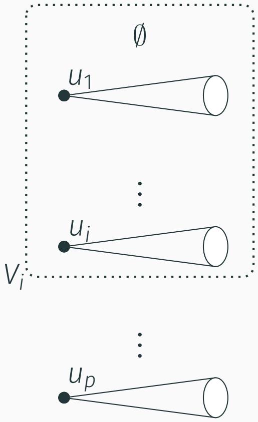

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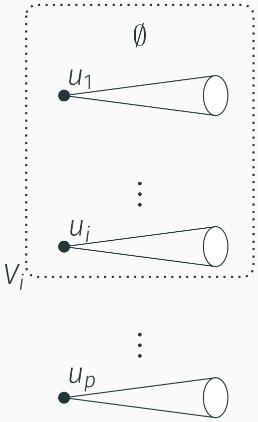

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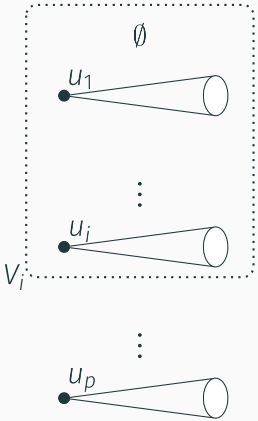

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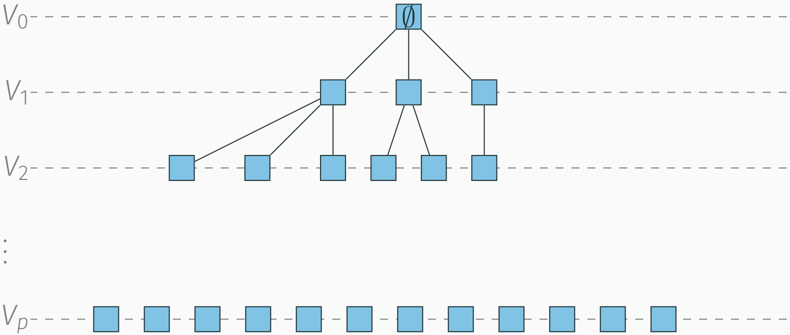
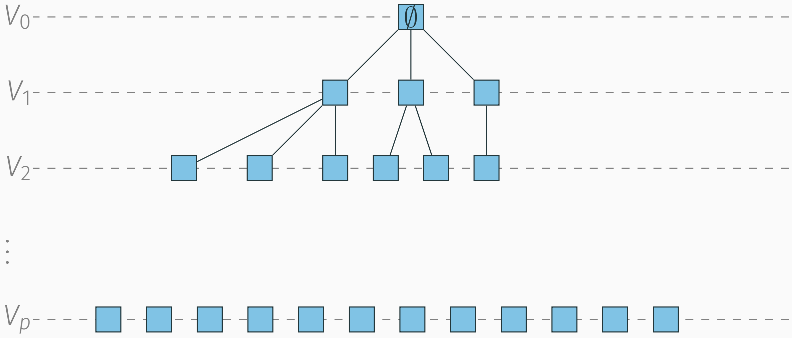
   extend each minimal DS of V_i
   to a minimal DS of V_{i+1}

Growing partial minimal dominating sets





Important properties:

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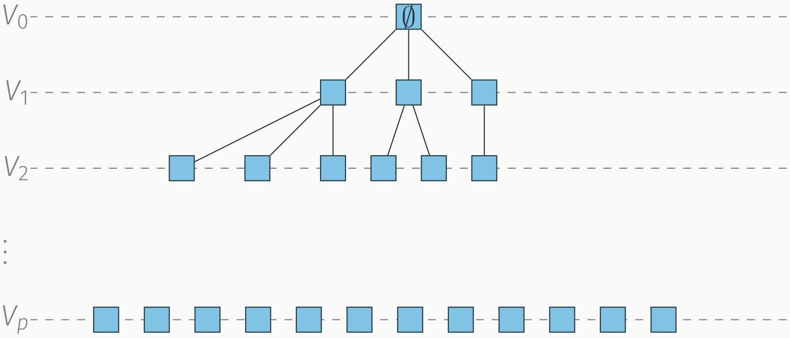

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- no cycle
- no leaf before level p

From V_i to V_{i+1}

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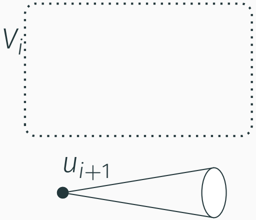

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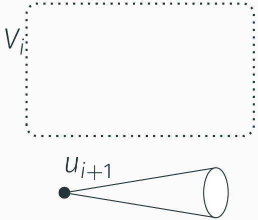

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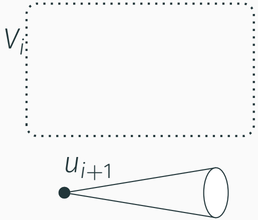

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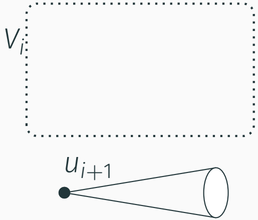

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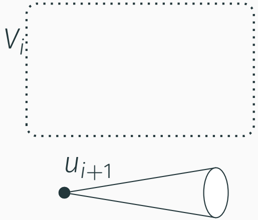

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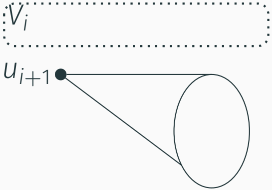
Lemma: $|\text{candidate extensions of } D| \leqslant |\text{minimal DS of } G|$
(so we can try them all even if only few *work*)

Which are the candidate extensions of D?

D: minimal DS of V_i

Candidate ext.: minimal X s.t. $D \cup X$ dominates V_{i+1}



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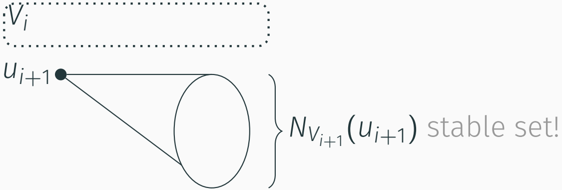

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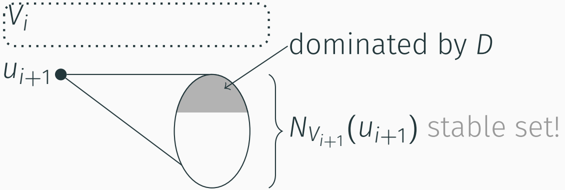

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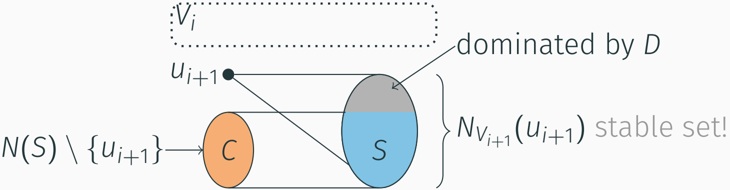

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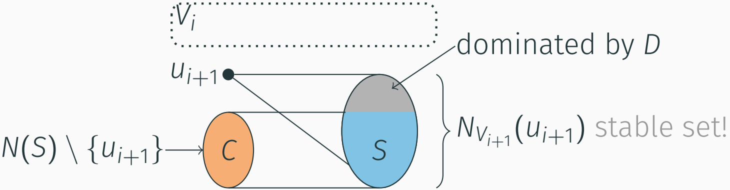

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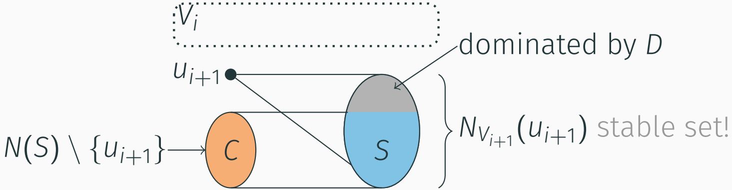

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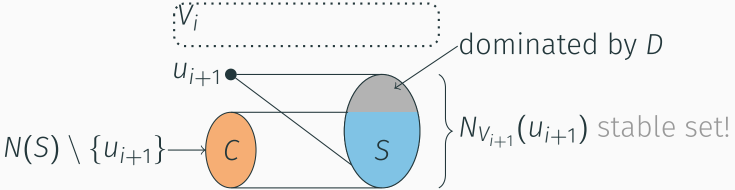

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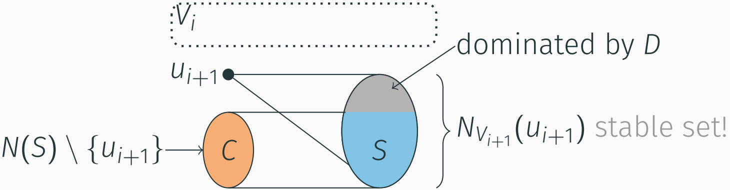
WHICH ARE THE CANDIDATE EXTENSIONS OF D?

D: minimal DS of V_i

Candidate ext.: minimal X s.t. $D \cup X$ dominates V_{i+1}



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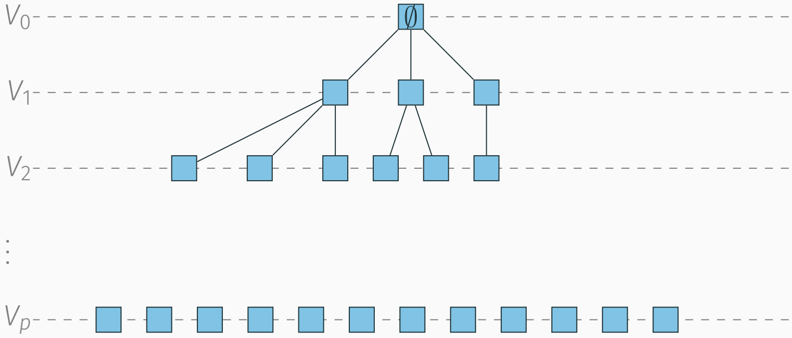
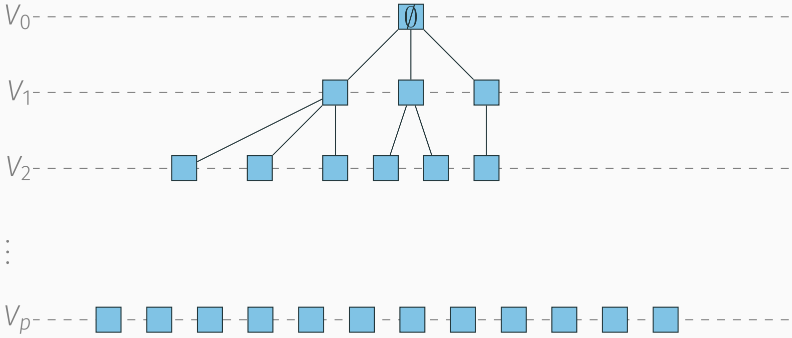
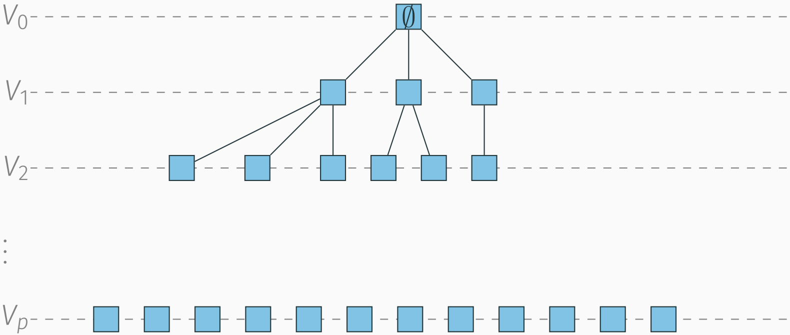
- or u_{i+1} not dom. by D: they should dom. u_{i+1} as well
  $\rightarrow$ irredundant $\{t\} \cup Q$ s.t. $\begin{cases} t \in N(u_{i+1}) \\ Q \subseteq C \text{ minimal DS of } \mathrm{Split}(C, S) \end{cases}$

The algorithm





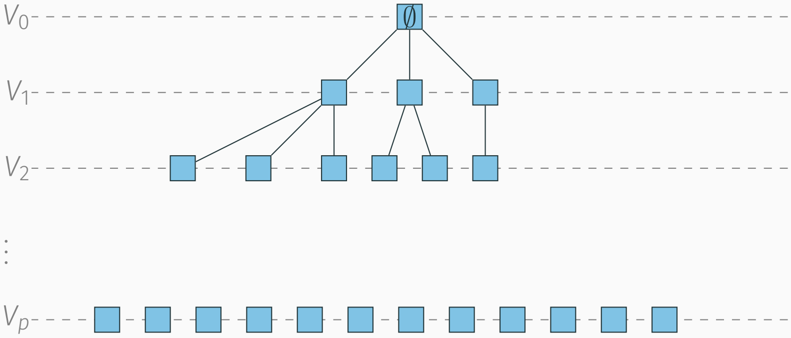
For each D minimal DS of V_i:



For each D minimal DS of V_i:

- compute all candidate extensions;

The algorithm



For each D minimal DS of V_i:

- compute all candidate extensions;
- only keep the $X \cup D$'s that are minimal and children of D.

Theorem (Bonamy, Defrain, Heinrich, R., 2018+)

The set $\mathcal{D}(G)$ of minimal dominating sets of any triangle-free graph G can be enumerated in time $\mathrm{poly}(|G|) \cdot |\mathcal{D}(G)|^2$ and polynomial space.

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Thank you!