Lean tree-cut decompositions
obstructions & algorithms

Jean-Florent Raymond
(TU Berlin)

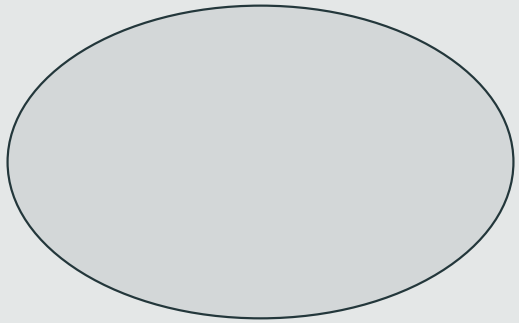
joint work with:

Archontia C. Giannopoulou
O-joung Kwon
Dimitrios M. Thilikos

1927: Menger's theorem

Theorem (Menger, 1927)

$\forall G$



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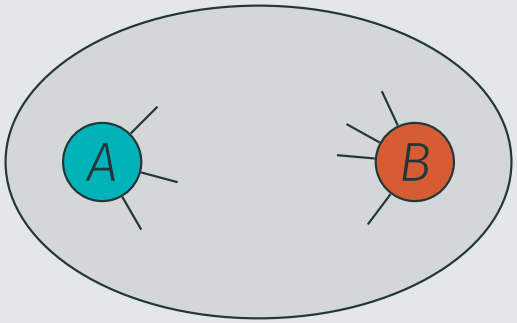

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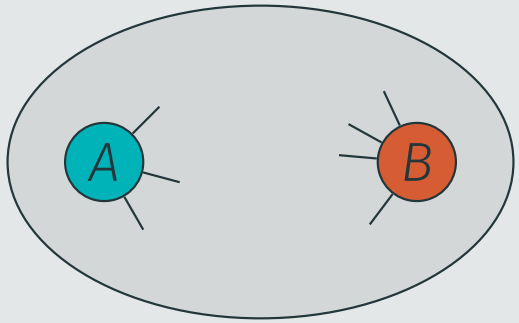

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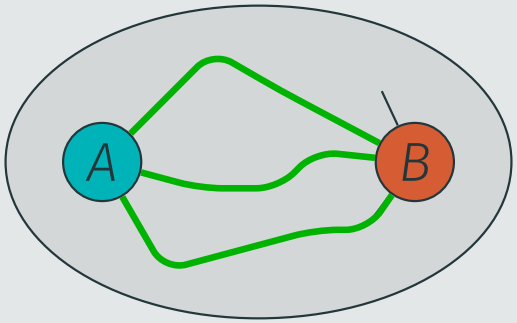

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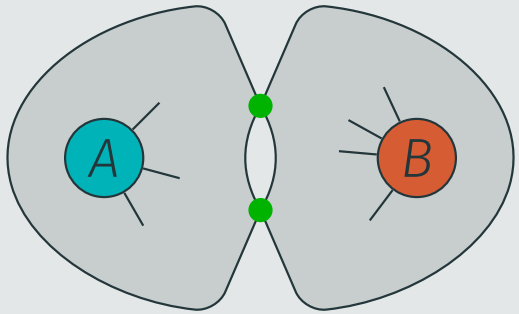

G has:

- *k vertex-disjoint A-B paths*
- *or an A-B separator of size $< k$*

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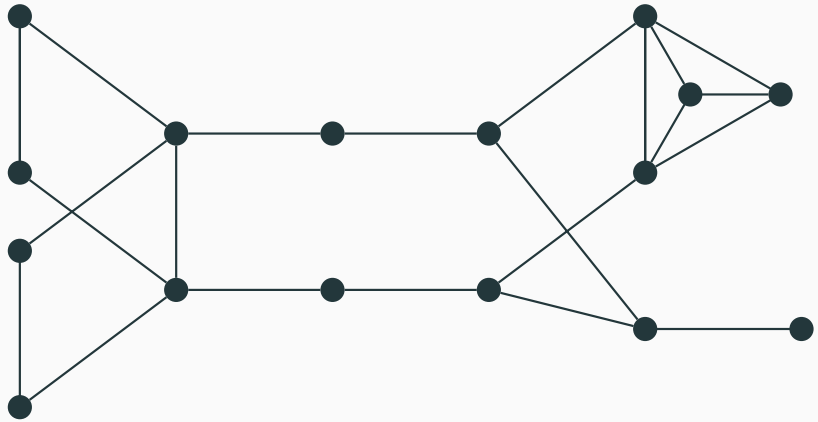

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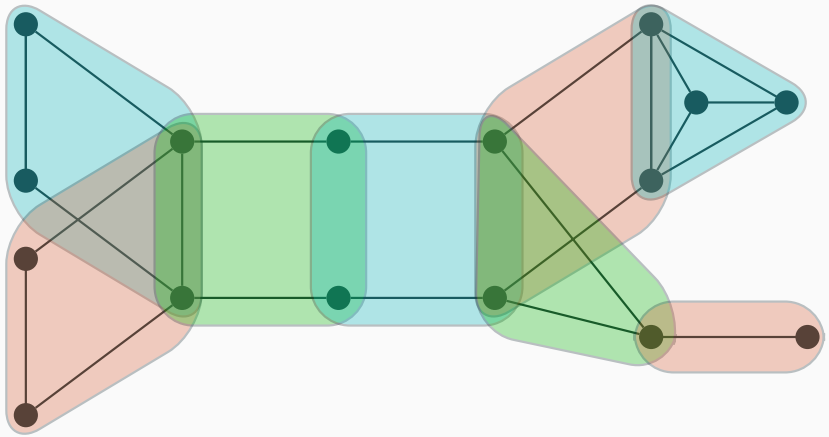

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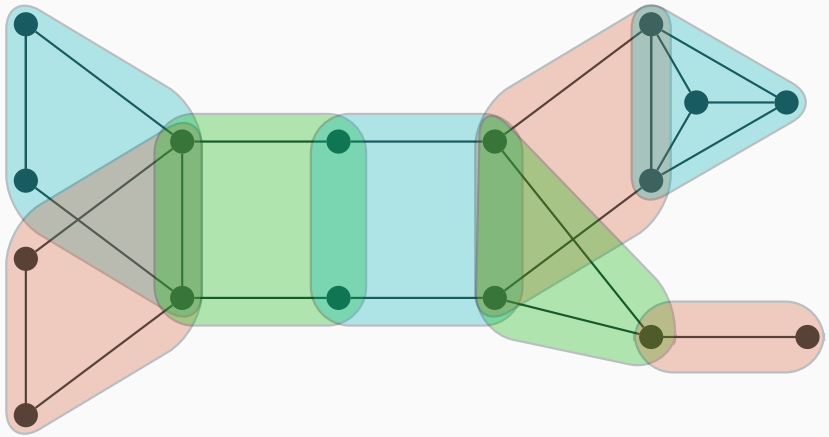

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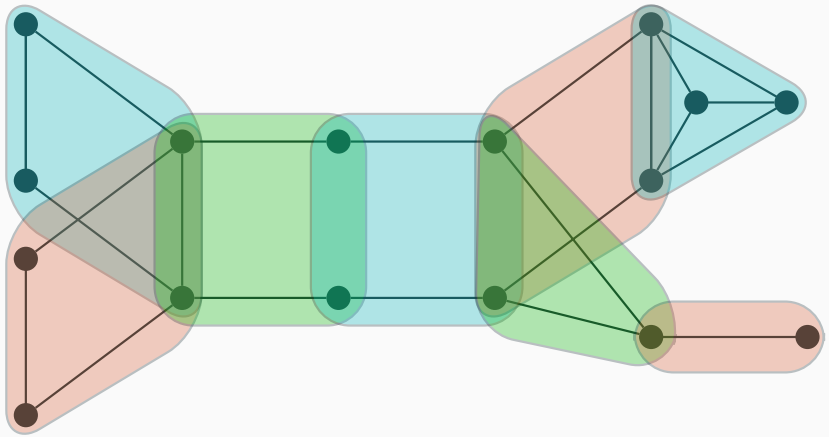

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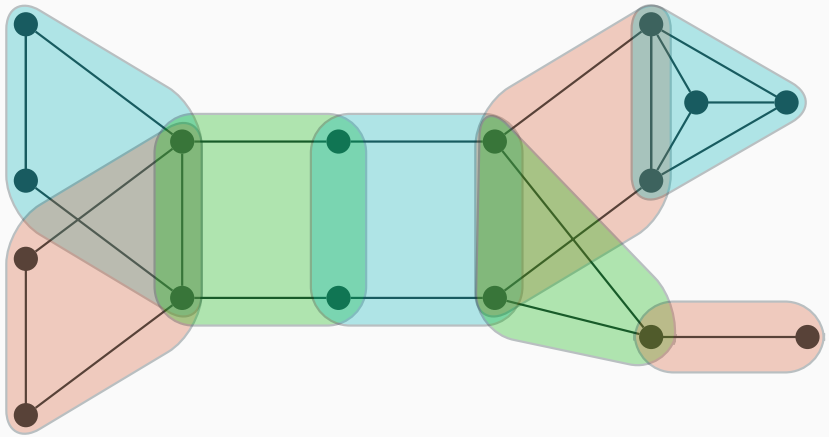

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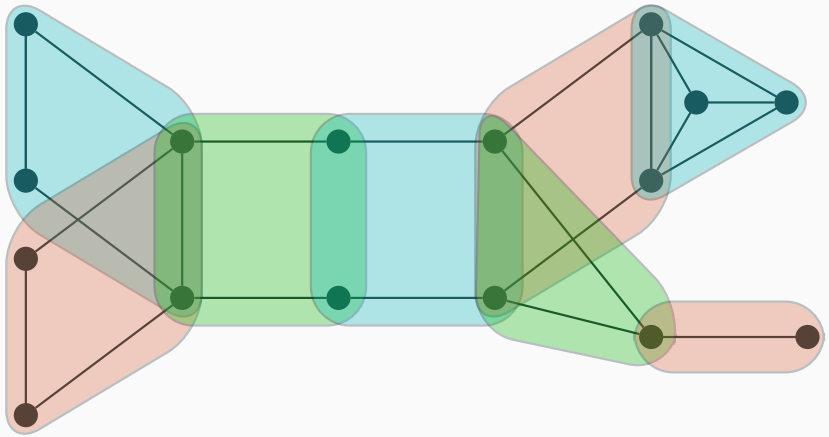

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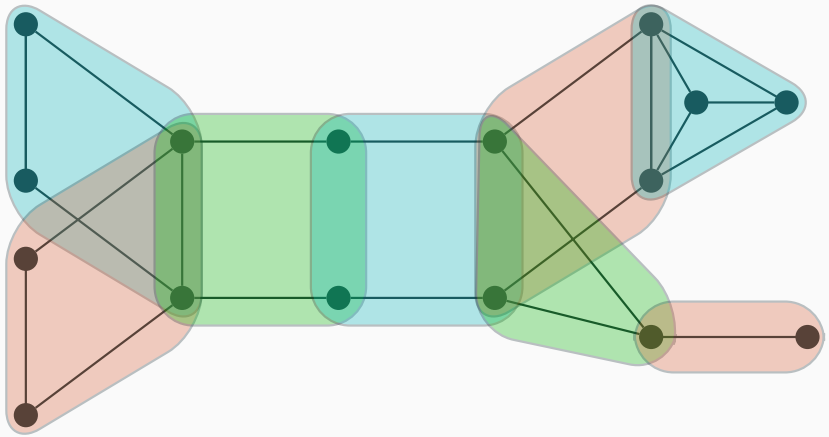

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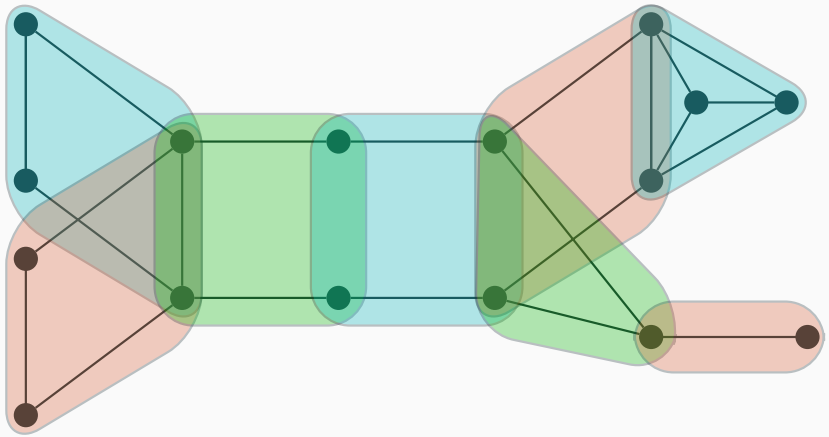

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- *or there is a node c between a and b with $|\text{bag}(c)| < |A|$.*

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In this talk: tree-cut width

Immersions

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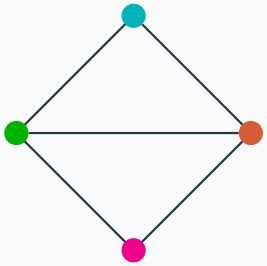

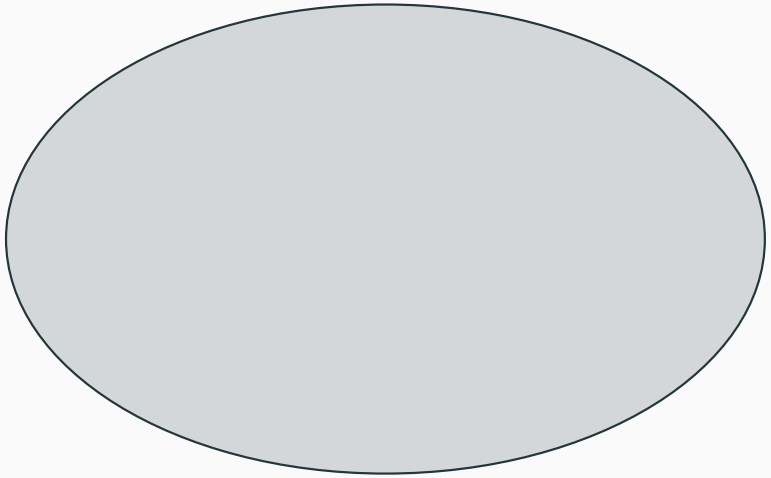

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Immersions

distinct vertices $\mapsto$ distinct vertices

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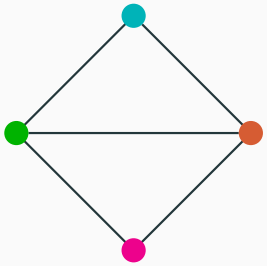

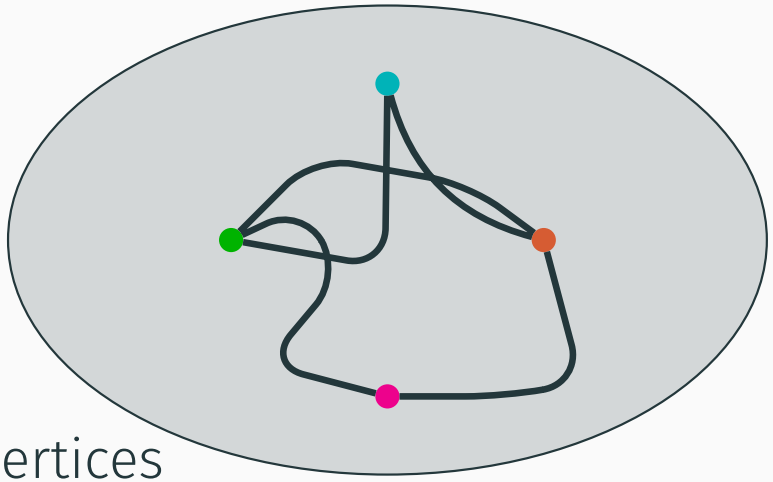

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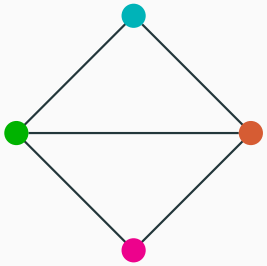

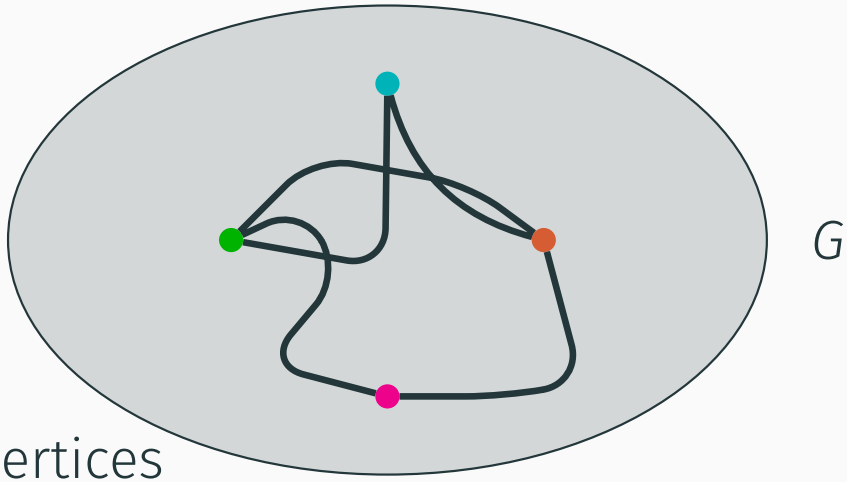

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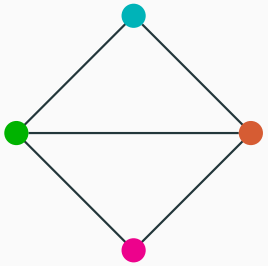

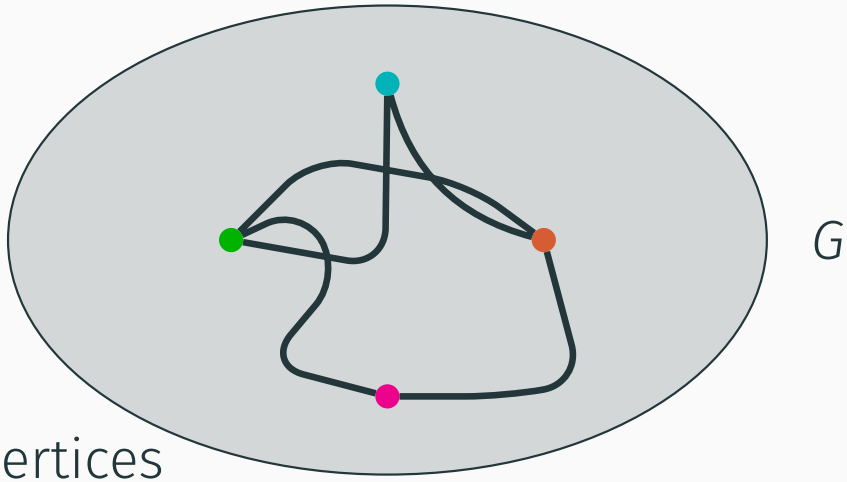

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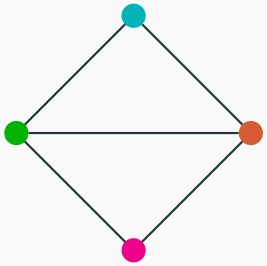

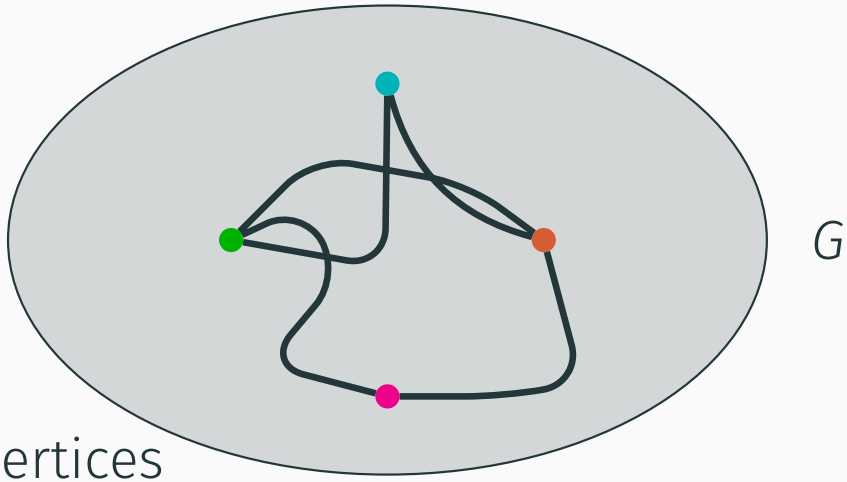

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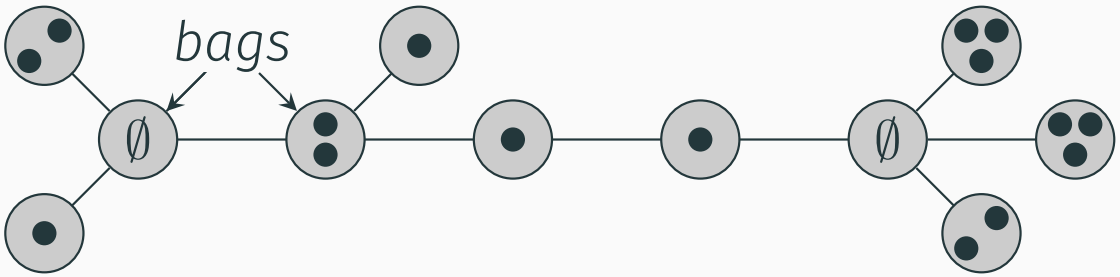
Immersions are much less studied than minors!

Tree-cut decompositions



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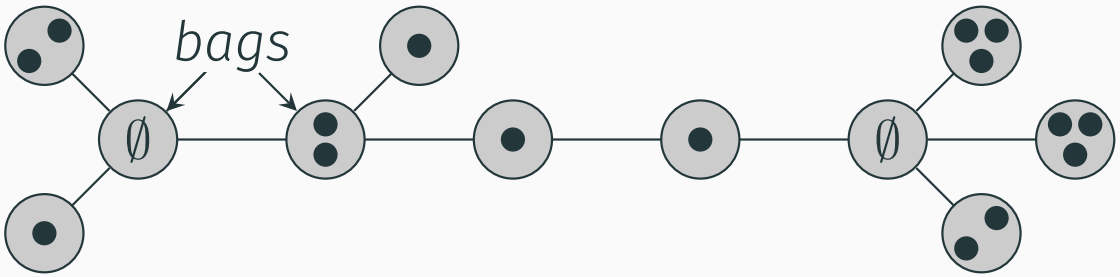

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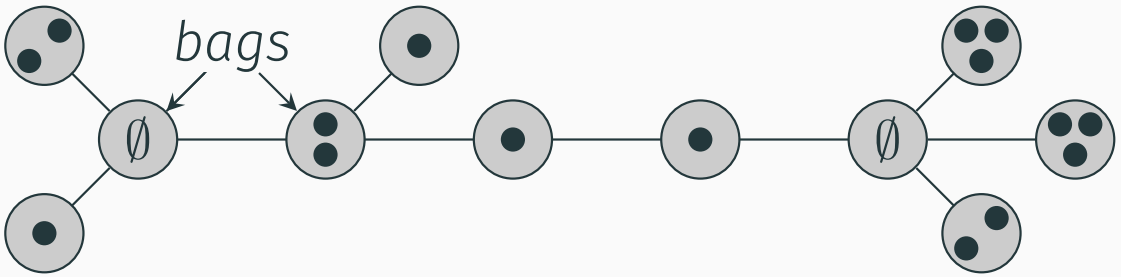

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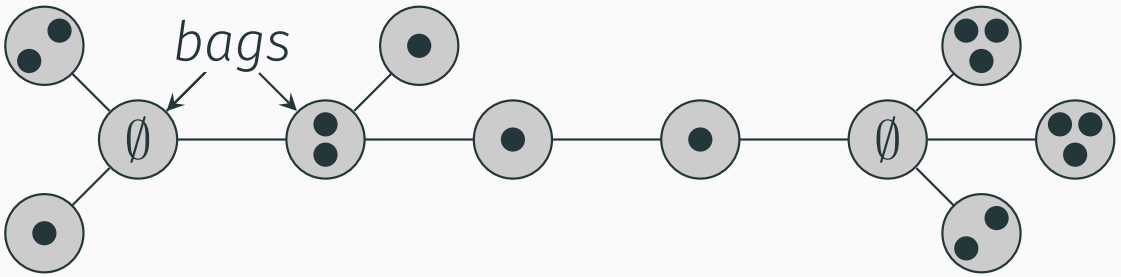

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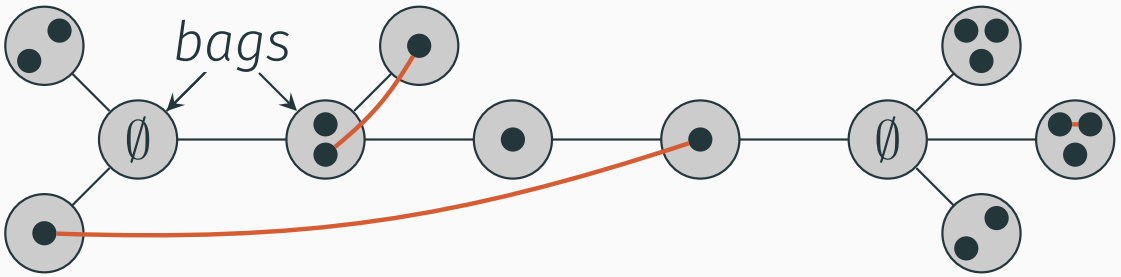

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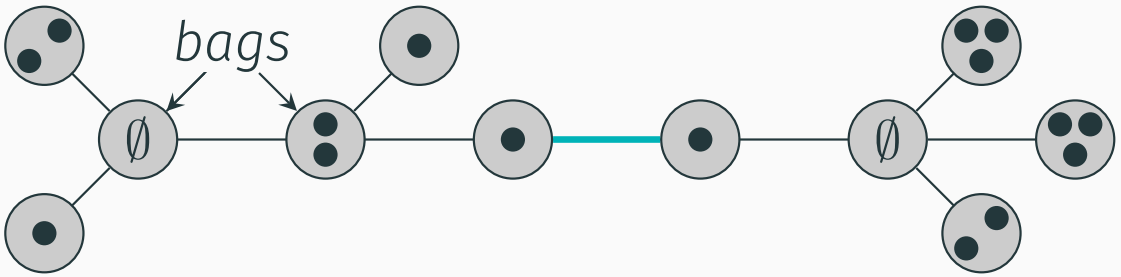

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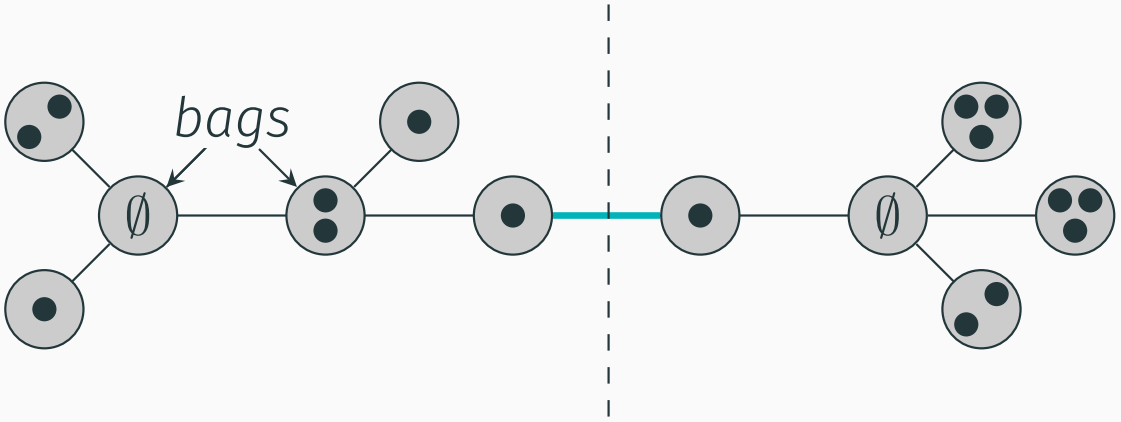

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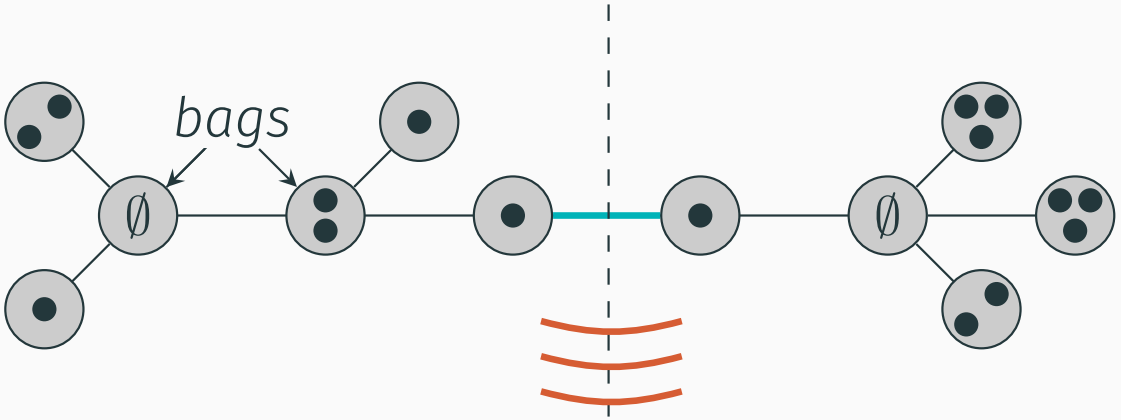

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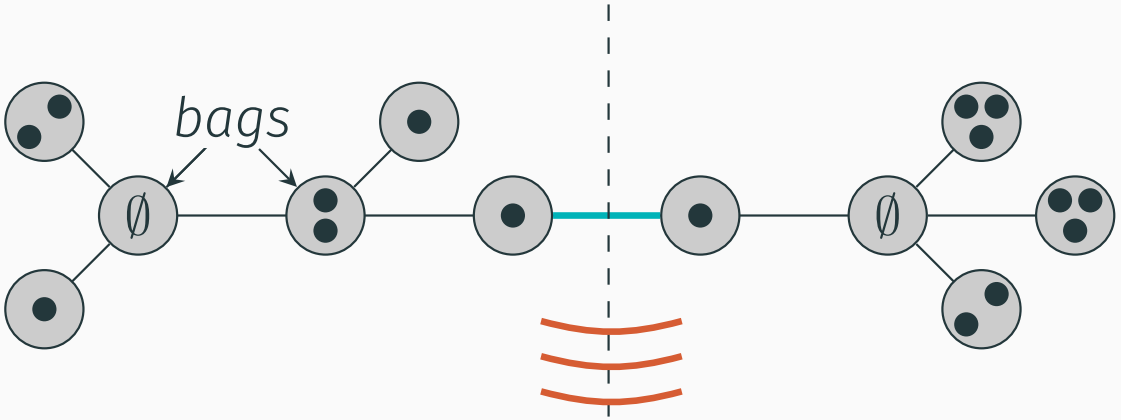

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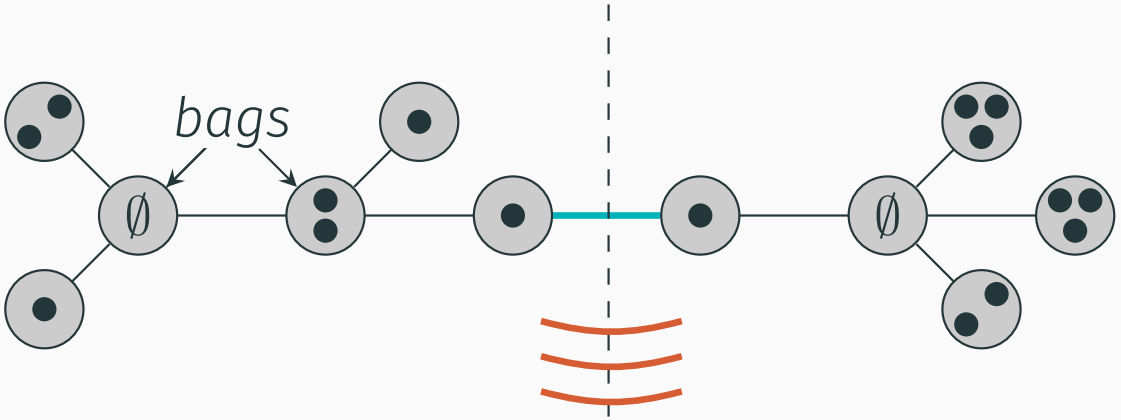

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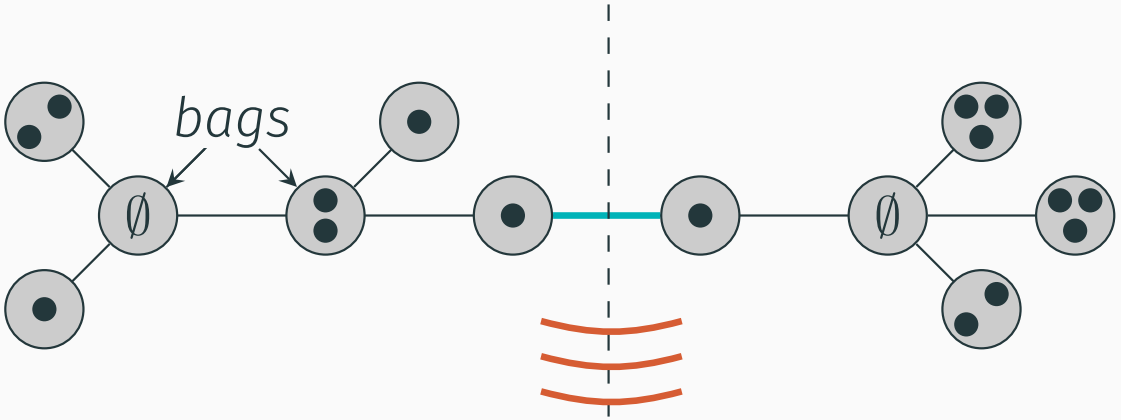

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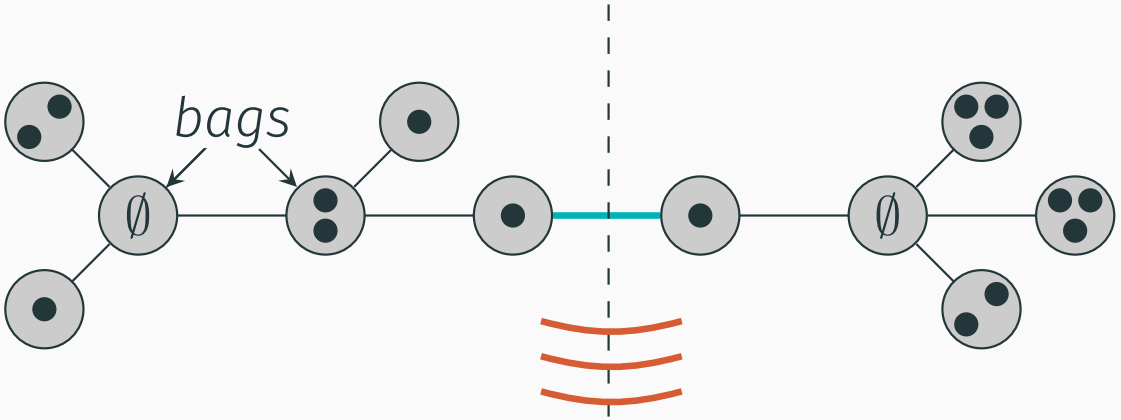

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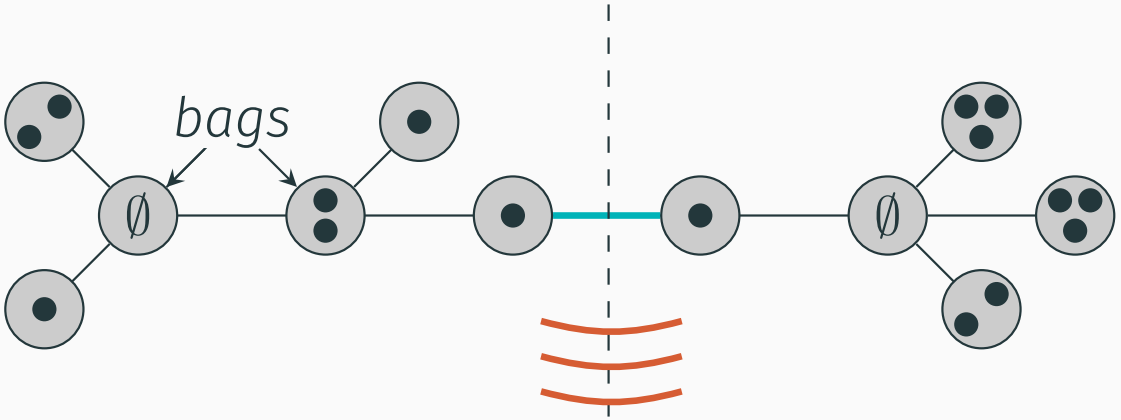

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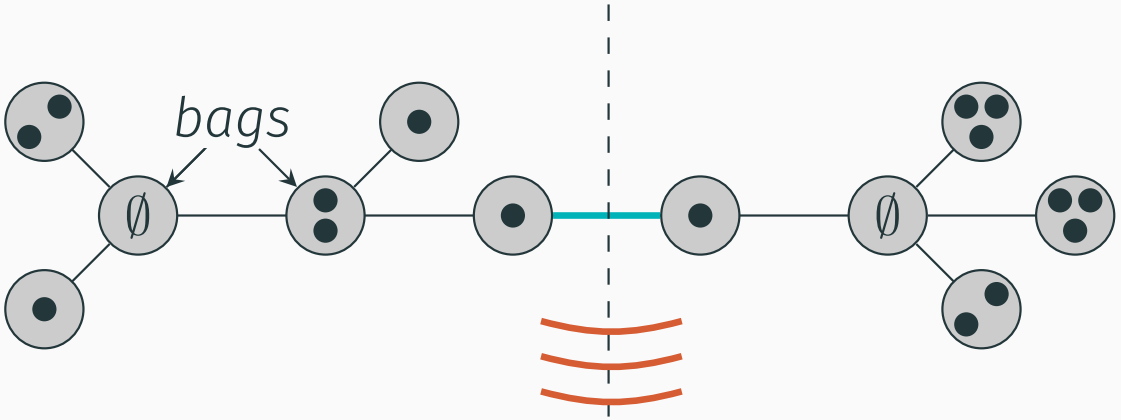

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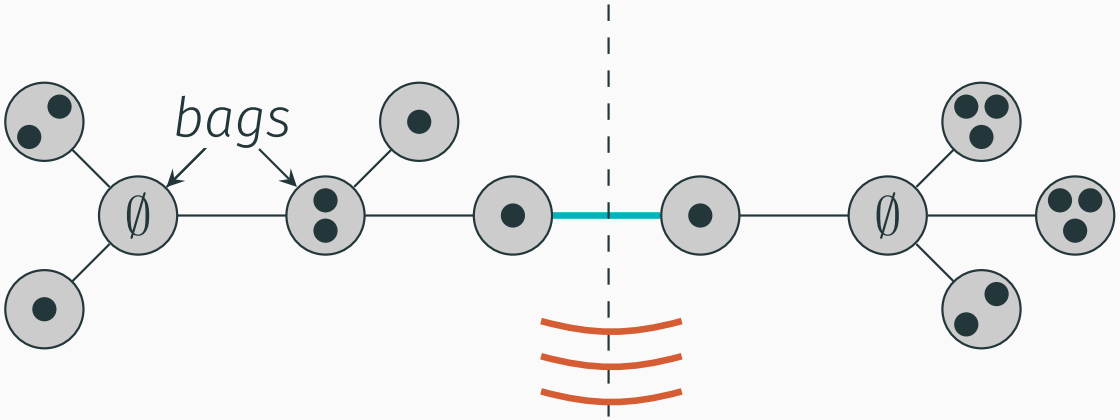
Main message: **tcw** $\sim$ edge version of **tw**, shows small edge separators

Tree-cut decompositions



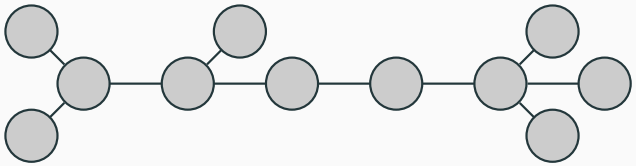
tree-cut decomposition:
partition of $V(G)$ into *bags*
+possibly some empty bags
organized along a tree T

- **adhesion of** $e \in V(T)$**:** edges of G between the two parts of $T - e$
- **width of the decomposition** for 3-edge-connected graphs**:**
  $\max\left\{\max_{r \in E(T)} |\mathrm{adh}(e)|, \quad \max_{v \in V(T)}(|\mathrm{bag}(v)| + \Delta(v))\right\}$
  grows with adhesion size, bag size, and degree in T
- **tcw**(G)**:** min width of a decomposition of G

Main message: **tcw** $\sim$ edge version of **tw**, shows small edge separators, well behaved wrt immersions

A Menger-like property of tree-cut width

A tree-cut decomposition $(T, \mathcal{X})$ is **lean** if:



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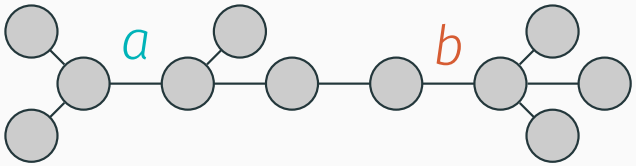

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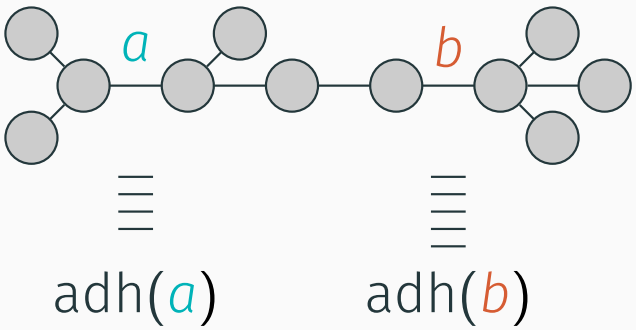

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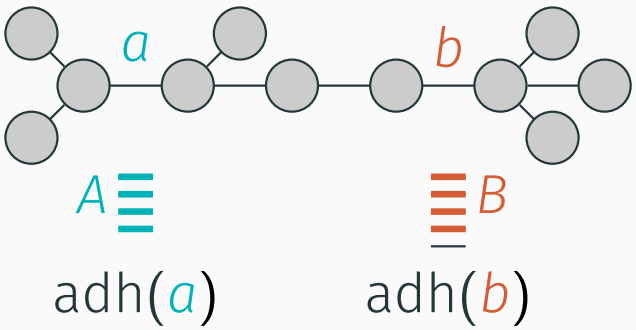

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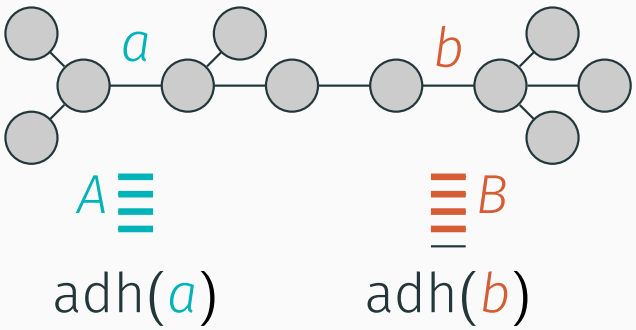

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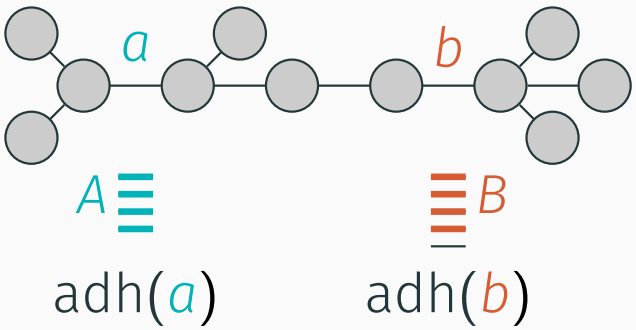

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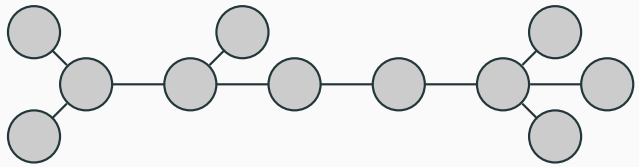
Theorem (Giannopoulou, Kwon, R., Thilikos, 2019)

Every G has a tree-cut decomposition of width $\mathbf{tcw}(G)$ *that is lean.*

Every graph has a lean optimal tree-cut decomposition

start with any
opt. tree-cut dec.



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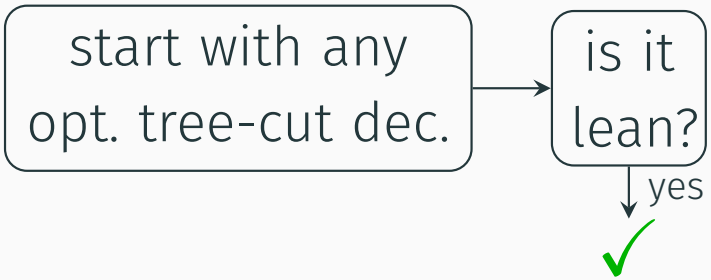

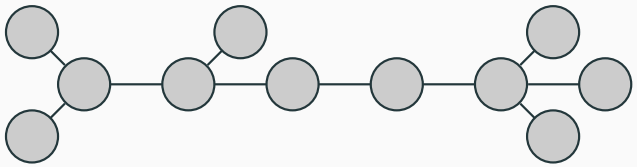

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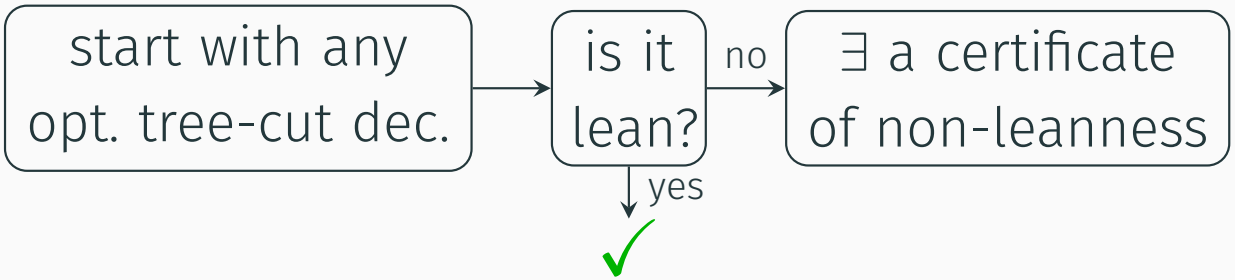

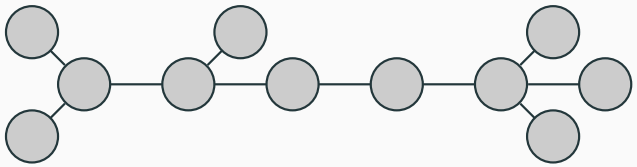

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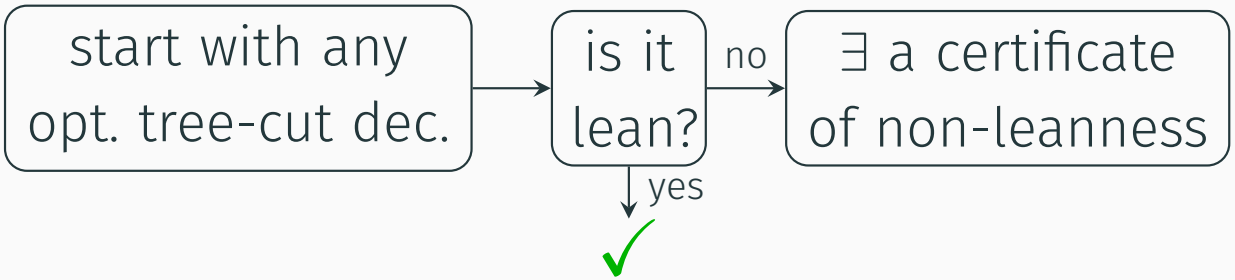

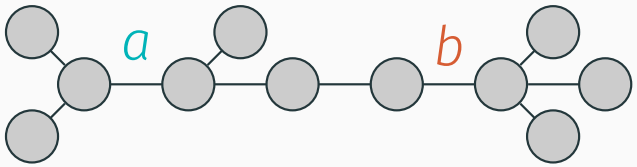

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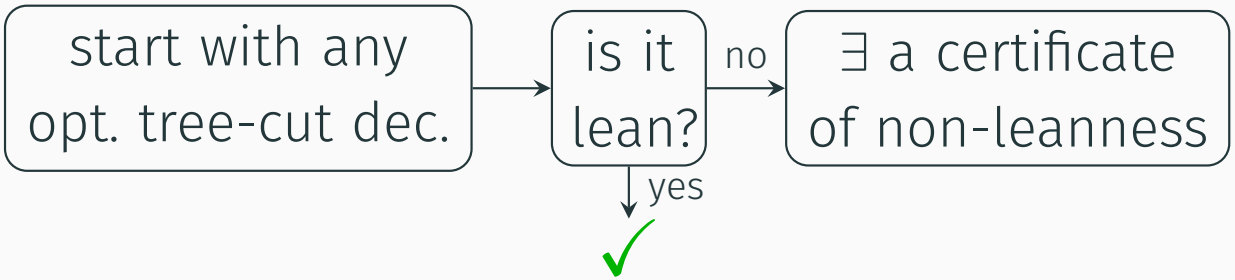

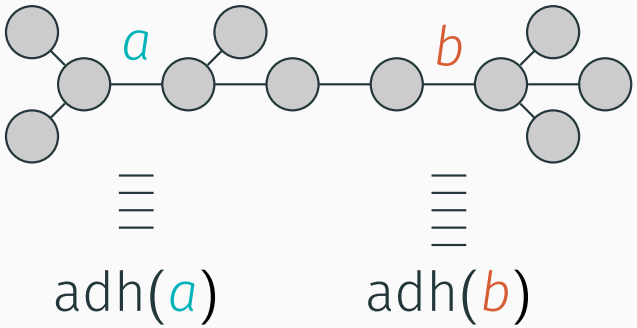

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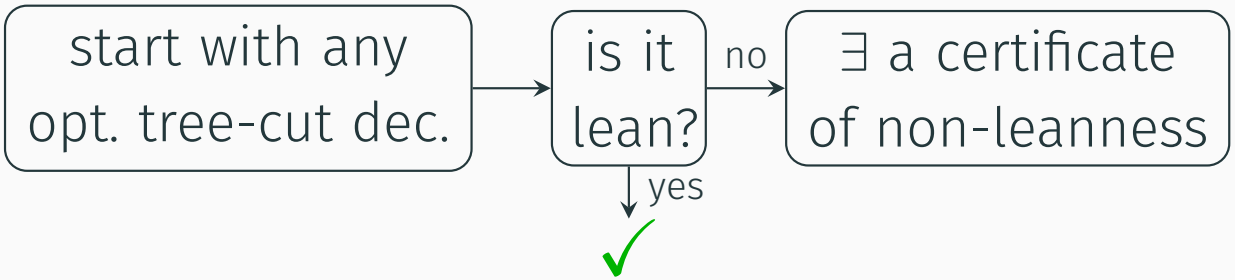

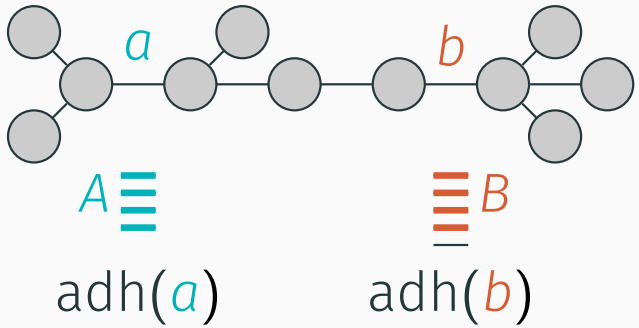

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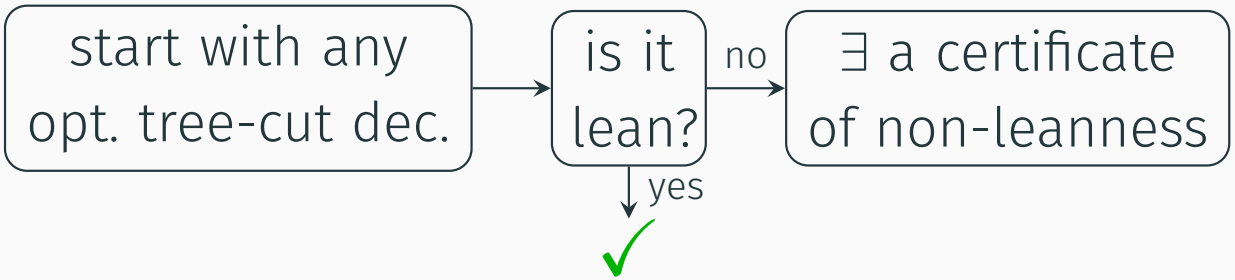

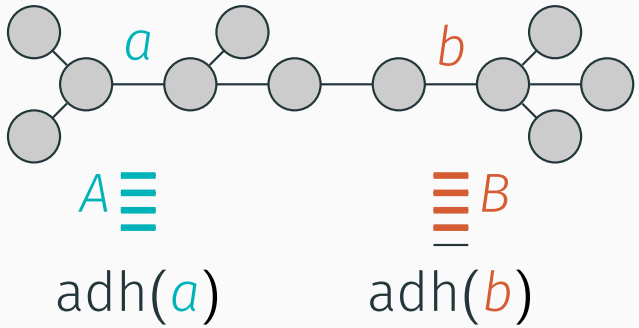

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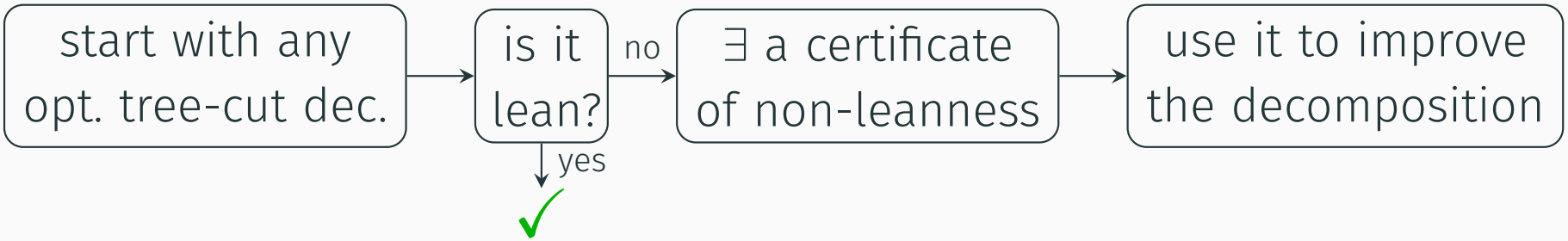

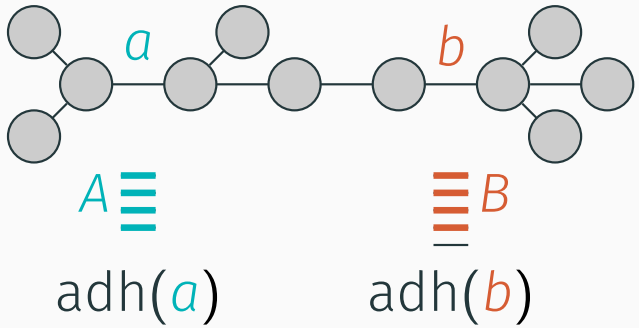

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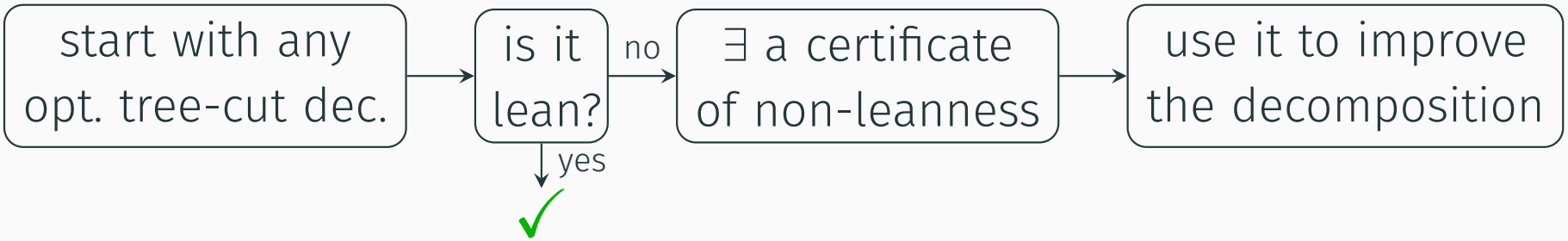

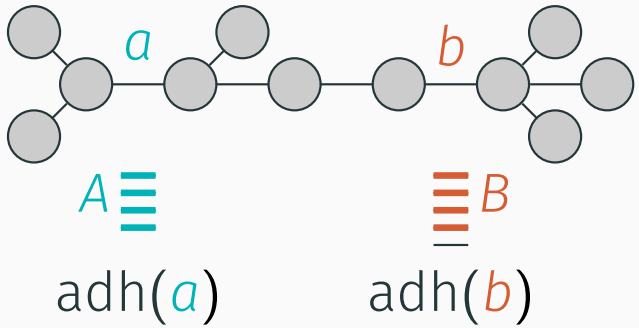

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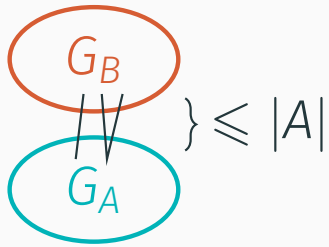

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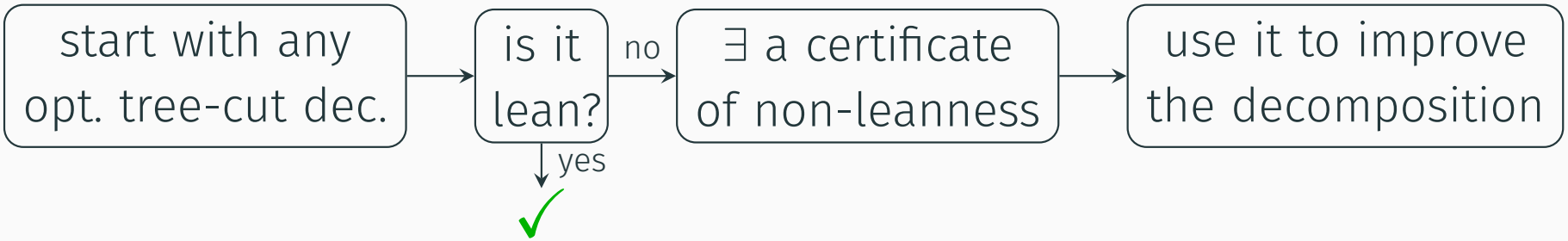

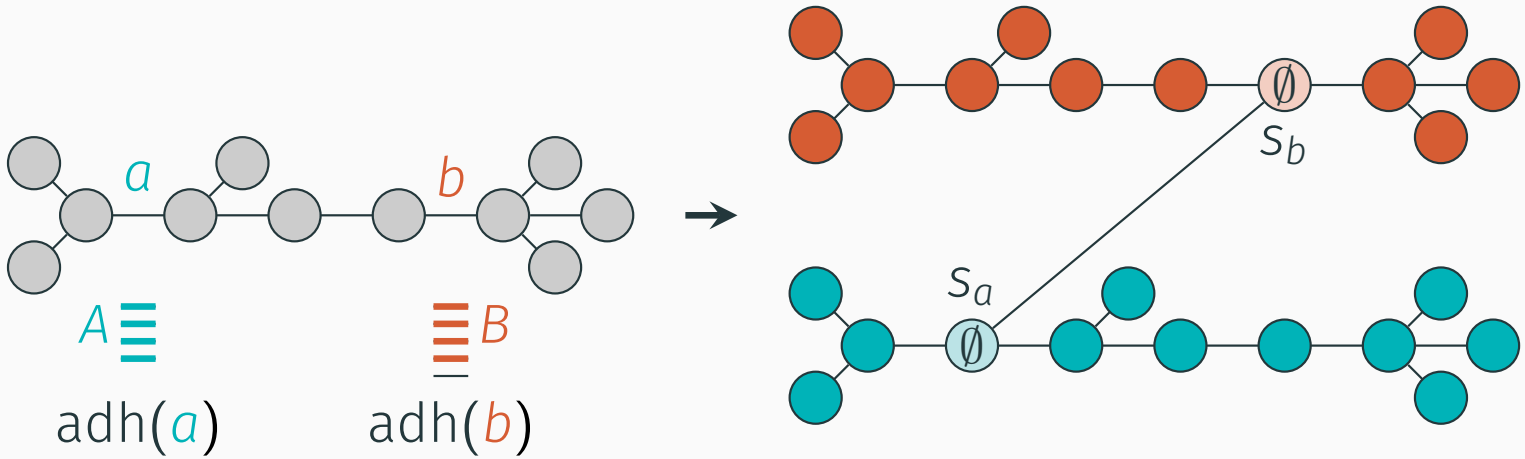

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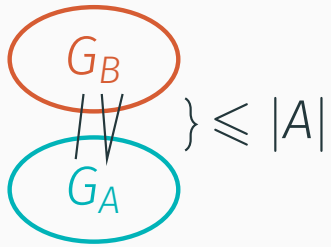

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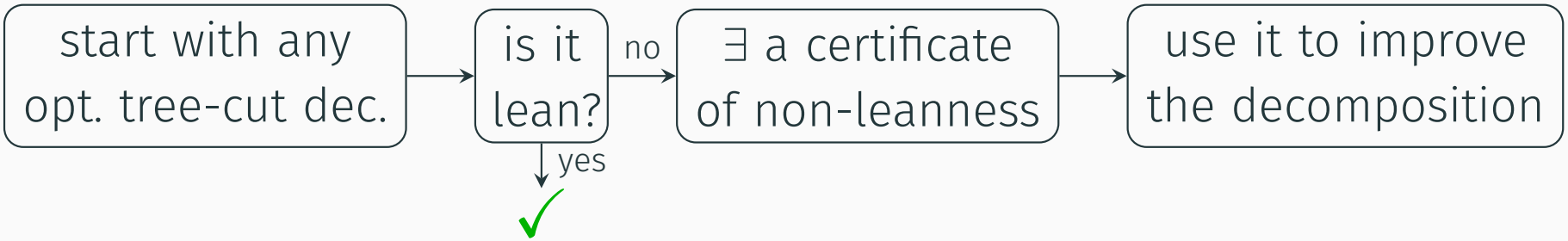

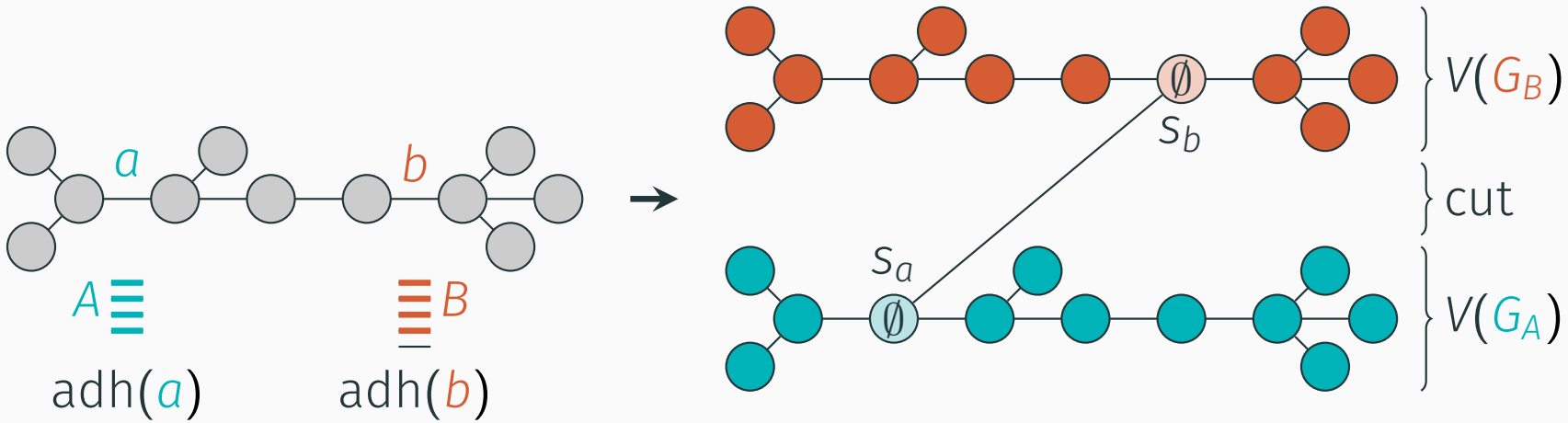

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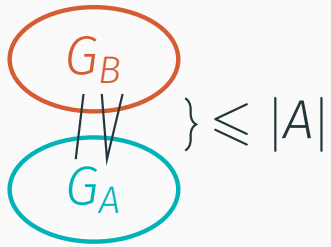

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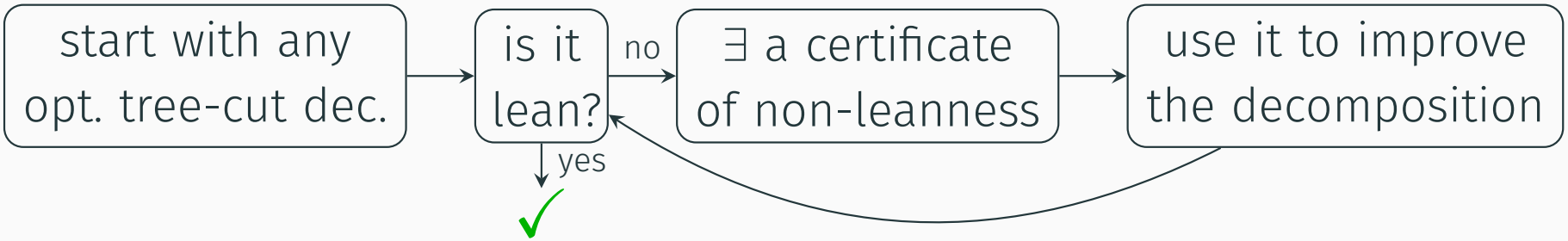

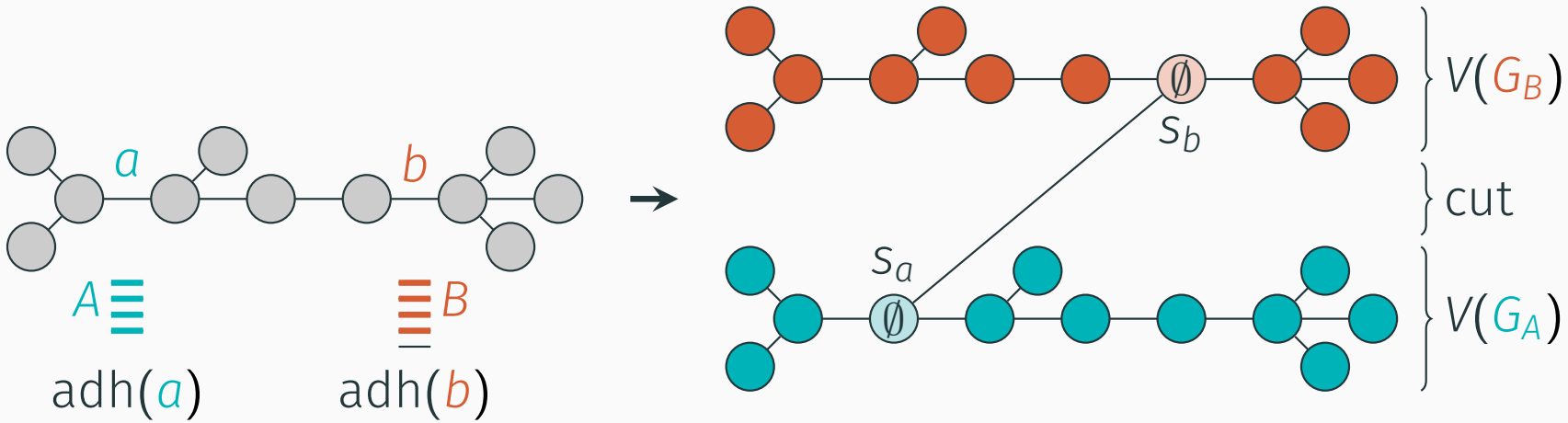

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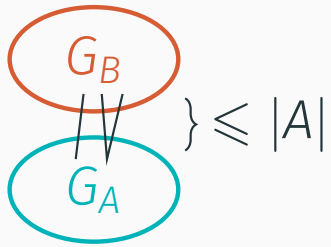

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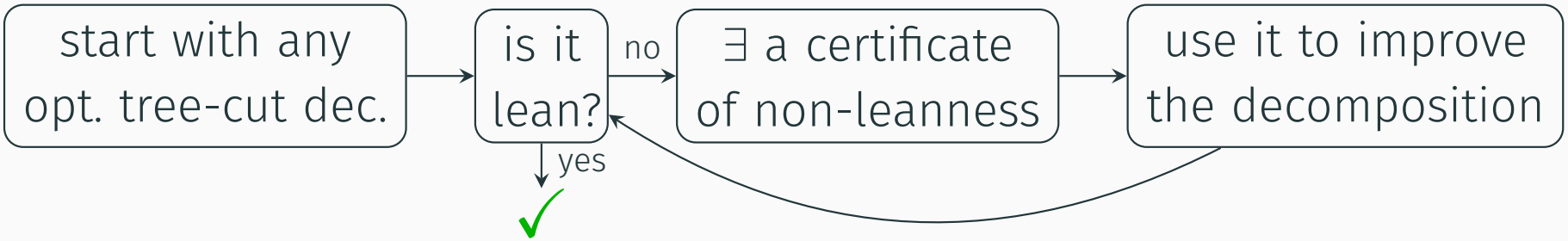

Two statements to prove:

- the tree-cut width does not increase
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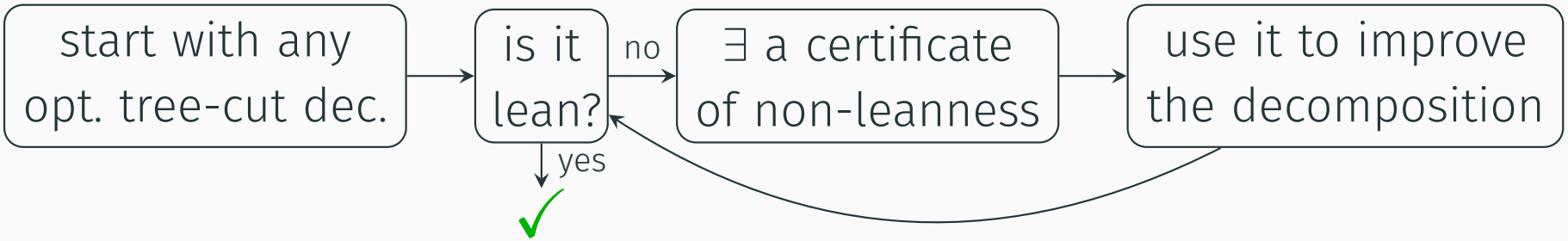

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Not included in this talk!

Application 1: obstructions bounds

obstruction to $\mathcal{F}$: minimal structure present in non-members of $\mathcal{F}$

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minor-obstructions to planarity: K_5 or $K_{3,3}$ (minor-minimal non-planar)

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- G minor-obstruction of $\mathcal{TW}_{\leqslant k} \Rightarrow |G| = 2^{2^{O(k^5)}}$ (Lagergren 1998)
- G minor-obstruction of $\mathcal{PW}_{\leqslant k} \Rightarrow |G| = 2^{O(k^4)}$ (idem)

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Theorem (Giannopoulou, Kwon, R., Thilikos, 2019)

One can construct an algorithm that runs in time $f(k) \cdot n$.

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- reduce the contribution of k in the algorithm complexity

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Thank you for your attention!