

A TIGHT ERDŐS-PÓSA FUNCTION FOR WHEEL MINORS

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Abstract. Let W_t denote the wheel on $t + 1$ vertices. We prove that for every integer $t \geq 3$ there is a constant $c = c(t)$ such that for every integer $k \geq 1$ and every graph G , either G has k vertex-disjoint subgraphs each containing W_t as a minor, or there is a subset X of at most $ck \log k$ vertices such that $G - X$ has no W_t minor. This is best possible, up to the value of c . We conjecture that the result remains true more generally if we replace W_t with any fixed planar graph H .

1. Introduction

Let H be a fixed graph. An H -model \mathcal{M} in a graph G is a collection $\{S_x \subseteq G : x \in V(H)\}$ of vertex-disjoint connected subgraphs of G such that S_x and S_y are linked by an edge in G for every edge $xy \in E(H)$. The *vertex set* $V(\mathcal{M})$ of \mathcal{M} is the union of the vertex sets of the subgraphs in the collection. Two H -models \mathcal{M} and \mathcal{M}' are *disjoint* if $V(\mathcal{M}) \cap V(\mathcal{M}') = \emptyset$.

Let $\nu_H(G)$ be the maximum number of pairwise disjoint H -models in G . Let $\tau_H(G)$ be the minimum size of a subset $X \subseteq V(G)$ such that $G - X$ has no H -model. Clearly, $\nu_H(G) \leq \tau_H(G)$. We say that the *Erdős-Pósa property holds for H -models* if there exists a *bounding function* $f : \mathbb{N} \rightarrow \mathbb{R}$ such that

$$\tau_H(G) \leq f(\nu_H(G))$$

for every graph G .

Robertson and Seymour [16] proved that the Erdős-Pósa property holds for H -models if and only if H is planar. Their original bounding function was exponential. However, this has been significantly improved by recent breakthrough results of Chekuri and Chuzhoy [3, 4] on the polynomial Grid Theorem.

Theorem 1.1 (Chekuri and Chuzhoy [3]). *There exist integers $a, b, c \geq 0$ such that for every planar graph H on h vertices, the Erdős-Pósa property holds for H -models with*

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bounding function

$$f(k) = ah^b \cdot k \log^c(k + 1).$$

If we consider H to be fixed and focus solely on the dependence on k —which is the point of view we take in this paper—[Theorem 1.1](#) gives a $O(k \log^c k)$ bounding function. This is remarkably close to being best possible: If H is planar with at least one cycle, then there is a $\Omega(k \log k)$ lower bound on bounding functions. This follows easily from the original lower bound of Erdős and Pósa for the case where H is a triangle [\[6\]](#). Alternatively, one can see this by considering n -vertex graphs G with treewidth $\Omega(n)$ and girth $\Omega(\log n)$ (which exist [\[13\]](#)), and notice that $\tau_H(G) = \Omega(n)$ (because removing one vertex decreases treewidth by at most one, and $G - X$ has treewidth $O(1)$ when $G - X$ has no H -minor, by the Grid Theorem) while $\nu_H(G) = O(n/\log n)$ (because each H -model contains a cycle). Thus, a $O(k \log^c k)$ bound is optimal, up to the value of c . While no explicit value for c is given in [\[3\]](#), a quick glance at the proof suggests that it is at least a double-digit integer. In this paper, we put forward the conjecture that a $O(k \log^c k)$ bound holds with $c = 1$.

Conjecture 1.2. *For every planar graph H , the Erdős-Pósa property holds for H -models with a $O(k \log k)$ bounding function.*

If true, [Conjecture 1.2](#) would completely settle the growth rate of the Erdős-Pósa functions for H -models for all planar graphs H (up to the constant factor depending on H). That is, if H is planar with at least one cycle, then the $O(k \log k)$ bound would match the $\Omega(k \log k)$ lower bound mentioned above. And if H is a forest, it is already known that the right order of magnitude is $O(k)$, see [\[9\]](#).

Going back to the $O(k \log^c k)$ bound of Chekuri and Chuzhoy [\[3\]](#), one could initially hope that a value of $c = 1$ could be obtained by optimizing the various steps of their proof. However, any constant c obtained using their general approach necessarily satisfies $c \geq 2$. This is because they obtain [Theorem 1.1](#) as a corollary from the following result.

Theorem 1.3 (Chekuri and Chuzhoy [\[3\]](#)). *There exist integers $a', b', c' \geq 0$ such that for all integers $r, k \geq 1$, every graph G of treewidth at least*

$$a' r^{b'} \cdot k \log^{c'}(k + 1)$$

has k vertex-disjoint subgraphs G_1, \dots, G_k , each of treewidth at least r .

Now, if we fix a planar graph H and if G is such that $\nu_H(G) < k$, then G cannot have k vertex-disjoint subgraphs each of treewidth at least r , where $r = r(H)$ is a constant such that every graph with treewidth at least r contains an H minor. Note that $r(H)$ exists by the Grid Theorem of Robertson and Seymour [\[16\]](#). Thus, the above theorem implies that G has treewidth $O(k \log^{c'} k)$. The authors of [\[3\]](#) then apply a standard divide-and-conquer approach on an optimal tree decomposition, and obtain a $O(k \log^c k)$ bound on $\tau_H(G)$ (see [\[3, Lemma 5.4\]](#)). This unfortunately results in an extra $\log k$ factor;

$c = c' + 1$. On the other hand, we must have $c' \geq 1$ in [Theorem 1.3](#), as shown again by n -vertex graphs with treewidth $\Omega(n)$ and girth $\Omega(\log n)$. Hence, $c \geq 2$. Therefore, one needs a different approach to prove [Conjecture 1.2](#).

As a side remark, it is natural to conjecture that we could take $c' = 1$ in [Theorem 1.3](#) (at least, if we forget about the precise dependence on r):

Conjecture 1.4. *There is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all integers $r, k \geq 1$, every graph G of treewidth at least*

$$f(r) \cdot k \log(k + 1)$$

has k vertex-disjoint subgraphs G_1, \dots, G_k , each of treewidth at least r .

As it turns out, this conjecture is implied by our [Conjecture 1.2](#): It suffices to take H to be the $r \times r$ -grid, which has treewidth r . Then either $\nu_H(G) \geq k$, in which case we are done, or $\nu_H(G) < k$, and then there is a subset X of $O(k \log k)$ vertices such that $G - X$ has no H -minor, and hence $G - X$ has treewidth at most $g(r)$ for some function g by the Grid Theorem. Adding X to all bags of an optimal tree decomposition of $G - X$, we deduce that G has treewidth $O(k \log k)$. Thus, this is another motivation to study [Conjecture 1.2](#).

While [Conjecture 1.2](#) remains open in general, it is known to hold for some specific graphs H . For example, the original Erdős-Pósa theorem [\[6\]](#) is simply the assertion that [Conjecture 1.2](#) holds when H is a triangle. This was recently extended to the case where H is an arbitrary cycle [\[7\]](#) (see also [\[1, 14\]](#) for related results). The conjecture also holds when H is a multigraph consisting of two vertices linked by a number of parallel edges [\[2\]](#).

Our main result is that [Conjecture 1.2](#) holds when H is a wheel. A *wheel* is a graph obtained from a cycle by adding a new vertex adjacent to all vertices of the cycle. We denote by W_t the wheel on $t + 1$ vertices.

Theorem 1.5. *For each integer $t \geq 3$, the Erdős-Pósa property holds for W_t -models with a $O(k \log k)$ bounding function.*

We remark that our main theorem implies all the aforementioned special cases. This is because the existence of a $O(k \log k)$ bounding function for H -models is preserved under taking minors of H (see [Lemma 2.7](#)). Our result also have the following consequence.

Corollary 1.6. *For every $t \in \mathbb{N}$ there is a function $g : \mathbb{N} \rightarrow \mathbb{N}$ with $g(k) = O(k \log k)$ such that every $(k \cdot W_t)$ -minor free graph has treewidth at most $g(k)$.*

The rest of the paper is organized as follows. In the next section we present some general lemmas about H -models. Since these lemmas are valid for arbitrary planar graphs H , they may be useful in attacking [Conjectures 1.2](#) and [1.4](#). In [Section 3](#), we specialize to

the case of wheels and prove our main theorem. We conclude with some open problems in [Section 4](#).

2. General Tools

In this paper, our graphs are simple (no parallel edges nor loops). Let H, G be two graphs. We let $|G|$ denote $|V(G)|$. We assume the reader is familiar with the notions of graph minors, tree decompositions, and treewidth (see Diestel [5] for an introduction to the area). We let $\text{tw}(G)$ denote the treewidth of G .

An H -transversal of G is a set X of vertices of G such that $G - X$ has no H -model. A graph is *minor-minimal* for a given property if it satisfies the property and none of its proper minors does.

We use the following results. The first is an extension of a classic result of Kostochka [11] and Thomason [18], where in addition the size of the K_t -model is logarithmic. (For definiteness, all logarithms are in base 2 in this paper.)

Theorem 2.1 ([12], see also [8, 17]). *There is a function $\varphi(t) = O(t\sqrt{\log t})$ such that, if an n -vertex graph has average degree at least $\varphi(t)$, then it contains a K_t -model on $O(\log n)$ vertices.*

The second is a theorem of Fomin, Lokshtanov, Misra and Saurabh [10], whose original purpose was to show that the algorithmic problem of finding a minimum-size H -transversal admits a polynomial-size kernel when H is planar.

Theorem 2.2 ([10]). *For every planar graph H , there is a polynomial π such that for every $k \in \mathbb{N}$, every graph G with $\tau_H(G) = k$ and minor-minimal with this property satisfies $|G| \leq \pi(k)$.*

2.1. Minimal counterexamples to the Erdős-Pósa property. Let H be a graph and let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a function. We say that a graph G is a *minimal counterexample* to the Erdős-Pósa property for H -models with bounding function f if the following properties hold:

- (i) $\tau_H(G) > f(\nu_H(G))$;
- (ii) subject to the above constraint, $\nu_H(G)$ is minimum;
- (iii) subject to the above constraints, $|G|$ is minimum;
- (iv) subject to the above constraints, $|E(G)|$ is minimum.

Notice that the two last requirements of the above definition imply that a minimal counterexample is a minor-minimal graph satisfying requirements (i) and (ii). The following lemma gives a bound on the size of minimal counterexamples.

Lemma 2.3. *Let H be a planar graph and let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a polynomial non-decreasing function. Then there is a polynomial ρ such that, for every minimal counterexample G to the Erdős-Pósa property for H -models with bounding function f , we have $|G| \leq \rho(\nu_H(G))$.*

Proof. Let $k := \nu_H(G)$. Let us first show that $\tau_H(G) = \lfloor f(k) \rfloor + 1$. Let $v \in V(G)$. Observe that $\tau_H(G) \leq \tau_H(G - v) + 1$. By minimality of G , $\tau_H(G - v) \leq f(\nu_H(G - v))$. As $G - v$ is a minor of G , we also have $\nu_H(G - v) \leq \nu_H(G)$. We deduce $\tau_H(G) \leq f(k) + 1$. Since G is a counterexample, we also have $\tau_H(G) > f(k)$. It follows that $\tau_H(G) = \lfloor f(k) \rfloor + 1$.

Now, $\nu_H(G') \leq \nu_H(G)$ holds for every proper minor G' of G , and thus $\tau_H(G') < \tau_H(G)$ (otherwise G would not be a minimal counterexample). Hence G is minor-minimal with the property that $\tau_H(G) = \lfloor f(k) \rfloor + 1$. By [Theorem 2.2](#) we obtain $|G| \leq \pi(\lfloor f(k) \rfloor + 1)$ where π is the polynomial given by that theorem. Therefore, it suffices to take $\rho : t \mapsto \pi(\lfloor f(t) \rfloor + 1)$. \square

Informally, the following result, originally proved in [\[9\]](#), states that if a graph G has a large H -minor-free induced subgraph with a small ‘boundary’, then there is a smaller graph G' where the values of ν_H and τ_H are the same.

Theorem 2.4 ([\[9\]](#)). *For every planar graph H , there is a computable function $g' : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every graph G , if J is an H -minor-free induced subgraph of G such that exactly p vertices of J have a neighbor in $V(G) \setminus V(J)$ and $|J| \geq g'(p)$, then there exists a graph G' such that $\tau_H(G') = \tau_H(G)$, $\nu_H(G') = \nu_H(G)$, and $|G'| < |G|$.*

We can use [Theorem 2.4](#) to upper bound the size of H -minor-free induced subgraphs in minimal counterexamples as follows.

Corollary 2.5. *For every planar graph H , there is a computable and non-decreasing function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that, if G is a minimal counterexample to the Erdős-Pósa property for H -models with bounding function f for some function $f : \mathbb{N} \rightarrow \mathbb{R}$, then every H -minor free induced subgraph J of G that has exactly p vertices with a neighbor in $V(G) \setminus V(J)$ satisfies $|J| < g(p)$.*

Proof. Let g' be the function in [Theorem 2.4](#). We define the function $g : \mathbb{N} \rightarrow \mathbb{N}$ as follows: $g(k) = \max_{i \in \{0, \dots, k\}} g'(i)$. Notice that $g(k) \geq g'(k)$ holds for every $k \in \mathbb{N}$ and that g is non-decreasing. Now, suppose that G is a graph having an H -minor free induced subgraph J with exactly p vertices having a neighbor in $V(G) \setminus V(J)$ and such that $|J| \geq g(p)$. Then, since $g(p) \geq g'(p)$, by [Theorem 2.4](#) there is a graph G' such that $\tau_H(G') = \tau_H(G)$, $\nu_H(G') = \nu_H(G)$, and $|G'| < |G|$. In particular, G cannot be a minimal counterexample to the Erdős-Pósa property for H -models for any bounding function f , a contradiction. \square

2.2. Interplays between treewidth and the Erdős-Pósa property. Given a planar graph H , the standard approach to show that H -models satisfy the Erdős-Pósa property is to first note that $k \cdot H$ (the disjoint union of k copies of H) is also planar. Thus, if $\nu_H(G) < k$ for a graph G , then the treewidth of G is bounded by a function of k and H , by the Grid Theorem [16]. Then one can use a tree decomposition of small width to find a small H -transversal of G . This was first used by Robertson and Seymour [16, Theorem 8.8] in their original proof (see also [19, Theorem 3] and the survey [15, Section 3]). It was subsequently used by several authors to obtain improved bounding functions, most notably by Chekuri and Chuzhoy [3] when deriving their [Theorem 1.1](#) from [Theorem 1.3](#).

As it was already mentioned in the introduction when discussing [Conjecture 1.4](#), the reverse direction holds as well: A bounding function for H -models translates directly to an upper bound on the treewidth of $(k \cdot H)$ -minor free graphs, up to an additive term depending only on H :

Lemma 2.6. *Let H be a planar graph, let f be a bounding function for H -models, and let $c = c(H)$ be a constant such that $\text{tw}(G) \leq c$ for every H -minor free graph G . Then, for every $k \geq 1$, every $(k \cdot H)$ -minor free graph G has treewidth at most $f(k - 1) + c$.*

Proof. Let G be a graph not containing $k \cdot H$ as a minor. Since $\nu_H(G) \leq k - 1$ and f is a bounding function for H -models, we deduce $\tau_H(G) \leq f(k - 1)$. That is, G has a set X of at most $f(k - 1)$ vertices such that $G - X$ is H -minor free. By definition of c , we have $\text{tw}(G - X) \leq c$. Then $\text{tw}(G) \leq c + |X| \leq c + f(k - 1)$, as desired. \square

Thus combining our main result with [Lemma 2.6](#) gives [Corollary 1.6](#) (stated in the introduction).

We also include the following lemma, which states that if H' is a minor of H , then a bounding function for H' -models can be easily obtained from a bounding function for H -models.

Lemma 2.7. *Let H be a fixed planar graph, let f be a bounding function for H -models, and let $c = c(H)$ be a constant such that $\text{tw}(G) \leq c$ for all H -minor free graphs G . If H' is a minor of H with q connected components, then $k \mapsto f(k) + (qk - 1)(c + 1)$ is a $O(f(k))$ bounding function for H' -models.*

Proof. Let G be a graph with $\nu_{H'}(G) \leq k$. As H' is a minor of H , we deduce $\nu_H(G) \leq k$. By definition of f , there is a set X of at most $f(k)$ vertices such that $G - X$ is H -minor free. Hence $\text{tw}(G - X) \leq c$. Theorem 8.8 in [16] provides the following upper-bound on τ in graphs of bounded treewidth:

$$\tau_{H'}(G - X) \leq (qk - 1)(c + 1).$$

Then, $\tau_{H'}(G) \leq \tau_{H'}(G - X) + |X| \leq (qk - 1)(c + 1) + f(k)$. Finally, since $f(k) \geq k$, we deduce $(qk - 1)(c + 1) + f(k) = O(f(k))$. \square

3. The Proof for Wheels

In this section, we prove our main theorem:

Theorem 1.5. *For each integer $t \geq 3$, the Erdős-Pósa property holds for W_t -models with a $O(k \log k)$ bounding function.*

Proof. To keep track of the dependencies between the constants that we use, we define them here. Recall that t denotes the number of spokes of the wheel that we are considering. Let φ and φ' be constants such that every n -vertex graph of average degree at least φ has a K_{t+1} -model on at most $\varphi' \log n$ vertices (both φ and φ' depend on t , see [Theorem 2.1](#)). Let $\alpha, \beta \geq 1$ be constants such that $\rho(n) \leq \alpha n^\beta$, for every $n \in \mathbb{N} \setminus \{0\}$, where ρ is the polynomial of [Lemma 2.3](#) for $H = W_t$. Let g denote the function from [Corollary 2.5](#) for $H = W_t$.

We then set

$$\begin{aligned} c_1 &= g(2t\varphi^2), & p &= g(2c_1\varphi^2), & c_2 &= 4p, \\ \sigma &= \max \{3\varphi'c_2, 2c_2 + tp, (2t^2p + 1)(c_2 + 2c_1\varphi^2)\}, & \text{and} \\ \gamma &= \sigma(\beta + \log \alpha). \end{aligned}$$

Observe that we have $t < c_1 < p < c_2$. Let $f(k) := \gamma \cdot k \log(k + 1)$, for every $k \in \mathbb{N}$. We show that the Erdős-Pósa property holds for W_t -models with bounding function f .

Arguing by contradiction, let G be a minimal counterexample to the Erdős-Pósa property for W_t -models with bounding function f . Let $k := \nu_{W_t}(G)$. Then $k \geq 1$ and $|G|$ is polynomial in k by [Lemma 2.3](#). That is, letting $n := |G|$, we have $n \leq \alpha k^\beta$, for the constants α and β defined above.

We first show that G cannot contain a W_t -model of logarithmic size.

Claim 3.1. *G has no W_t -model of size at most $\sigma \log n$.*

Proof. Towards a contradiction, we consider a W_t -model \mathcal{M} of size at most $\sigma \log n$. Notice that $\log n \leq (\beta + \log \alpha) \log(k + 1)$ can be deduced from the aforementioned upper-bound on n . Since $\nu_{W_t}(G - V(\mathcal{M})) \leq k - 1$, by minimality of G ,

$$\begin{aligned} \tau_{W_t}(G) &\leq |V(\mathcal{M})| + \tau_{W_t}(G - V(\mathcal{M})) \\ &\leq \sigma \log n + f(k - 1) \\ &\leq \gamma \log(k + 1) + \gamma(k - 1) \log k \\ &\leq \gamma k \log(k + 1) \leq f(k). \end{aligned}$$

However, this contradicts the fact that G is a minimal counterexample to the Erdős-Pósa property for W_t -models with bounding function f . \square

Note that since $\nu_{W_t}(G) \geq 1$, we have $n \geq t + 1 \geq 2$, and thus $\log n \geq 1$ (recall that all logarithms are in base 2). Thus, [Claim 3.1](#) implies in particular that G has no W_t -model of size at most σ .

Let \mathcal{C} be a maximum-size collection of vertex-disjoint cycles in G whose lengths are in the interval $[c_1, c_2]$. Let \mathcal{P} be a maximum-size collection of vertex-disjoint paths of length p in $G - V(\mathcal{C})$, where $V(\mathcal{C}) := \bigcup_{C \in \mathcal{C}} V(C)$. (In this paper, the length of a path is defined as its number of edges.) Finally, let \mathcal{R} be the collection of components of $G - (V(\mathcal{C}) \cup V(\mathcal{P}))$ and let $V(\mathcal{R}) := V(G) - (V(\mathcal{C}) \cup V(\mathcal{P}))$, where $V(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} V(P)$. We point out that the cycles in \mathcal{C} and the paths in \mathcal{P} are subgraphs of G but not necessarily induced subgraphs of G , while the components in \mathcal{R} are induced subgraphs of G . We call the elements of $\mathcal{C} \cup \mathcal{P} \cup \mathcal{R}$ *pieces*.

Observe that, by maximality of \mathcal{P} , every path in a piece of \mathcal{R} has length at most $p - 1$. This implies that each such piece is W_t -minor free. Indeed, observe that if such a piece R has a W_t -model then R contains a subgraph consisting of a cycle C and a rooted tree T such that T has at most t leaves, $V(T) \cap V(C) = \emptyset$, and the leaves of T collectively have at least t neighbours in C . The cycle C has at most p vertices, and each root-to-leaf path in T has at most p vertices. Thus, this gives a W_t -model with at most $(t + 1)p$ vertices. However, this contradicts [Claim 3.1](#) since $(t + 1)p \leq \sigma$.

Similarly, each piece in \mathcal{C} (in \mathcal{P} , respectively) has at most c_2 (p , respectively) vertices, and these vertices induce a W_t -minor free subgraph of G ; otherwise, there would exist a W_t -model of size at most c_2 (resp. p), again a contradiction to [Claim 3.1](#) since $p \leq c_2 \leq \sigma$. These facts will be used often in the rest of the proof.

We say that two distinct pieces K and K' *touch* if some edge of G links some vertex of K to some vertex of K' . Note that, by construction, two distinct pieces in \mathcal{R} cannot touch. A piece is said to be *central* if it is a cycle in \mathcal{C} , a path in \mathcal{P} , or a piece in \mathcal{R} that touches at least 2φ other pieces. In the next paragraph, we define two auxiliary simple graphs H_s (for small degrees) and H_b (for big degrees) that model how the central pieces are connected through the noncentral pieces. To keep track of the correspondence between the edges of H_s and the noncentral pieces, we put labels on some of these edges.

Initialize both H_s and H_b to the graph whose set of vertices is the set of central pieces and whose set of edges is empty. For each pair of central pieces that touch in G , add an (unlabeled) edge between the corresponding vertices in both H_s and H_b .

Next, while there is some noncentral piece $R \in \mathcal{R}$ that touches two central pieces K and K' that are not yet adjacent in H_b , call \mathcal{Z}_R the set of central pieces that touch R and do the following:

- (1) add all (unlabeled) edges to H_b between pieces of \mathcal{Z}_R (not already present in H_b). This creates a clique on vertex set \mathcal{Z}_R in H_b , some of whose edges might have already been there before.
- (2) choose a piece $K \in \mathcal{Z}_R$ such that the number of newly added edges of H_b incident to K is maximum. Add to H_s every edge that links K to another piece of \mathcal{Z}_R (not already present in H_s), and label it with R . This creates a star centered at K in H_s with all its edges labeled with the noncentral piece R .

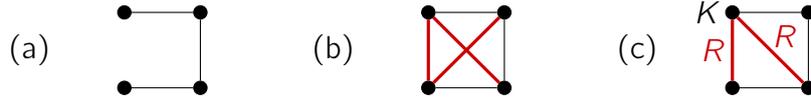


Figure 1. Construction of H_s and H_b . (a): the vertices of a set \mathcal{Z}_R in H_s and H_b before step (1). (b): $H_b[\mathcal{Z}_R]$ after step (1). (c): $H_s[\mathcal{Z}_R]$ after step (2).

The edges added during these steps are depicted in thick red lines in the example of Figure 1. By construction, H_s is a subgraph of H_b . These graphs have the following two crucial properties.

Claim 3.2. *If H_s has a W_t -model of size q , then G has a W_t -model of size at most $3c_2q$.*

Proof. Suppose that H_s has a W_t -model of size q . Then there exists a subgraph $M_s \subseteq H_s$ with q vertices that can be contracted to W_t . We may assume that the average degree of M_s is at most that of W_t , and hence at most 4. That is, $|E(M_s)| \leq 2|M_s|$. From the subgraph M_s , we construct a subgraph $M \subseteq G$ that can be contracted to M_s , and thus also to W_t .

First, for each central piece $K \in V(M_s) \cap (\mathcal{C} \cup \mathcal{P})$, we add all its vertices to M , as well as $|K| - 1$ edges from K in such a way that the subgraph of M induced by $V(K)$ is connected. For each central piece $K \in V(M_s) \cap \mathcal{R}$, we choose some vertex $v_K \in K$ and add it to M . This creates at most $c_2|M_s| = c_2q$ vertices in M .

Second, for each unlabeled edge KK' of M_s with $K, K' \in V(M_s) \cap (\mathcal{C} \cup \mathcal{P})$, we choose some edge of G linking K to K' and add it to M . This does not create any new vertex in M .

Third, for each edge KK' of M_s that has not been considered so far, we add to M a path linking some vertex of $V(M) \cap V(K)$ to some vertex of $V(M) \cap V(K')$, as follows. If the edge KK' is not labeled, then exactly one of its endpoints is a central piece in \mathcal{R} , say K . The path we add to M links v_K to some vertex of K' and is a subgraph of K , except for the last edge and last vertex. Thus, this path has at most $p - 1$ internal vertices. If the edge KK' is labeled with the noncentral piece $R \in \mathcal{R}$, then this edge is part of a star in M_s whose edges are all labeled with R . We may assume without loss of generality that K is the center of this star. In this case, the path we add to M links

some vertex of K to some vertex of K' and has all its internal vertices in R . Thus, this path has at most p internal vertices.

In total, the addition of these paths to M creates at most $p|E(M_s)| \leq 2c_2|M_s| = 2c_2q$ new vertices in M . The resulting subgraph M has at most $c_2|M_s| + p|E(M_s)| \leq 3c_2q$ vertices. By construction, M can be contracted to M_s , as desired. \square

Claim 3.3. *The average degree of H_b is at most φ times the average degree of H_s . The degree of each central piece of \mathcal{R} in H_s is at least 2φ .*

Proof. First, note that edges that appear in H_b but not in H_s must be labeled. Let $R \in \mathcal{R}$ be a noncentral piece, and let r be the number of pieces in $\mathcal{C} \cup \mathcal{P}$ it touches. By definition of noncentral pieces, $r < 2\varphi$. When R is treated in the algorithm used to construct H_b and H_s , if q new edges are added to H_b , then one of the pieces touched by R is incident to at least $2q/r > q/\varphi$ of these new edges and thus at least q/φ new edges are added to H_s . This proves the first part of the claim.

By definition, a piece K of \mathcal{R} is central if it touches at least 2φ other pieces. As two pieces of \mathcal{R} cannot touch, K touches at least 2φ pieces from $\mathcal{C} \cup \mathcal{P}$, that is, at least 2φ other central pieces. Then in the first step of the construction of H_s , all edges have been added from K to these pieces. \square

If the average degree of H_s is at least φ , then by definition of φ and φ' at the beginning of the proof, H_s has a K_{t+1} -model of size at most $\varphi' \log |H_s|$, and thus in particular a W_t -model of size at most $\varphi' \log |H_s|$. By [Claim 3.2](#), this gives a W_t -model of size at most $3\varphi'c_2 \log |H_s| \leq 3\varphi'c_2 \log n$ in G , a contradiction to [Claim 3.1](#) since $3\varphi'c_2 \leq \sigma$.

Thus, the average degree of H_s is smaller than φ . Hence, by [Claim 3.3](#), the average degree of H_b is smaller than φ^2 . Then strictly more than half of the central pieces have degree less than 2φ in H_s (otherwise at least half of the vertices of H_s have degree at least 2φ , a contradiction to the fact that H_s has average degree less than φ). Similarly, strictly more than half of the central pieces have degree less than $2\varphi^2$ in H_b . Thus there is a central piece whose degree in H_s is less than 2φ , and whose degree in H_b is less than $2\varphi^2$. Choose such a piece K . By [Claim 3.3](#) (second part of the statement), K is either in \mathcal{C} or in \mathcal{P} .

In the rest of the proof we use the fact that K has degree less than $2\varphi^2$ in H_b to find a W_t -model of size at most $\sigma \log n$, contradicting [Claim 3.1](#).

If K is the unique central piece in $\mathcal{C} \cup \mathcal{P}$, then $V(K)$ is a W_t -transversal of G since each piece in \mathcal{R} is W_t -minor free. Thus $\tau_{W_t}(G) \leq |K| \leq c_2 \leq f(k)$, contradicting the fact that G is a counterexample.

For each central piece K' adjacent to K in H_b , we consider the collection $\mathcal{R}_{K,K'}$ of all noncentral pieces $R \in \mathcal{R}$ that touch both K and K' ($\mathcal{R}_{K,K'}$ might be empty). Then we consider the subgraph $G_{K'}$ of G induced by $V(K) \cup V(K') \cup V(\mathcal{R}_{K,K'})$.

Let q be an integer equal to t if $K \in \mathcal{C}$, to c_1 if $K \in \mathcal{P}$.

Our next goal is to show that for every central piece K' adjacent to K in H_b , there exists a set of strictly less than q vertices that separates K from K' in $G_{K'}$. Thus fix a piece K' adjacent to K in H_b . By Menger's theorem, it suffices to show that the maximum number of vertex-disjoint K - K' paths in $G_{K'}$ is strictly less than q . Assume for contradiction that $G_{K'}$ contains q vertex-disjoint K - K' paths.

By taking the paths to be as short as possible, we may assume that only their endpoints are in K and K' , all their internal vertices are in pieces in $\mathcal{R}_{K,K'}$, and each such path intersects at most one piece in $\mathcal{R}_{K,K'}$ and thus has length at most $p + 1$.

Assume first that $K \in \mathcal{C}$, and so $q = t$. In this case $G_{K'}$ contains a small W_t -model as follows. Let T be a smallest tree in K' containing all the endpoints of our paths in K' . The center vertex of the wheel is then modeled by the union of T and the t K - K' paths (minus their endpoints in K). If $K' \in \mathcal{C} \cup \mathcal{P}$, then obviously $|V(T)| \leq c_2$, and the model thus has at most $2c_2 + tp$ vertices. If $K' \in \mathcal{R}$, then $|V(T)| \leq tp$ since each path in K' has length at most $p - 1$; moreover, $\mathcal{R}_{K,K'}$ is empty in this case, implying that the model has at most $c_2 + tp$ vertices. Therefore, in both cases the resulting model has at most $2c_2 + tp$ vertices, which contradicts [Claim 3.1](#) since $2c_2 + tp \leq \sigma$.

Assume now that $K \in \mathcal{P}$. Since $t < c_1$, by the previous case we may assume that $K' \in \mathcal{P} \cup \mathcal{R}$. Since there are $q = c_1$ vertex-disjoint K - K' paths in $G_{K'}$, two of these paths intersect K on two vertices that are at distance at least $c_1 - 1$ on the path K , which allows us to construct a cycle in $G_{K'}$ of length at least c_1 and at most $4p$: The cycle might use all the vertices of K and at most p vertices of K' , which is at most $2p$ vertices, and might intersect at most two pieces of $\mathcal{R}_{K,K'}$, using at most p vertices in each of them. This is a contradiction to the maximality of \mathcal{C} : The length of this cycle is in the interval $[c_1, c_2]$ and yet the cycle is vertex disjoint from all cycles in \mathcal{C} .

Therefore, for each K' adjacent to K in H_b , there exists a set $X(K')$ with less than q vertices meeting all the K - K' paths in $G_{K'}$.

Let $X := \bigcup_{K'} X(K')$ where the union is taken over all central pieces K' adjacent to K in H_b . Note that $|X| \leq 2q\varphi^2$ since there are at most $2\varphi^2$ such pieces K' and for every K' we have $|X(K')| \leq q$.

We also note that X separates K from all other central pieces in G . To see this, let K'' be a central piece distinct from K and let Q be a K'' - K path in G . Let K' be the last

central piece that Q meets before reaching K . Then Q contains a $K'-K$ path that is contained in $G_{K'}$, which must contain a vertex from X .

Let J be the union of the components of $G - X$ that intersect K . Observe that $V(K)$ is not completely included in X : If $K \in \mathcal{C}$, then $|K| \geq c_1 > 2t\varphi^2 \geq |X|$, and if $K \in \mathcal{P}$, then $|K| = p > 2c_1\varphi^2 \geq |X|$. Thus J is not empty. Note also that X separates J from the rest of the graph.

Suppose that the subgraph G' of G induced by $X \cup V(J)$ is W_t -minor free. Thus, by [Corollary 2.5](#), $|G'| < g(|X|)$. We deduce

$$\begin{aligned} |X| + |J| &< g(|X|) && \text{since } |G'| = |X| + |J| \\ |K| &< g(|X|) && \text{since } |J| \geq |K| - |X| \\ |K| &< g(2q\varphi^2) && \text{since } g \text{ is non-decreasing.} \end{aligned}$$

Hence, if $K \in \mathcal{C}$, then $c_1 \leq |K| < g(2t\varphi^2)$, and if $K \in \mathcal{P}$, then $p \leq |K| < g(2c_1\varphi^2)$. Since $c_1 = g(2t\varphi^2)$ and $p = g(2c_1\varphi^2)$, we get a contradiction in both cases.

Thus, we may assume that G' contains a W_t -model. Let M be a subgraph of G' containing W_t as a minor with $|V(M)| + |E(M)|$ minimum. (We remark that here we take M to be a subgraph instead of just a model as before because we will need to consider the edges of that subgraph in the proof.) To finish the proof, it is now enough to prove that M has at most $\sigma \log n$ vertices, since by [Claim 3.1](#) this will give us the desired contradiction.

Let $R(J) := J[V(\mathcal{R})]$. Thus $R(J)$ consists of a number of disjoint pieces or subgraphs of pieces of \mathcal{R} . Note that M might use all vertices of $V(K) \cup X$ (which is fine); what we need to prove is that it cannot use too many vertices of $R(J)$.

First, suppose that M is fully contained in some piece of \mathcal{R} . Since the vertices of M can be covered with $2t$ paths, and each path in the piece has length less than p , it follows that $|M| \leq 2tp \leq \sigma$ and we are done.

Thus we may assume that M is not contained in some piece of \mathcal{R} , and thus in particular M is not contained in $R(J)$ (since M is connected). By the above remark, we also know that each component of $M[V(R(J))]$ contains at most $2tp$ vertices. Since M has maximum degree at most t (by minimality of $|V(M)| + |E(M)|$), there are at most $t|V(K) \cup X|$ edges of M with one endpoint in $V(K) \cup X$ and the other in $R(J)$. Hence M intersects $R(J)$ on at most $2t^2p|V(K) \cup X|$ vertices. Therefore, M has at most $2t^2p|V(K) \cup X| + |V(K) \cup X|$ vertices. Since $|V(K) \cup X| \leq |K| + |X| \leq c_2 + 2c_1\varphi^2$, we deduce that $|M| \leq (2t^2p + 1)(c_2 + 2c_1\varphi^2) \leq \sigma$, as desired. \square

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4. Conclusion

One obvious extension of our result for wheels would be to prove it for all planar graphs. Note that the first steps of our proof work for any such H : Starting with G a minimal counterexample for some bounding function and some value k , we have that G has $n \leq k^{O(1)}$ vertices. Thus, in order to get a contradiction, it is enough to show that there is a $O(\log n)$ -size H -model in G . Unfortunately, the rest of our proof is specific to wheels and does not generalize.

Let us mention another possible extension of our result. Strengthening the $O(k \log k)$ bound from [7], Mousset, Noever, Škorić, and Weissenberger [14] recently showed that there is a constant $c > 0$ such that for every $\ell \geq 3$, models of the ℓ -cycle C_ℓ have the Erdős-Pósa property with bounding function $ck \log k + \ell k$. In particular, the constant c in front of the $k \log k$ term is independent of ℓ . We expect that a similar property holds for wheels:

Conjecture 4.1. *There is a constant $c > 0$ and a function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that for all integers $t \geq 3$, W_t -models have the Erdős-Pósa property with bounding function*

$$ck \log k + g(t)k.$$

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