Sur quelques généralisations polynomiales de la décomposition modulaire.

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Outline of the Thesis

Part I. Generalizations of Modular Decomposition

- Homogeneous relations and modular decomposition.
- Umodular decomposition: a new point of view.

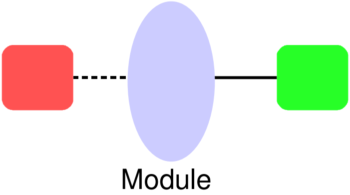
Part II. Efficient Algorithms

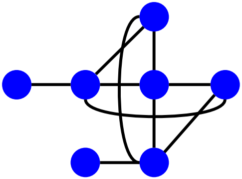
- Overlap Components.
- NLC-2 graphs recognition algorithm.

Outline

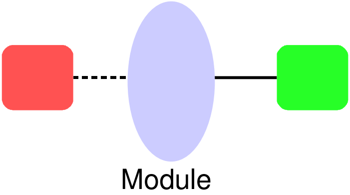
Basic definitions

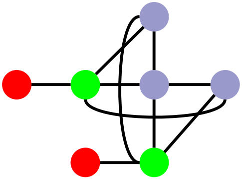
Modules and Modular decomposition





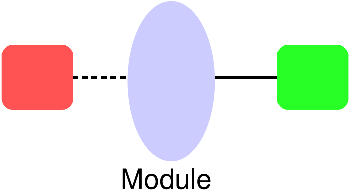
Modules and Modular decomposition

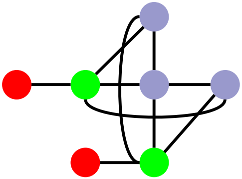




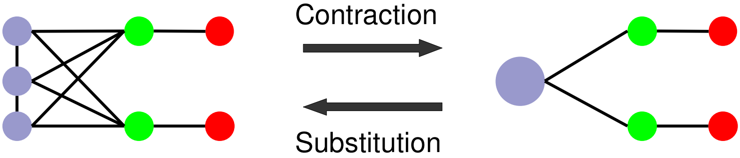
Basic definitions

Modules and Modular decomposition





Substitution / Contraction



Generalizing

Why and How ?

Modular decomposition

- Social sciences,
- Bioinformatics,
- Computer science
- ...

Known generalizations

Role coloring:Everett & Borgatti'91
proven NP-complete by Fiala & Paulusma'05 that this problem

Desired properties of the generalizations

- Polynomial computation
- Good structural properties
- Decomposition tree
- Compact encoding of the family
- ...

Module

A *module* is a set of vertices which have the same neighborhood outside.

Summary

Module

A *module* is a set of vertices which have the same neighborhood outside.

Role

A *"role"* in a graph is a set of vertices which plays the same role.

Summary

Module

A *module* is a set of vertices which have the same neighborhood outside.

Homogeneous Relations

Homogeneous relation is something in between...

Role

A *"role"* in a graph is a set of vertices which plays the same role.

Homogeneous Relations

Homogeneous Relations

Definition

Let X be a finite set. A Homogeneous Relation is a collection of triples on X, noted $H(a|b,c)$ fullfiling the following properties:

1. **Reflexivity**: $H(a|x,x)$,
2. **Symmetry**: $H(a|x,y) \equiv H(a|y,x)$ and
3. **Transitivity**: $H(a|x,y)$ **and** $H(a|y,z) \Rightarrow H(a|x,z)$

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$H(a|b, c)$

a is said to be homogeneous with respect to b and c,
or
a does not distinguish b from c.

An example

$X = \{a, b, c, d\}$

Let H be defined as follows:

$$
\begin{aligned}
&H(a|c, d), H(a|b, b),\\
&H(b|a, c), H(b|c, d), H(b|a, d),\\
&H(c|a, a), H(c|b, b), H(c|d, d),\\
&H(d|b, c), H(d|a, a).
\end{aligned}
$$

Homogeneous relation ∼ Equivalence relations

To each element x of X, thanks to the transitivity property we can associate an equivalence relation H_x defined on $X \setminus \{x\}$

Homogeneous Relations: Representation

Equivalence relation

$$\begin{aligned} H_a &= \{b\}, \{c, d\} \\ H_b &= \{a, c, d\} \\ H_c &= \{a\}, \{b\}, \{d\} \\ H_d &= \{a\}, \{b, c\} \end{aligned}$$

Matrix representation

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 2 & 0 \end{pmatrix} \end{array}$$

Graphic Homogeneous Relations

Graphic

A homogeneous relations H is graphic if there exists a graph G s.t.

$$\forall v \text{ of } V(G),\ H_v = N(v), \overline{N(v)}$$

Theorem

A homogeneous relation H *is graphic iff* $\forall\ x, y, z \in X,\ H$ *does not contain:*

1. $H(x|y,z) \wedge H(y|x,z) \wedge \overline{H(z|x,y)}$
2. $\overline{H(x|y,z)} \wedge \overline{H(y|x,z)} \wedge \overline{H(z|x,y)}$

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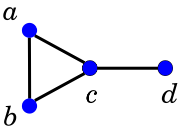


$$\begin{aligned} H_a &= \{b,c\},\{d\} \\ H_b &= \{a,c\},\{d\} \\ H_c &= \{a,b,d\} \\ H_d &= \{a,b\},\{c\} \end{aligned}$$

Local Congruence

Maximum number of classes associated to an element.

Example

$$
\begin{aligned}
H_a &= \{b\},\{c,d\} \\
H_b &= \{a,c,d\} \\
H_c &= \{a\},\{b\},\{c\} \\
H_d &= \{a\},\{b,c\}
\end{aligned}
$$

Modules

Definition

A Module in a Homogeneous relation H is a set M such that:

$$\forall m, m' \in M \text{ and } \forall x \in X \setminus M \text{ we have:}$$

$$H(x|mm')$$

Family of modules

$\mathscr{M}_H$: family of modules.

Example

$H_a = \{b\}, \{c, d\}; H_b = \{a, c, d\}; H_c = \{a\}, \{b\}, \{d\}; H_d = \{a\}, \{b, c\}.$

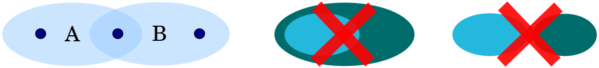
The modules are $\{a\}, \{b\}, \{c\}, \{d\}, \{a, b, c, d\}$ and $\{c, d\}$.

Basic Properties

Definition (Overlap)

Let A and B be subsets of X. A overlaps B if:

$$A \between B \equiv A \setminus B \neq \emptyset \textbf{ and } B \setminus A \neq \emptyset \textbf{ and } A \cap B \neq \emptyset$$



Proposition (Intersecting family)

Let H be a homogeneous relation on X, and let M and M' modules of H s.t. $M \between M'$ then:

$$M \cap M' \in \mathscr{M}_H \textbf{ and } M \cup M' \in \mathscr{M}_H$$

Theorem (Gabow'95)

$\mathscr{M}_H$ can be stored in space $O(n^2)$

Results on Homogeneous Relations

Modular Decomposition

On Arbitrary Homogeneous relations:

Primality	$O(n^2)$
Decomposition algorithm:	$O(n^3)$

On *good* Homogeneous relations

Primality	$O(n^2)$
Decomposition algorithm:	$O(n^2)$

Where n is the cardinality of the ground set X.

Good Homogeneous Relations

The modules family on good homogeneous relations forms a weakly partitive family.

Umodules

Umodules

Definition

Let H be a homogeneous relation defined on X, a Umodule U is a set such that:

$$\forall u, u' \in U \text{ and } \forall x, x' \in X \setminus U :$$

$$H(u|xx') \Longleftrightarrow H(u'|xx')$$

Umodules

Definition

Let H be a homogeneous relation defined on X, a Umodule U is a set such that:

$$\forall u, u' \in U \text{ and } \forall x, x' \in X \setminus U :$$

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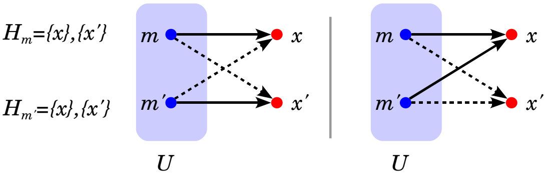


We have $\overline{H(m|xx')}$ and $\overline{H(m'|xx')}$

Basic properties

$\mathscr{U}_{\mathrm{H}}$ is the family of umodules.

Proposition (Union closed)
Let U *and* U' *be two umodules of* H *such that* $\mathrm{U} \oplus \mathrm{U}'$ *then:*

$$\mathrm{U} \cup \mathrm{U}' \in \mathscr{U}_{\mathrm{H}}$$

Crossing families

Definition (Cross)

Let A and B be two subsets of X. A crosses B if:

$$A \overset{\bullet}{\between} B \equiv A \between B \text{ and } A \cup B \neq X$$

Definition (Crossing family)

Let X be a finite set and $\mathscr{F}$ be a family of subset. $\mathscr{F}$ is said to be crossing if:

$$\forall A, B \in \mathscr{F} \text{ such that } A \overset{\bullet}{\between} B$$
$$A \cup B \textbf{ and } A \cap B \textbf{ belong to } \mathscr{F}.$$

*Homogeneous relations of Local Congruence 2 (*LC2*)*

Proposition

Let H *be a homogeneous relation of Local Congruence* 2 *(*$\mathrm{LC2}$*)and :* $\mathscr{U}_{\mathrm{H}}$ *is a crossing family.*

Homogeneous relations of Local Congruence 2 (LC2)

Proposition

Let H *be a homogeneous relation of Local Congruence* 2 (LC2)*and :* $\mathscr{U}_{\mathrm{H}}$ *is a* *crossing* *family.*

Sketch of Proof

$\cup$: from the previous proposition.

$\cap$: Let A and B be two umodules. By hypothesis we have:

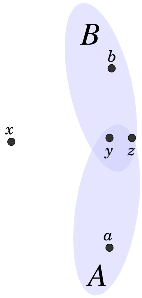
$$\mathrm{H}(a|x,b) \iff \mathrm{H}(y|x,b) \iff \mathrm{H}(z|x,b)$$

$$\mathrm{H}(b|x,a) \iff \mathrm{H}(y|x,a) \iff \mathrm{H}(z|x,a)$$

we obtain:

$$\mathrm{H}(y|a,b) \iff \mathrm{H}(z|a,b)$$

□



Homogeneous relations of Local Congruence 2 (LC2)

Proposition

Let H be a homogeneous relation of Local Congruence 2 (LC2)and : $\mathscr{U}_H$ is a crossing family.

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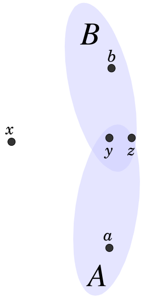


□

Theorem (Gabow'95 & Bernath'04)

Crossing families defined on a ground set X can be stored in $O(n^2)$ space.

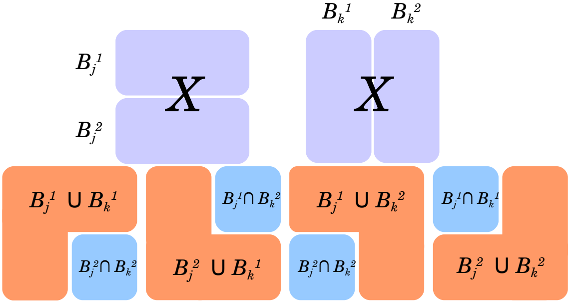
Bipartitive families

Let X be a finite set, and let $\mathscr{B} = \{\{B_1^1, B_1^2\}, \ldots, \{B_l^1, B_l^2\}\}$ be a set of bipartitions of X.

Definition (Bipartitive families – Cunningham & Edmonds'80)

$\mathscr{B}$ is a **bipartitive family** if for all overlapping bipartitions $\{B_k^1, B_k^2\}$ and $\{B_j^1, B_j^2\}$ we have:

$$\left.\begin{array}{ccc} \{B_k^1 \cup B_j^1, B_k^2 \cap B_j^2\} & , & \{B_k^1 \cup B_j^2, B_k^2 \cap B_j^1\} \\ \\ \{B_k^2 \cup B_j^1, B_k^1 \cap B_j^2\} & , & \{B_k^2 \cup B_j^2, B_k^1 \cap B_j^1\} \end{array}\right\} \in \mathscr{B}$$



Bipartitive families

Theorem (Cunningham & Edmonds'80)

Let $\mathscr{B}$ be a bipartitive family defined on X There exists a unique unrooted tree encoding $\mathscr{B}$. Its size is $O(n)$.

Self complemented families

Definition

Let H be a Homogeneous Relation defined on X. H is said to be self-complemented iff:

$$\forall \mathsf{U} \in \mathscr{U}_{\mathrm{H}},\ \mathsf{X} \setminus \mathsf{U} \textbf{ belongs to } \mathscr{U}_{\mathrm{H}}$$

Theorem

Let $\mathscr{U}_{\mathrm{H}}$ be self-complemented then $\mathscr{U}_{\mathrm{H}}$ form a bipartitive family.

Self complemented families

4 Points condition

Let H be a homogeneous relation on X. For all x, x', m, m' of X we have:

- $H(m|xx') \wedge H(m'|xx') \wedge H(x|mm') \Rightarrow H(x'|mm')$
- $\overline{H(m|xx')} \wedge \overline{H(m'|xx')} \wedge \overline{H(x|mm')} \Rightarrow \overline{H(x'|mm')}$

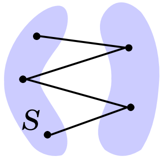
Proposition

Let H be a Homogeneous relation fullfiling the **4 points condition** *then $\mathscr{U}_H$ is self-complemented.*

Definition (Seidel switch)

Let $G = (V, E)$ be a undirected loopless graph, and $S \subseteq V$, A Seidel switch on G is the graph obtained by removing all the edges between S and $\bar{S}$, and adding all the missing edges.

Schema



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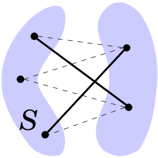

Seidel switch on graphs

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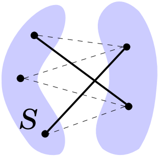

Definition (Pointed Seidel switch)

The pointed Seidel switch: $S = N(v)$

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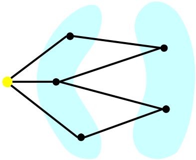

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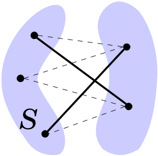

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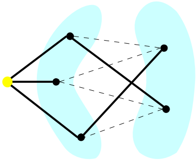

Definition (Seidel switch on Homogeneous Relations)

Let H be a Homogeneous relation of local congruence 2 defined on X, the Seidel switch at an element s is defined in the following way:

$$\forall x \in X \setminus \{s\},\ H(s) = \begin{cases} H(s)^1_x = (H^1_x \Delta H^j_s) \setminus \{s\} \\ \\ H(s)^2_x = (H^2_x \Delta H^j_s) \setminus \{s\} \end{cases}$$

... on Homogeneous Relations

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Theorem

Let H be a $LC2$ Homogeneous relation s.t. $\mathscr{U}_H$ is self-complemented. Let s an element of X, and let $U \subseteq X$ s.t. $s \in U$. Then

U *is a* **umodule** *of* H

$$\iff$$

$M = \bar{U}$ *is a module of* $H(s)$ *(Homogeneous relation on* $X - s$*).*

Algorithmic consequences

Theorem

Given a Self-complemented LC2 Homogeneous relation H *on* X*, its decomposition tree can be obtained in linear time.*

Algorithmic consequences

Theorem

Given a Self-complemented LC2 Homogeneous relation H on X, its decomposition tree can be obtained in linear time.

Sketch of Proof

- Pick an element s of X.
- Seidel switch at x.
- Compute modular decomposition of $\mathrm{H}(x)$.
- Add x carefully.

□

Umodules & Undirected graphs

Definition (Bi-Joins de Montgolfier & Rao'05)

Let $G = (V, E)$ a graph, a bi-join in G is a bipartition V_1, V_2 of V, s.t. $V_1 = \{V_{1,1}, V_{1,2}\}$ and $V_2 = \{V_{2,1}, V_{2,2}\}$ and $V_{1,i}$ is completely connected to $V_{2,i}$ and $V_{1,i}$ is completely disconnected from $V_{2,j}$.

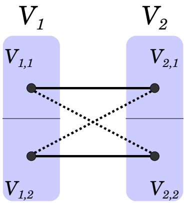
Self complement

The bi-joins of a graph are self-complemented.

Bipartitivity

Bi-joins of a graph form a bipartitive family.
There is a unique decomposition tree.

Schema

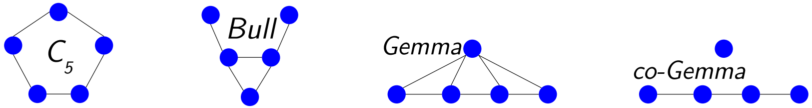


Completely decomposable graphs

Theorem (de Montgolfier & Rao'05)

The graphs completely decomposable w.r.t. Bi-join decomposition are the graphs without C_5, Bull, Gemma and co-Gemma as **induced subgraphs***.*

Forbidden Subgraphs



Computation - de Montgolfier & Rao'05

Decomposition Algorithm	*Complexity*
(1) Choose a vertex v, proceed to a Seidel switch $G * v$	$O(n+m)$
(2) Compute modular decomposition of $(G * v) \setminus v$	$O(n+m)$
(3) Turn the modular decomposition tree of $(G * v) \setminus v$ into the bi-join decomposition tree of G	$O(n+m)$

Computation - de Montgolfier & Rao'05

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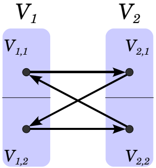
Completely Decomposable graph Recognition	
(1) Choose a vertex v, proceed to a Seidel switch $G * v$	$O(n+m)$
(2) Check if $(G * v) \setminus v$ is a cograph	$O(n+m)$

Tournaments

Umodules in tournaments



Locally transitive tournaments

A tournament $T = (V, A)$ is locally locally if for each vertex v

$T[N^+(v)]$ and $T[N^-(v)]$ are transitive tournaments.

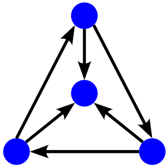
Completely decomposable tournaments

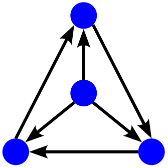
Completely decomposable tournaments are exactly locally transitive tournaments.

Forbidden characterization

A tournament $T = (V, A)$ is completely decomposable w.r.t. umodular decomposition

$$\iff$$





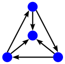
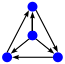
Sketch of Proof

A tournament is completely decomposable w.r.t. modular decomposition iff it is a transitive tournament. i.e. does not contain a $\overrightarrow{C_3}$

We then check that only these graphs can produce a $\overrightarrow{C_3}$, after a Seidel switch

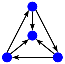
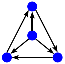
Simple Recognition Algorithm

Naive approach

- To check in $O(n^4)$ time if T contains  or  as induced sub-tournaments.
- To check for each vertex v if $T[N^+(v)]$ and $T[N^-(v)]$ are transitive tournaments. We obtain a $O(n^3)$ time algorithm.

Simple Recognition Algorithm

Naive approach

- To check in $O(n^4)$ time if T contains  or  as induced sub-tournaments.
- To check for each vertex v if $T[N^+(v)]$ and $T[N^-(v)]$ are transitive tournaments. We obtain a $O(n^3)$ time algorithm.

Linear time algorithm

1. Pick a vertex v and check $T[N^+(v)]$ (A) and $T[N^-(v)]$ (B) are transitive tournaments
2. Check that the edges between A and B do not contain a forbidden configuration.

A Simple Recognition Algorithm

Proposition (Locally Transitive Tournament)

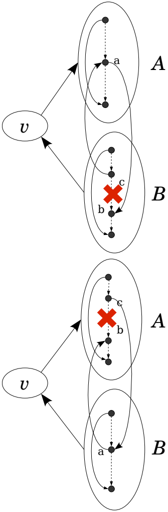
Let $T = (V, A)$ a tournament, T is locally transitive iff:

(i) $T[N^+(v)]$ *and* $T[N^-(v)]$ *are transitive tournaments,*

(ii) *If a vertex* $a \in T[N^+(v)]$ *has an outgoing neighbor* $b \in T[N^-(v)]$ *and an ingoing neighbor* $c \in T[N^-(v)]$ *then* $(b, c) \in A$.

(iii) *If a vertex* $a \in T[N^-(v)]$ *has an outgoing neighbor* $b \in T[N^+(v)]$ *and an ingoing neighbor* $c \in T[N^+(v)]$ *then* $(b, c) \in A$.

The second step of the algorithm is equivalent to check the previous proposition.



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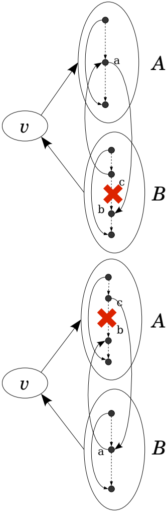

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Complexity

1. The first step is done in linear time.
2. the second step is done $O(1)$ per edge between A and B. Every edge is considered only once. Thus overall complexity is $O(n^2)$.

Isomorphism testing & Feedback Vertex Set

Isomorphism

Thanks to the unicity of the structure obtained, we are able to decide in linear time if two completely decomposable tournaments are isomorph.

Feedback Vertex Set

The Feedback Vertex Set problem is polynomial on completely decomposable tournaments.

Algorithmic results

Primality testing : $O(n^3)$

Umodular decomposition : $O(n^5)$

Overlap Components

Overlap components

The problem

Let X be a finite set, and let $\mathscr{F}=\{X_1, \ldots, X_t\}$ be a family of subsets of X
input: $\mathscr{F}$
output: Overlap Components of $\mathscr{F}$.

Size of the data is $|X| + \sum_{i=1}^{t} |X_i|$,
$n = |X|$ and $f = \sum_{i=1}^{t} |X_i|$.

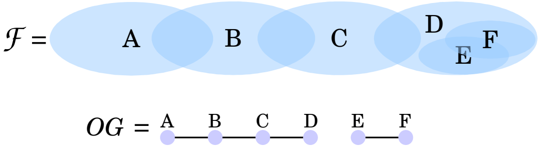
Overlap graph

Let $OG = (\mathscr{F}, E)$ be the overlap graph of $\mathscr{F}$. $uv \in E$ iff $u \between v$.

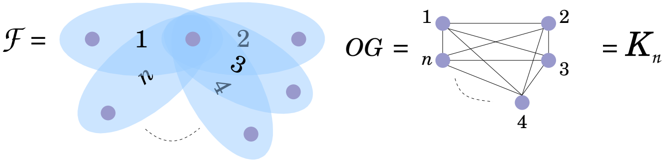
Overlap component

The overlap components of $\mathscr{F}$ are the connected components of OG.

Examples



A pathologic example



A first idea

Naive approach

First compute OG and then output the connected components.
But OG is not necessarily linear in the size of $\mathscr{F}$.

Dahlhaus's algorithm

Linear time and space algorithm to find overlap components of $\mathscr{F}$ in $O(n + f)$

Our result

A drastic simplification of Dahlhaus's algorithm.

Output a spanning subgraph of OG in time $O(n + f)$.

Homogeneous Relations

Homogeneous relations

- Characterize *"digraphic"* and *"oriented"* homogeneous relations.
- Improve modular decomposition algorithm:
  1. **Conjecture:** a $O(n + m)$ algorithm
  2. a $O(n^2)$ algorithm for arbitrary Homogeneous relations.

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Umodular decomposition

- Improve the $O(n^5)$ decomposition algorithm.
- Corresponding decomposition for directed and oriented graphs.
- Necessary and sufficient condition to characterize self-complemented families.
- Investigate **Seidel minor** properties.

Overlap Component and related problems

Overlap component

- Overlap-k component.
- recognition specific properties of the overlap graph in linear time:
  1. Bipartite,
  2. Chain, tree
  3. ...

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Partition refinement

- To "implement" *Least Common Ancestor* (LCA) with partition refinement techniques.
- Dynamic partition refinement.

NLC-*2 Graphs*

NLC-*width*

- Improve recognition algorithm to $O(n.m)$.
- What about NLC-3 graphs ?
- Is NLC-k a FPT problem ?

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Clique-width

- Clique-width $\geqslant 4$?
- Is clique-width FPT ?

Sagolun תודה Takk

Merci Thank you