NLC-2 graph recognition and isomorphism

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Abstract

NLC-width is a variant of clique-width with many application in graph algorithmic. This paper is devoted to graphs of NLC-width two. After giving new structural properties of the class, we propose a $O(n^2m)$-time algorithm, improving Johansson's algorithm [14]. Moreover, our alogrithm is simple to understand. The above properties and algorithm allow us to propose a robust $O(n^2m)$-time isomorphism algorithm for NLC-2 graphs. As far as we know, it is the first polynomial-time algorithm.

1 Introduction

NLC-width is a graph parameter introduced by Wanke [16]. This notion is tightly related to clique-width introduced by Courcelle *et al.* [2]. Both parameters were introduced to generalise the well known tree-width. The motivation on research about such *width* parameter is that, when the width (NLC-, clique- or tree-width) is bounded by a constant, then many NP-complete problems can be solved in polynomial (even linear) time, if the decomposition is provided.

Such parameters give insights on graph structural properties. Unfortunately, finding the minimum NLC-width of the graph was shown to be NP-hard by Gurski *et al.* [12]. Some results however are known. Let NLC-k be the class of graph of NLC width bounded by k. NLC-1 is exactly the class of cographs. Probe-cographs, bi-cographs and weak-bisplit graphs [9] belong to NLC-2. Johansson [14] proved that recognising NLC-2 graphs is polynomial and provided an $O(n^4 \log(n))$ recognition algorithm. Complexity for recognition of NLC-k, $k \geq 3$, is still unknown.

In this paper we improve Johansson's result down to $O(n^2m)$. Our approach relies on graph decompositions. We establish the tight links that exist between NLC-2 graphs and the so-called modular decomposition, split decomposition, and bi-join decomposition.

NLC-2 can be defined as a graph colouring problem. Unlike NLC-k classes, for $k \geq 3$, *recolouring* is useless for prime NLC-2 graphs. That allow us to propose a canonical decomposition of bi-coloured NLC-2 graphs, defined as certain bi-coloured split operations. This decomposition can be computed in $O(nm)$ time if the colouring is provided. If a graph is *prime*, there using split and bi-join decompositions, we show that there is at most $O(n)$ colourings to check. Finally, modular decomposition properties allow to reduce NLC-2 graph decomposition to prime NLC-2 graph decomposition. Section 3 explains this $O(n^2m)$-time decomposition algorithm.

In Section 4 is proposed an isomorphism algorithm. Using modular, split and bi-join decompositions and the canonical NLC-2 decomposition, isomorphism between two NLC-2 graphs can be tested in $O(n^2m)$ time.

2 Preliminaries

A graph $G = (V, E)$ is pair of a set of *vertices* V and a set of *edges* E. For a graph G, $V(G)$ denote its set of vertices, $E(G)$ its set of edges, $n(G) = |V(G)|$ and $m(G) = |E(G)|$ (or V, E, n and m if

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the graph is clear in the context). $N(x) = \{y \in V : \{x,y\} \in E\}$ denotes the *neighbourhood* of the vertex x, and $N[x] = N(v) \cup \{v\}$. For $W \subseteq V$, $G[W] = (W, E \cap W^2)$ denote the *graph induced by* W. Let A and B be two disjoint subsets of V. Then we note $A \circled{1} B$ if for all $(a,b) \in A \times B$, then $\{a,b\} \in E$, and we note $A \circled{0} B$ if for all $(a,b) \in A \times B$, then $\{a,b\} \notin E$. Two graphs $G = (V,E)$ and $G' = (V',E')$ are *isomorphic* (noted $G \simeq G'$) if there is a bijection $\varphi : V \to V'$ such that $\{x,y\} \in E \Leftrightarrow \{\varphi(x), \varphi(y)\} \in E'$, for all $u, v \in V$.

A *k-labelling* (or *labelling*) is a function $l : V \to \{1, \dots, k\}$. A *$k$-labelled graph* is a pair of a graph $G = (V,E)$ and a k-labelling l on V. It is denoted by (G,l) or by (V,E,l). Two labelled graphs (V,E,l) and (V',E',l') are isomorphic if there is a bijection $\varphi : V \to V'$ such that $\{u,v\} \in E \Leftrightarrow \{\varphi(x), \varphi(y)\} \in E'$ and $l(u) = l'(\varphi(u))$ for all $u, v \in V$.

NLC-k classes. Let k be a positive integer. The class of NLC-k graphs is defined recursively by the following operations.

- For all $i \in \{1, \dots, k\}$, $\cdot(i)$ is in NLC-k, where $\cdot(i)$ is the graph with one vertex labelled i.
- Let $G_1 = (V_1, E_1, l_1)$ and $G_2 = (V_2, E_2, l_2)$ be NLC-k and let $S \subseteq \{1, \dots, k\}^2$. Then $G_1 \times_S G_2$ is in NLC-k, where $G_1 \times_S G_2 = (V, E, l)$ with $V = V_1 \cup V_2$,
$$E = E_1 \cup E_2 \cup \{\{u,v\} : u \in V_1, v \in V_2, (l_1(u), l_2(v)) \in S\}$$
$$\text{and for all } u \in V,\ l(u) = \begin{cases} l_1(u) \text{ if } u \in V_1 \\ l_2(u) \text{ if } u \in V_2. \end{cases}$$
- Let $R : \{1, \dots, k\} \to \{1, \dots, k\}$ and $G = (V,E,l)$ be NLC-k. Then $\rho_R(G)$ is in NLC-k, where $\rho_R(G) = (V, E, l')$ such that $l'(u) = R(l(u))$ for all $u \in V$.

A graph is NLC-k if there is a k-labelling of G such that (G,l) is in NLC-k. A k-labelled graph is *NLC-k ρ-free* if it can be constructed without the ρ_R operation.

Modules and modular decomposition. A *module* in a graph is a non-empty subset $X \subseteq V$ such that for all $u \in V \setminus X$, then either $N(u) \cap X = \emptyset$ or $X \subseteq N(u)$. A module is *trivial* if $|X| \in \{1, |V|\}$. A graph is *prime* (w.r.t. modular decomposition) if all its modules are trivial. Two sets X and X' *overlap* if $X \cap X'$,$X \setminus X'$ and $X' \setminus X$ are non-empty. A module X is *strong* if there is no module X' such that X and X' overlap. Let $\mathcal{M}'(G)$ be the set of modules, let $\mathcal{M}(G)$ be the set of strong modules of G, and let $\mathcal{P}(G) = \{M_1, \dots, M_k\}$ be the maximal (w.r.t. inclusion) members of $\mathcal{M}(G) \setminus \{V\}$.

Theorem 1. *[11] Let $G = (V,E)$ be a graph such that $|V| \geq 2$. Then:*

- *if G is not connected, then $\mathcal{P}(G)$ is the set of connected components of G,*
- *if $\overline{G}$ is not connected, then $\mathcal{P}(G)$ is the set of connected components of $\overline{G}$,*
- *if G and $\overline{G}$ are connected, then $\mathcal{P}(G)$ is a partition of V and is formed with the maximal members of $\mathcal{M}' \setminus \{V\}$.*

In all cases, $\mathcal{P}(G)$ is a partition of V, and G can be decomposed into $G[M_1], \dots, G[M_k]$. The *characteristic graph* G^* of a graph G is the graph of vertex set $\mathcal{P}(G)$ and two $P, P' \in \mathcal{P}(G)$ are adjacent if there is an edge between P and P' in G (and so there is no non-edges since P and P' are two modules). The recursive decomposition of a graph by this operation gives the *modular decomposition* of the graph, and can be represented by a rooted tree, called the *modular decomposition tree*. It can be computed in linear time [15]. The nodes of the modular decomposition tree are exactly the strong modules, so in the following we make no distinction between the modular decomposition of G and $\mathcal{M}(G)$. Note that $|\mathcal{M}(G)| \leq 2 \times n - 1$. For $M \in \mathcal{M}(G)$, let $G_M = G[M]$ and G^*_M its characteristic graph.

Lemma 2. *[14] Let G be a graph. G is NLC-k if and only if every characteristic graph in the modular decomposition of G is NLC-k.*

Moreover, a NLC-k expression for G can be easily constructed from the modular decomposition and from NLC-k expressions of prime graphs. On prime graphs, NLC-2 recognition is easier:

Lemma 3. *[14] Let G be a prime graph. Then G is NLC-2 if and only if there is a 2-labelling l such that (G, l) is NLC-2 ρ-free.*

Bi-partitive family. A *bipartition* of V is a pair $\{X, Y\}$ such that $X \cap Y = \emptyset$, $X \cup Y = V$ and X and Y are both non empty. Two bipartitions $\{X, Y\}$ and $\{X', Y'\}$ *overlap* if $X \cap Y$, $X \cap Y'$, $X' \cap Y$ and $X' \cap Y'$ are non empty. A family $\mathcal{F}$ of bipartitions of V is *bipartitive* if (1) for all $v \in V$, $\{\{v\}, V \setminus \{v\}\} \in \mathcal{F}$ and (2) for all $\{X, Y\}$ and $\{X', Y'\}$ in $\mathcal{F}$ such that $\{X, Y\}$ and $\{X', Y'\}$ overlap, then $\{X \cap X', Y \cup Y'\}$, $\{X \cap Y', Y \cup X'\}$, $\{Y \cap X', X \cup Y'\}$, $\{Y \cap Y', X \cup X'\}$ and $\{X \Delta X', X \Delta Y'\}$ are in $\mathcal{F}$ (where $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$). Bipartitive families are very close to partitive families [1], which generalise properties of modules in a graph.

A member $\{X, Y\}$ of a bipartitive family $\mathcal{F}$ is *strong* if there is no $\{X', Y'\}$ such that $\{X, Y\}$ and $\{X', Y'\}$ overlap. Let T be a tree. For an edge e in the tree, $\{C^1_e, C^2_e\}$ denote the bipartition of leaves of T such that two leaves are in the same set if and only if the path between them avoids e. Similarly, for an internal node α, $\{C^1_\alpha, \ldots, C^{d(\alpha)}_\alpha\}$ denote the partition of leaves of T such that two leaves are in the same set if and only if the path between them avoid α.

Theorem 4. *[3] Let $\mathcal{F}$ be a bipartitive family on V. Then there is an unique unrooted tree T, called the* representative tree *of $\mathcal{F}$, such that the set of leaves of T is V, the internal nodes of T are labelled* `degenerate` *or* `prime`*, and*

- for every edge e of T, $\{C^1_e, C^2_e\}$ is a strong member of $\mathcal{F}$, and there is no other strong member in $\mathcal{F}$,

- for every node α labelled `degenerate`*, and for every $\emptyset \subsetneq I \subsetneq \{1, \ldots, d(\alpha)\}$, $\{\cup_{i \in I} C^i_\alpha, V \setminus \cup_{i \in I} C^i_\alpha\}$ is in $\mathcal{F}$, and there is no other member in $\mathcal{F}$.*

Split decomposition. A *split* in a graph $G = (V, E)$ is a bipartition $\{X, Y\}$ of V such that the set of vertices in X having a neighbour in Y have the same neighbourhood in Y (*i.e.*, for all $u, v \in X$ such that $N(u) \cap Y \neq \emptyset$ and $N(v) \cap Y \neq \emptyset$, then $N(u) \cap Y = N(v) \cap Y$). A *co-split* in a graph G is a split in $\overline{G}$. The family of split in a connected graph is a bipartitive family [4]. The split decomposition tree is the representative tree of the family of splits, and can be computed in linear time [5]. Let α be an internal node of the split decomposition tree of a connected graph G. For all $i \in \{1, \ldots, d(\alpha)\}$ let $v_i \in C^i_\alpha$ such that $N(v_i) \setminus C^i_\alpha \neq \emptyset$. Since G is connected, such a v_i always exists. $G[\{v_1, \ldots, v_{d(\alpha)}\}]$ denote the *characteristic graph* of α. The characteristic graph of a `degenerate` node is a complete graph or a star [4].

Bi-join decomposition. A *bi-join* in a graph is a bipartition $\{X, Y\}$ such that for all $u, v \in X$, $\{N(u) \cap Y, Y \setminus N(u)\} = \{N(v) \cap Y, Y \setminus N(v)\}$. The family of bi-joins in a graph is bipartitive. The *bi-join decomposition tree* is the representative tree of the family of bi-joins, and can be computed in linear time [7, 8]. Let α be an internal node of the bi-join decomposition tree of a graph G. For all $i \in \{1, \ldots, d(\alpha)\}$ let $v_i \in C^i_\alpha$. $G[\{v_1, \ldots, v_{d(\alpha)}\}]$ denote the *characteristic graph* of α. The characteristic graph of a `degenerate` node is a complete bipartite graph or a disjoint union of two complete graphs [7, 8].

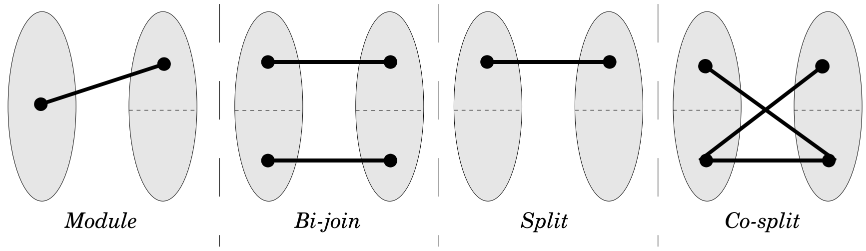

Figure 1: A module, a bi-join, a split and a co-split

3 Recognition of NLC-2 graphs

3.1 NLC-2 ρ-free canonical decomposition

In this section, $G = (V, E, l)$ is a 2-labelled graph such that every mono-coloured module (*i.e.* a module M such that $\forall v, v' \in M$, $l(v) = l(v')$) has size 1. A couple (X, Y) is a *cut* if $X \cup Y = V$, $X \cap Y = \emptyset$, $X \neq \emptyset$ and $Y \neq \emptyset$. Let $S \subseteq \{1, 2\} \times \{1, 2\}$. A cut (X, Y) is a S-*cut* of G if for all $u \in X$ and $v \in Y$, then $\{u, v\} \in E$ if and only if $(l(u), l(v)) \in S$. For $S \subseteq \{1, 2\} \times \{1, 2\}$ let $\mathcal{F}_S(G)$ be the set of S-cut of G.

Definition 5 (Symmetry)**.** We say that $S \in \{1, 2\} \times \{1, 2\}$ is *symmetric* if $(1, 2) \in S \iff (2, 1) \in S$, otherwise we say that S is *non-symmetric*.

Definition 6 (Degenerate property)**.** A family $\mathcal{F}$ of cuts has the *degenerate property* if there is a partition $\mathcal{P}$ of V such that for all $\emptyset \subsetneq \mathcal{X} \subsetneq \mathcal{P}$, $(\bigcup_{X \in \mathcal{X}} X, \bigcup_{Y \in \mathcal{P} \setminus \mathcal{X}} Y)$ is in $\mathcal{F}$, and there is no others cut in $\mathcal{F}$.

Lemma 7. *For every symmetric $S \subseteq \{1, 2\} \times \{1, 2\}$, $\mathcal{F}_S(G)$ has the degenerate property.*

Proof. The family $\mathcal{F}_{\{\}}(G)$ has the degenerate property since (X, Y) is a $\{\}$-cut if and only if there is no edges between X and Y ($\mathcal{P}$ is exactly the connected components). For $W \subseteq V$, let $G|W = (V, E\Delta W^2, l)$. For $i \in \{1, 2\}$ let $V_i = \{v \in V : l(v) = i\}$. Let $G_1 = G|V_1$, $G_2 = G|V_2$ and $G_{12} = (G|V_1)|V_2$.

- $\mathcal{F}_{\{(1,1)\}}(G) = \mathcal{F}_{\{\}}(G_1)$, $\mathcal{F}_{\{(2,2)\}}(G) = \mathcal{F}_{\{\}}(G_2)$, $\mathcal{F}_{\{(1,1),(2,2)\}}(G) = \mathcal{F}_{\{\}}(G_{12})$,
- $\mathcal{F}_{\{(1,1),(1,2),(2,1),(2,2)\}}(G) = \mathcal{F}_{\{\}}(\overline{G})$, $\mathcal{F}_{\{(1,2),(2,1),(2,2)\}}(G) = \mathcal{F}_{\{\}}(\overline{G_1})$, $\mathcal{F}_{\{(1,1),(1,2),(2,1)\}}(G) = \mathcal{F}_{\{\}}(\overline{G_2})$, $\mathcal{F}_{\{(1,2),(2,1)\}}(G) = \mathcal{F}_{\{\}}(\overline{G_{12}})$.

Thus for every symmetric $S \subseteq \{1, 2\} \times \{1, 2\}$, $\mathcal{F}_S(G)$ has the degenerate property. $\square$

Definition 8 (Linear property)**.** A family $\mathcal{F}$ of cuts has the *linear property* if for all (X, Y) and (X', Y') in $\mathcal{F}$, either $X \subseteq X'$ or $X' \subseteq X$.

Lemma 9. *For every non-symmetric $S \subseteq \{1, 2\} \times \{1, 2\}$, $\mathcal{F}_S(G)$ has the linear property.*

Proof. Case $S = \{(1, 2)\}$: suppose that $X \setminus X'$ and $X' \setminus X$ are both non-empty. Then if $u \in X \setminus X'$ is labelled 1 and $v \in X' \setminus X$ is labelled 2, u and v has to be adjacent and non-adjacent, contradiction. Thus $X \setminus X'$ and $X' \setminus X$ are mono-coloured. Now suppose w.l.o.g. that all vertices in $X\Delta X'$ are labelled 1. Then $X\Delta X'$ is adjacent to all vertices labelled 2 in $Y \cap Y'$ and non adjacent to all vertices labelled 1 in $Y \cap Y'$. Moreover $X\Delta X'$ is non adjacent to all vertices in $X \cap X'$. Thus $X\Delta X'$ is a mono-coloured module, and $|X\Delta X'| \geq 2$. Contradiction. For others non-symmetric S, we bring back to case $\{(1, 2)\}$ like in the proof of lemma 7. $\square$

Input: A 2-labelled graph $G = (V, E, l)$
Output: A NLC-2 ρ-free decomposition tree, or fail if G is not NLC-2 ρ-free
if $|V| = 1$ **then return** *the leaf* $\cdot(l(v))$ (where $V = \{v\}$)
Let $\mathcal{S}$ be the set of subsets of $\{1,2\} \times \{1,2\}$ and σ be the lexicographic order of $\mathcal{S}$
foreach $S \in \mathcal{S}$ *w.r.t.* σ **do**
  Compute $\mathcal{P}_S(G)$, and $\mathcal{P}'_S(G)$ if S is non-symmetric (see algorithm 2)
  if $|\mathcal{P}_S(G)| > 1$ **then**
    Create a new $\times_S$ node β
    foreach $P \in \mathcal{P}_S(G)$ *(w.r.t.* $\mathcal{P}'_S(G)$ *if* S *is non-symmetric)* **do**
      make NLC-2 ρ-free decomposition tree of $G[P]$ be a child of β.
    return *the tree rooted at* β
fail with *Not NLC-2 ρ-free*

Algorithm 1: Computation of the NLC-2 ρ-free canonical decomposition tree

For $S \subseteq \{1,2\} \times \{1,2\}$, let $\mathcal{P}_S(G)$ denote the unique partition of V such that (1) for all $(X,Y) \in \mathcal{F}_S(G)$ and $P \in \mathcal{P}_S(G)$, $P \subseteq X$ or $P \subseteq Y$, and (2) for all $P, P' \in \mathcal{P}$, $P \neq P'$, there is a $(X,Y) \in \mathcal{F}_S(G)$ such that $P \subseteq X$ and $P' \subseteq Y$, or $P \subseteq Y$ and $P' \subseteq X$. For a non-symmetric $S \in \{1,2\} \times \{1,2\}$, let $\mathcal{P}'_S(G) = (P_1, \ldots, P_k)$ denote the unique ordering of elements in $\mathcal{P}_S(G)$ such that for all $(X,Y) \in \mathcal{F}_S(G)$, there is a l such that $X = \cup_{i \in \{1,\ldots,l\}} P_i$.

Lemma 10. *If G is in NLC-2 ρ-free, then there is a $S \subseteq \{1,2\} \times \{1,2\}$ such that $\mathcal{F}_S(G)$ is non-empty.*

Proof. If G is NLC-2 ρ-free, then there is a $S \subseteq \{1,2\} \times \{1,2\}$, and two graphs G_1 and G_2 such that $G = G_1 \times_S G_2$. Thus $(V(G_1), V(G_2)) \in \mathcal{F}_S(G)$ and $\mathcal{F}_S(G)$ is non empty. □

Lemma 11. *Let $G = (V, E, l)$ 2-labelled graph and let $S \subseteq \{1,2\} \times \{1,2\}$. If G is NLC-2 ρ-free and has no mono-coloured non-trivial module, then for all $P \in \mathcal{P}_S(G)$, $G[P]$ has no mono-coloured non-trivial module.*

Proof. If M is a mono-coloured module of $G[P]$, then M is a mono-coloured module of G. Contradiction. □

Lemma 12. *Let $G = (V, E, l)$ 2-labelled graph and let $S \subseteq \{1,2\} \times \{1,2\}$. Then G is NLC-2 ρ-free if and only if for all $P \in \mathcal{P}_S(G)$, $G[P]$ is NLC-2 ρ-free.*

Proof. The "only if" is immediate. Now suppose that for all $P \in \mathcal{P}_S(G)$, $G[P]$ is NLC-2 ρ-free. If S is symmetric, let $\mathcal{P}_S(G) = \{P_1, \ldots, P_{|\mathcal{P}_S(G)|}\}$. Then $G = ((G[P_1] \times_S G[P_2]) \times_S \ldots \times_S G[P_{|\mathcal{P}_S(G)|}]$, and G is NLC-2 ρ-free. Otherwise, if S is non-symmetric, let $\mathcal{P}'_S(G) = (P_1, \ldots, P_{|\mathcal{P}_S(G)|})$. Then $G = ((G[P_1] \times_S G[P_2]) \times_S \ldots \times_S G[P_{|\mathcal{P}_S(G)|}]$, and G is NLC-2 ρ-free. □

The *NLC-2 ρ-free decomposition tree* of a 2-labelled graph G is a rooted tree such that the leaves are the vertices of G, and the internal nodes are labelled by $\times_S$, with $S \subseteq \{1,2\} \times \{1,2\}$. An internal node is `degenerated` if S is symmetric, and `linear` if S is non-symmetric. By lemmas 10, 11 and 12, G is NLC-2 ρ-free if and only if it has a NLC-2 ρ-free decomposition tree. This decomposition tree is not unique. But we can define a *canonical decomposition tree* if we fix a total order on the subsets of $\{1,2\} \times \{1,2\}$ (for example, the lexicographic order). If two graphs are isomorphic, then they have the same canonical decomposition tree. Algorithm 1 computes the canonical decomposition tree of a 2-labelled prime graph, or fails if G is not NLC-2 ρ-free.

Algorithm 2 computes $\mathcal{P}_S$ and $\mathcal{P}'_S$ for a 2-labelled prime graph G and $S \subseteq \{1,2\} \times \{1,2\}$ in linear time. We need some additional definitions for this algorithm and its proof of correctness. A

Input: A 2-labelled graph G, and $S \subseteq \{1,2\} \times \{1,2\}$
Output: $\mathcal{P}_S$ if S is symmetric, $\mathcal{P}'_S$ if S is non-symmetric
$V_i \leftarrow \{v : v \in V \text{ and } l(v) = i\}$;
if $(1,1) \in S$ **then** $\mathcal{C}_1 \leftarrow$ co-connected components of $G[V_1]$;
else $\mathcal{C}_1 \leftarrow$ connected components of $G[V_1]$;
if $(2,2) \in S$ **then** $\mathcal{C}_2 \leftarrow$ co-connected components of $G[V_2]$;
else $\mathcal{C}_2 \leftarrow$ connected components of $G[V_2]$;
$\mathcal{B} = (\mathcal{C}_1, \mathcal{C}_2, E_j, E_m) \leftarrow$ the bipartite trigraph between the elements of $\mathcal{C}_1$ and $\mathcal{C}_2$;
if $S \cap \{(1,2),(2,1)\} = \emptyset$ **then**
  return *connected components of* $(\mathcal{C}_1, \mathcal{C}_2, E_j \cup E_m)$
else if $S \cap \{(1,2),(2,1)\} = \{(1,2),(2,1)\}$ **then**
  return *connected components of the bi-complement of* $(\mathcal{C}_1, \mathcal{C}_2, E_j)$
else Search all semi-joins of $\mathcal{B}$ (see appendix) ;

Algorithm 2: Computation of $\mathcal{P}_S$ and $\mathcal{P}'_S$

bipartite graph is a triplet (X, Y, E) such that $E \subseteq X \times Y$. The *bi-complement* of a bipartite graph (X, Y, E) is the bipartite graph $(X, Y, (X \times Y) \setminus E)$. A *bipartite trigraph (BT)* is a bipartite graph with two types of edges: the *join* edges and the *mixed* edges. It is denoted by $\mathcal{B} = (X, Y, E_j, E_m)$ where E_j are the set of *join* edges, and E_m the set of *mixed* edges. A *BT-module* in a BT is a $M \subseteq X$ or $M \subseteq Y$ such that M is a module in (X, Y, E_j) and there is no *mixed* edges between M and $(X \cup Y) \setminus M$. For $v \in X \cup Y$, let $N_j(v) = \{u \in X \cup Y : \{u,v\} \in E_j\}$ and $N_m(v) = \{u \in X \cup Y : \{u,v\} \in E_m\}$. Let $d_j(v) = |N_j(v)|$ and $d_m(v) = |N_m(v)|$. A *semi-join* in a BT (X, Y, E_j, E_m) is a cut (A, B) of $X \cup Y$, such that there is no edges between $A \cap Y$ and $B \cap X$, and there is only *join* edges between $A \cap X$ and $B \cap Y$.

In algorithm 2, $\mathcal{B}$ is obtained from the graph G. Vertices of X correspond to subsets of vertices labelled 1 in G, and vertices of Y correspond to subsets of vertices labelled 2. There is a *join* edge between M and M' in $\mathcal{B}$ if M ① M' in G, and there is a *mixed* edge between $M \in X$ and $M' \in Y$ in $\mathcal{B}$ if there is at least an edge and a non-edge between M and M' in G. Such a graph $\mathcal{B}$ can easily be built in linear time from a given graph G. It suffices to consider a list and an array bounded by the number of component in G with the same colour. The following lemmas are close to observations in [9], but deal with BT instead of bipartite graphs (proofs are given in appendix).

Lemma 13. *Let $G = (X, Y, E_j, E_m)$ be a BT such that every BT-module has size 1. Let $(x_1, \ldots, x_{|X|})$ be X sorted by $(d_j(x), d_m(x))$ in lexicographic decreasing order. If (A, B) is a semi-join of G, then there is a $k \in \{0, \ldots, |X|\}$ such that $A \cap X = \{x_1, \ldots, x_k\}$.*

Lemma 14. *Let $k \in \{0, \ldots, |X|\}$ and $k' \in \{0, \ldots, |Y|\}$. Then $(A, (X \cup Y) \setminus A)$, where $A = \{x_1, \ldots, x_k, y_1, \ldots, y_{k'}\}$, is a semi-join of G if and only if $\sum_{i=1}^{k} d_j(x_i) - \sum_{i=1}^{k'} d_j(y_i) = k \times (|Y| - k')$ and $\sum_{i=1}^{k} d_m(x_i) - \sum_{i=1}^{k'} d_m(y_i) = 0$.*

Theorem 15. *Algorithm 2 is correct and runs in linear time.*

Proof. **Correctness:** Suppose that (A, B) is a S-cut. If $(1,1) \notin S$, then there is no edge between $A \cap V_1$ and $B \cap V_1$, thus (A, B) cannot cut a component $\mathcal{C}_1$ (and similarly for $(1,1) \in S$, and for $\mathcal{C}_2$). Now we work on the BT $\mathcal{B} = (\mathcal{C}_1, \mathcal{C}_2, E_j, E_m)$. If $S \cap \{(1,2),(2,1)\} = \emptyset$, then S-cuts correspond exactly to connected components of $\mathcal{B}$, and if $S \cap \{(1,2),(2,1)\} = \{(1,2),(2,1)\}$ then S-cuts correspond exactly to connected components of the BT of $\overline{G}$, which is $(\mathcal{C}_1, \mathcal{C}_2, (\mathcal{C}_1 \times \mathcal{C}_2) \setminus (E_j \cup E_m), E_m)$. Finally, if S is non-symmetric, S-cuts correspond to semi-joins of $\mathcal{B}$ (see appendix).

Complexity: It is well admitted that we can perform a BFS on a graph or its complement in linear time [13, 6]. The instructions on lines [2-5,8] can be done with a BFS on a graph or its complement. It is easy to see that we can do a BFS on the bi-complement in linear time (like a BFS on a complement graph, with two vertex lists for X and Y), so instruction line 10 can be done in linear time. Finally, the operations at line 11 are done in linear time (see appendix). $\square$

These results can be summarized as:

Theorem 16. *Algorithm 1 computes the canonical NLC-2 ρ-free decomposition tree of a 2-labelled graph in $O(nm)$ time.*

3.2 NLC-2 decomposition of a prime graph

In this section, G is an unlabelled prime (w.r.t. modular decomposition) graph, with $|V| \geq 3$.

Definition 17 (2-*bimodule*)**.** A bipartition $\{X, Y\}$ of V is a 2-*bimodule* if X can be partitioned into X_1 and X_2, and Y into Y_1 and Y_2 such that for all $(i,j) \in \{1,2\} \times \{1,2\}$, then either X_i ⓪ Y_j or X_i ① Y_j. It is easy to see that if $\{X, Y\}$ is a 2-bimodule if and only if $\{X, Y\}$ is a split, a co-split or a bi-join. Moreover, if $\min(|X|, |Y|) > 1$ then $\{X, Y\}$ cannot be both of them in the same time (since G is prime).

Let $l : V \to \{1, 2\}$ be a 2-labelling. Then $s(l)$ denote the 2-labelling on V such that for all $v \in V$, $s(l)(v) = 1$ if and only if $l(v) = 2$.

Definition 18 (Labelling induced by a 2-bimodule)**.** Let $\{X, Y\}$ be a 2-bimodule. We define the *labelling $l : V \to \{1, 2\}$ of G induced by* $\{X, Y\}$. If $|X| = |Y| = 1$, then $l(x) = 1$ and $l(y) = 2$, where $X = \{x\}$ and $Y = \{y\}$. If $|X| = 1$, then $l(v) = 1$ iff $v \in N[x]$. Similarly if $|Y| = 1$, then $l(v) = 1$ iff $v \in N[y]$. Now we suppose $\min(|X|, |Y|) > 1$. If $\{X, Y\}$ is a split, then the set of vertices in X with a neighbour Y and the set of vertices in Y with a neighbour in X is labelled 1, others vertices are labelled 2. If $\{X, Y\}$ is a co-split, then a labelling of G induced by $\{X, Y\}$ is a labelling of $\overline{G}$ induced by the split $\{X, Y\}$. Finally if $\{X, Y\}$ is a bi-join, l is such that $\{v \in X : l(v) = 1\}$ is a join with $\{v \in Y : l(v) = 1\}$ and $\{v \in X : l(v) = 2\}$ is a join with $\{v \in Y : l(v) = 2\}$. Note that if $\{X, Y\}$ is a bi-join, then there is two possibles labelling l_1 and l_2, with $l_1 = s(l_2)$. If $\{X, Y\}$ is a 2-bimodule of G and l a labelling induced by $\{X, Y\}$, then every mono-coloured module has size 1 (since G is prime and $|V| \geq 3$).

Definition 19 (Good 2-bimodule)**.** A 2-bimodule $\{X, Y\}$ is *good* if the graph G with the labelling induced by $\{X, Y\}$ is NLC-2 ρ-free. The following proposition comes immediately from lemma 3.

Proposition 20. *G is NLC-2 if and only if G has a good* 2*-bimodule.*

Lemma 21. *If G has a good* 2*-bimodule $\{X, Y\}$ which is a split, then G has a good* 2*-bimodule which is a strong split.*

Proof. There is a node α in the split decomposition tree and $\emptyset \subsetneq I \subsetneq \{1, \ldots, d(\alpha)\}$ such that $\{X, Y\} = \{\cup_{i \in I} C_\alpha^i, \cup_{i \notin I} C_\alpha^i\}$. Let $l : V \to \{1, 2\}$ be the labelling of G induced by $\{X, Y\}$. For all $i \in \{1, \ldots, d(\alpha)\}$, $(G[C_\alpha^i], l|_{C_\alpha^i})$ is NLC-2 ρ-free (where $l|_W$ is the function l restricted at W).

Let l' be the 2-labelling of V such that for all i, and $v \in C_\alpha^i$, $l(v) = 1$ if and only if v has a neighbour outside of C_α^i. For all i, either $l|_{C_\alpha^i} = l'|_{C_\alpha^i}$, or $\forall v \in C_\alpha^i$, $l(v) = 2$. Then for all i, $(G[C_\alpha^i], l'|_{C_\alpha^i})$ is NLC-2 ρ-free, and thus (G, l') is NLC-2 ρ-free. Since there is a dominating vertex in the characteristic graph of α, there is a j such that the labelling induced by the strong split $\{C_\alpha^j, V \setminus C_\alpha^j\}$ is l'. Thus the strong split $\{C_\alpha^j, V \setminus C_\alpha^j\}$ is good. $\square$

Input: A graph G
Result: Yes iff G is NLC-2
$\mathcal{S} \leftarrow$ the set of strong splits, co-splits and bi-joins of G ;
foreach $\{X, Y\} \in \mathcal{S}$ **do**
  $l \leftarrow$ the labelling of G induced by $\{X, Y\}$;
  if $(G[X], G[Y], l)$ *is NLC-2 ρ-free* **then return** *Yes* ;
return *No* ;

Algorithm 3: Recognition of prime NLC-2 graphs

Previous lemma on $\overline{G}$ say that if G has a good 2-bimodule $\{X, Y\}$ which is a co-split, then G has a good 2-bimodule which is a strong co-split. The following lemma is similar to Lemma 21.

Lemma 22. *If G has a good 2-bimodule $\{X, Y\}$ which is a bi-join, then G has a good 2-bimodule which is a strong bi-join.*

Theorem 23. *Algorithm 3 recognises prime NLC-2 graphs, and its time complexity is $O(n^2m)$.*

Proof. Trivially if the algorithm return Yes, then G is NLC-2. On the other hand, by proposition 20, and lemmas 21 and 22, if G is NLC-2, then it has a good strong 2-bimodule and the algorithm returns Yes.

The set $\mathcal{S}$ can be computed using algorithms for computing split decomposition on G and $\overline{G}$, and bi-join decomposition on G. Note that it is not required to use a linear time algorithm for split decomposition [5]: some simpler algorithms run in $O(n^2m)$ [4, 10]. [7, 8] show that bi-join decomposition can be computed in linear time, using a reduction to modular decomposition. But there also, modular decomposition algorithms simpler than [15] may be used. The set $\mathcal{S}$ has $O(n)$ elements. Testing if a 2-bimodule is good takes $O(nm)$ using algorithm 1. So total running time is $O(n^2m)$. □

3.3 NLC-2 decomposition

Using lemma 2, modular decomposition and algorithm 3, we get:

Theorem 24. *NLC-2 graphs can be recognised in $O(n^2m)$, and a NLC-2 expression can be generated in the same time.*

4 Graph isomorphism on NLC-2 graphs

4.1 Graph Isomorphism on NLC-2 ρ-free prime graphs

The following propositions are direct consequences of properties (linear and degenerate) of S-cuts.

Proposition 25. *Consider a symmetric $S \in \{1,2\} \times \{1,2\}$. Two graphs G and H are isomorphic if and only if there is a bijection π between $\mathcal{P}_S(G)$ and $\mathcal{P}_S(H)$ such that for all $P \in \mathcal{P}_S(G)$, $G[P]$ is isomorphic to $H[\pi(P)]$.*

Proposition 26. *Let a non-symmetric $S \in \{1,2\} \times \{1,2\}$ and let G and H be two graphs. Let $\mathcal{P}'_S(G) = (P_1, \ldots, P_k)$ and $\mathcal{P}'_S(H) = (P'_1, \ldots, P'_{k'})$ then G and H are isomorphic if and only if $k = k'$ and for all $i \in \{1, \ldots, k\}$, $G[P_i]$ is isomorphic to $H[P'_i]$.*

By the previous 2 propositions, two NLC-2 ρ-free 2-labelled graphs G and H are isomorphic if and only if there is an isomorphism between their canonical NLC-2 ρ-free decomposition tree which respects the order of children of `linear` nodes. This isomorphism can be tested in linear time, thus isomorphism of NLC-2 ρ-free graphs can be done in $O(nm)$ time.

Input: Two prime NLC-2 graphs G and H
Result: Yes if $G \simeq H$, No otherwise
$\mathcal{S} \leftarrow$ the set of strong splits, co-splits and bi-joins of G ;
$\mathcal{S}' \leftarrow$ the set of strong splits, co-splits and bi-joins of H ;
if *there is no good* 2*-bimodule in* $\mathcal{S}$ **then fail with** "G is not NLC-2";
$\{X, Y\} \leftarrow$ a good 2-bimodule in $\mathcal{S}$;
$l \leftarrow$ the labelling of G induced by $\{X, Y\}$;
foreach $\{X', Y'\} \in \mathcal{S}'$ *such that* $\{X', Y'\}$ *is good* **do**
    $l' \leftarrow$ the labelling of H induced by $\{X', Y'\}$;
    if $|X| > 1$ **and** $|Y| > 1$ **and** $\{X, Y\}$ *is a bi-join* **then**
        if $(G, l) \simeq (H, l')$ **or** $(G, l) \simeq (H, s(l'))$ **then return** *Yes ;*
    else if $(G, l) \simeq (H, l')$ **then return** *Yes ;*
return *No* ;

Algorithm 4: Isomorphism for prime NLC-2 graphs

4.2 Graph isomorphism on prime NLC-2 graphs

Theorem 27. *Algorithm 4 test isomorphism between two prime NLC-2 graphs in time $O(n^2m)$.*

Proof. If the algorithm returns "yes", then trivially $G \simeq H$. On the other hand suppose that $G \simeq H$ and let $\pi : V(G) \to V(H)$ be a bijection such that $\{u, v\} \in E(G)$ iff $(\pi(u), \pi(v)) \in E(H)$. Then $\{X', Y'\}$ with $X' = \pi(X)$ and $Y' = \pi(Y)$ is a good 2-bimodule if H. If $\min(|X|, |Y|) > 1$ and $\{X', Y'\}$ is a bi-join, then by definition there is two labelling induced by $\{X, Y\}$, and $(G, l) \simeq (H, l')$ or $(G, l) \simeq (H, s(l'))$. Otherwise the labelling is unique and $(G, l) \simeq (H, l')$.

The sets $\mathcal{S}$ and $\mathcal{S}'$ can be computed in $O(n^2)$ time using linear time algorithms for computing split decomposition on G and $\overline{G}$, and bi-join decomposition on G. The sets $\mathcal{S}$ and $\mathcal{S}'$ have $O(n)$ elements. Test if a 2-bimodule is good take $O(nm)$ using algorithm 1, and test if two 2-labelled prime graphs are isomorphic take also $O(nm)$. Thus the total running time is $O(n^2m)$. □

4.3 Graph isomorphism on NLC-2 graphs

It is easy to show that graph isomorphism on prime NLC-2 graphs with an additional labels into $\{1, \ldots, q\}$ can be done in $O(n^2m)$ time. For that, we add the additional label of v at the leaf corresponding to v in the NLC-2 ρ-free decomposition tree.

We show that we can do graph isomorphism on NLC-2 graphs in time $O(n^2m)$, using the modular decomposition and algorithm 4. Let $\mathcal{M}(G)$ and $\mathcal{M}(H)$ be the modular decomposition of G and H. For $M \in \mathcal{M}(G)$, let G_M be $G[M]$, and for $M \in \mathcal{M}(H)$, let H_M be $H[M]$. Let G^*_M be the characteristic graph of G_M (note that $|V(G^*_M)|$ is the number of children of M in the modular decomposition tree). Let $\mathcal{M}_{(i,*)} = \{M \in \mathcal{M}(G) \cup \mathcal{M}(H) : |M| = i\}$, let $\mathcal{M}_{(*,j)} = \{M \in \mathcal{M}(G) \cup \mathcal{M}(H) : |V(G^*_M)| = j\}$ and let $\mathcal{M}_{(i,j)} = \mathcal{M}_{(i,*)} \cap \mathcal{M}_{(*,j)}$. Note that $\sum_{j=1}^{n} (\mathcal{M}_{(*,j)} \times j)$ is the number of vertices in G plus the number of edges in the modular decomposition tree, and thus is at most $3n - 2$.

Theorem 28. *Algorithm 5 tests isomorphism between two NLC-2 graphs in time $O(n^2m)$.*

Proof. The correctness comes from the fact that at each step, for all $M, M' \in \mathcal{M}(G) \cup \mathcal{M}(H)$ such that $l(M)$ and $l(M')$ are set, G_M and $G_{M'}$ are isomorphic if and only if $l(M) = l(M')$. The

Input: Two NLC-2 graphs G and H
Result: Yes if $G \simeq H$, No otherwise
for *every* $M \in \mathcal{M}(G) \cup \mathcal{M}(H)$ *such that* $|M| = 1$ **do** $l(M) \leftarrow 1$;
for i *from* 2 *to* n **do**
  for j *from* 2 *to* i **do**
    Compute the partition $\mathcal{P}$ of $\mathcal{M}_{(i,j)}$ such that M and M' are in the same class of $\mathcal{P}$ if and only if $(G^*_M, l) \simeq (G^*_{M'}, l)$. ;
    foreach $P \in \mathcal{P}$ **do**
      $a \leftarrow$ a new label (an integer not in $\mathrm{Img}(l)$) ;
      For all $M \in P$, $l(M) \leftarrow a$;

Algorithm 5: Isomorphism on NLC-2 graphs

total time $f(n, m)$ of this algorithm is $O(n^2 m)$ since ("big Oh" is omitted):

$$\begin{aligned} f(n,m) &\leq \sum_i \sum_j \left(j^2 m |\mathcal{M}_{(i,j)}|^2\right) \leq m \sum_j \left(j^2 \sum_i \left(|\mathcal{M}_{(i,j)}|^2\right)\right) \\ &\leq m \sum_j \left(j^2 |\mathcal{M}_{(*,j)}|^2\right) \leq m \sum_j \left(\left(j|\mathcal{M}_{(*,j)}|\right)^2\right) \leq n^2 m. \end{aligned}$$

□

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Appendix

A.1 Proof of lemma 13

Let $G = (X, Y, E_j, E_m)$ be a BT such that every BT-module has size 1. Let $(x_1, \ldots, x_{|X|})$ be X sorted by $(d_j(x), d_m(x))$ in lexicographic decreasing order. If (A, B) is a semi-join of G, then there is a $k \in \{0, \ldots, |X|\}$ such that $A \cap X = \{x_1, \ldots, x_k\}$.

Proof. For all $v \in A \cap X$, $d_j(v) \geq |B \cap Y|$, and for all $v \in B \cap X$, $d_j(v) \leq |B \cap Y|$. Moreover, if there is a $v \in B \cap X$ with $d_j(v) = |B \cap Y|$, then $d_m(v) = 0$. Let $C = \{v \in X : d_j(v) = |B \cap Y| \text{ and } d_m(v) = 0\}$. Then C is a BT-module of G, and thus $|C| \leq 1$. Every vertex in $A \cap X \setminus C$ are before every vertex in $B \cap X \setminus C$ in the ordering. Moreover, if $|C| > 0$, then vertices in $A \cap X \setminus C$ are before the vertex in C, and vertices in $B \cap X \setminus C$ are after the vertex in C in the ordering. □

A.2 Proof of lemma 14

Let $k \in \{0, \ldots, |X|\}$ and $k' \in \{0, \ldots, |Y|\}$. Then $(A, (X \cup Y) \setminus A)$, where $A = \{x_1, \ldots, x_k, y_1, \ldots, y_{k'}\}$, is a semi-join of G if and only if $\sum_{i=1}^{k} d_j(x_i) - \sum_{i=1}^{k'} d_j(y_i) = k \times (|Y| - k')$ and $\sum_{i=1}^{k} d_m(x_i) - \sum_{i=1}^{k'} d_m(y_i) = 0$.

Proof. The "If" part is by definition. Now let us consider the "Only if" part. Let us assume that the degree condition holds. We will denote a the number of join edges between $A \cap X$ and $B \cap Y$, b the number of join edges between $A \cap X$ and $A \cap Y$, and c the number of mixed edges between $A \cap X$ and $A \cap Y$. Note that $a \leq k(|Y| - k')$, $a + b = \sum_{i=1}^{k} d_j(x_i)$ and $b \leq \sum_{i=1}^{k'} d_j(y_i)$, thus $a \geq k(|Y| - k')$. So we have $a = k(|Y| - k')$, and $\sum_{i=1}^{k'} d_j(y_i) - b = 0$. In other words, there is only join edges between $A \cap X$ and $B \cap Y$, and there is no join edges between $A \cap Y$ and $B \cap X$. Now since there is only join edges between $A \cap X$ and $B \cap Y$, $c = \sum_{i=1}^{k} d_m(x_i) = \sum_{i=1}^{k'} d_m(y_i)$, thus there is no mixed edges between $A \cap Y$ and $B \cap X$. □

A.3 Algorithm to compute $\mathcal{P}'_S$ when S is non-symmetric

Proof. **Correctness:** Algorithm 6 generates all the semi-joins of $\mathcal{B}$. At any time, $s_j = \sum_{i=1}^{k} d_j(x_i)$, $s_m = \sum_{i=1}^{k} d_m(x_i)$, $s'_j = \sum_{i=1}^{k'} d_j(y_i)$ and $s'_m = \sum_{i=1}^{k'} d_m(y_i)$. In $\mathcal{B}$, every BT-module has size 1, otherwise there is a mono-coloured module in G of size at least 2. If (A, B) is a semi-join, then by lemma 13 on $(\mathcal{C}_1, \mathcal{C}_2, E_j, E_m)$ and $(\mathcal{C}_2, \mathcal{C}_1, E_j, E_m)$, there is a a and b such that $A \cap \mathcal{C}_1 = \{x_1, \ldots, x_a\}$ and $A \cap \mathcal{C}_2 = \{y_1, \ldots, y_b\}$. At any time, $(A', (\mathcal{C}_1 \cup \mathcal{C}_2) \setminus A')$ with $A' = \{x_1, \ldots, x_l, y_1, \ldots, y_{l'}\}$ is the last semi-join found. At $k = a$, the **while** line 12 will stop when $s_j - s'_j = k \times (|\mathcal{C}_2| - k')$ since for every $v \in A \cap \mathcal{C}_2$, $d_j(v) \leq k$, and $s'_j + k \times (|\mathcal{C}_2| - k')$ decrease with k'. Moreover, when the **while** loop stops, $s_m = s'_m$ since s'_m increase with k'. Thus if $b \neq k'$, then $\{y_{k'+1}, \ldots y_b\}$ is a BT-module and $b = k' + 1$ (since every BT-module has size 1). In all cases the algorithm finds (A, B), and adds the partition in $\mathcal{P}'$.

Complexity: As we see in proof of theorem 15, every instruction lines [2-5] can be done in linear time, and clearly every instruction lines [6-22] can be done in linear time, thus the total running time is $O(n + m)$. □

Input: A 2-labelled graph G, and a non-symmetric $S \subseteq \{1,2\} \times \{1,2\}$
Output: $\mathcal{P}'_S$

$V_i \leftarrow \{v : v \in V \text{ and } l(v) = i\}$;
if $(1,1) \in S$ **then** $\mathcal{C}_1 \leftarrow$ co-connected components of $G[V_1]$;
else $\mathcal{C}_1 \leftarrow$ connected components of $G[V_1]$;
if $(2,2) \in S$ **then** $\mathcal{C}_2 \leftarrow$ co-connected components of $G[V_2]$;
else $\mathcal{C}_2 \leftarrow$ connected components of $G[V_2]$;
$\mathcal{B} = (\mathcal{C}_1, \mathcal{C}_2, E_j, E_m) \leftarrow$ the bipartite trigraph between the elements of $\mathcal{C}_1$ and $\mathcal{C}_2$;
$(x_1, \ldots, x_{|\mathcal{C}_1|}) \leftarrow \mathcal{C}_1$ sorted by lexicographic order on $(-d_j(v), -d_m(v))$;
$(y_1, \ldots, y_{|\mathcal{C}_2|}) \leftarrow \mathcal{C}_2$ sorted by lexicographic order on $(d_j(v), d_m(v))$;
$\mathcal{P}' \leftarrow ()$; $l \leftarrow 0$; $l' \leftarrow 0$; $k' \leftarrow 0$; $k \leftarrow 0$;
$s_j \leftarrow 0$; $s_m \leftarrow 0$; $s'_j \leftarrow 0$; $s'_m \leftarrow 0$;
while $k \leq |\mathcal{C}_1|$ **do**
  while $s_j - s'_j < k \times (|\mathcal{C}_2| - k')$ **or** *(* $s_j - s'_j = k \times (|\mathcal{C}_2| - k')$ **and** $s_m > s'_m$*)* **do**
    $k' \leftarrow k' + 1$; $s'_j \leftarrow s'_j + d_j(y_{k'})$; $s'_m \leftarrow s'_m + d_m(y_{k'})$;
  if $s_j - s'_j = k \times (|\mathcal{C}_2| - k')$ **and** $s_m = s'_m$ **then**
    add $\{x_{l+1}, \ldots, x_k\} \cup \{y_{l'+1} \ldots, y_{k'}\}$ at the end of $\mathcal{P}'$; $l \leftarrow k$; $l' \leftarrow k'$;
    if $s_j - s'_j - d_j(y_{k+1}) = k \times (|\mathcal{C}_2| - k' - 1)$ **and** $s_m = s'_m + d_m(y_{k+1})$ **then**
      $k' \leftarrow k' + 1$; $s'_j \leftarrow s'_j + d_j(y_{k'})$; $s'_m \leftarrow s'_m + d_m(y_{k'})$;
      add $\{y_{k'}\}$ at the end of $\mathcal{P}'$; $l' \leftarrow k'$;
  $k \leftarrow k + 1$; $s_j \leftarrow s_j + d_j(x_k)$; $s_m \leftarrow s_m + d_m(x_k)$;
remove $\emptyset$ form $\mathcal{P}'$, if any ;
if $(2,1) \in S$ **then** reverse $\mathcal{P}'$;
return $\mathcal{P}'$

Algorithm 6: Computation of $\mathcal{P}'_S$ when S is non-symmetric