Potential Maximal Cliques enumeration in graphs

Pierre Bergé Vincent Limouzy Nicolas Schivre

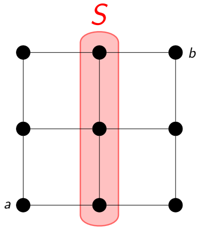
Université Clermont Auvergne

JGA, November 2025

Definitions and properties

(a, b)-Separator

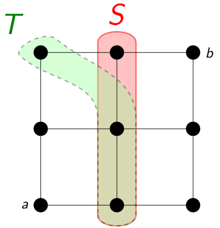
Vertices set S such that $G \setminus S$ is disconnected and a and b are in differents connected components. S is ***minimal*** if every subset is not an (a, b)-separator.
Two separators S and T are ***parallel*** if there exists a connected component C from $G \setminus S$ such that $T \subseteq S \cup C$.



Definitions and properties

(a, b)-Separator

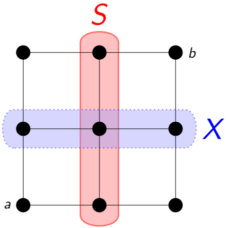
Vertices set S such that $G \setminus S$ is disconnected and a and b are in differents connected components. S is ***minimal*** if every subset is not an (a, b)-separator.
Two separators S and T are ***parallel*** if there exists a connected component C from $G \setminus S$ such that $T \subseteq S \cup C$.



Definitions and properties

(a, b)-Separator

Vertices set S such that $G \setminus S$ is disconnected and a and b are in differents connected components. S is ***minimal*** if every subset is not an (a, b)-separator.
Two separators S and T are ***parallel*** if there exists a connected component C from $G \setminus S$ such that $T \subseteq S \cup C$.



Definitions and properties

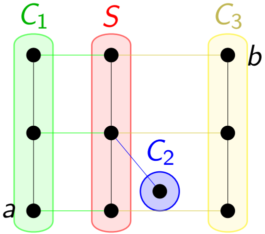
Full component

A set S admits a ***full component*** C if $\forall v \in S, N(v) \cap C \neq \emptyset$.

Definitions and properties

Full component

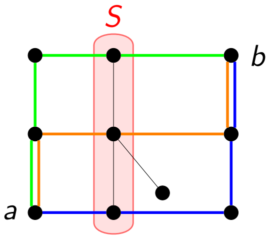
A set S admits a ***full component*** C if $\forall v \in S, N(v) \cap C \neq \emptyset$.



Definitions and properties

Full component

A set S admits a ***full component*** C if $\forall v \in S, N(v) \cap C \neq \emptyset$.



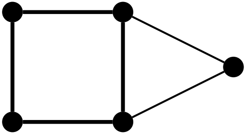
Lemmas (separators characterisations)

- Folklore : For all graph G with $a, b \in V$, and all (a, b)-separator $S \subset V$ of G, S is minimal iff $\forall v \in S$, there is a path from a to b which intersect S only in v.
- A minimal separator admits at least 2 full components.

Chordal graphs

Chordal graphs

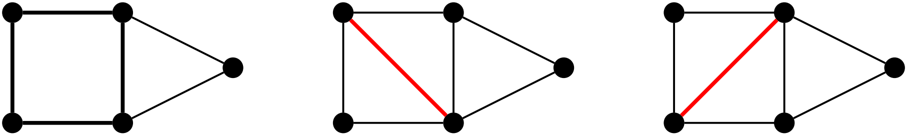
A graph G is said ***chordal*** if it does not have any induced cycle of length at least 4 (also called hole).



Chordal graphs

Chordal graphs

A graph G is said ***chordal*** if it does not have any induced cycle of length at least 4 (also called hole).

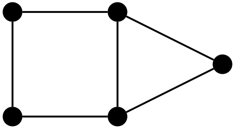


Triangulation

A ***triangulation*** is an edges addition to the graph that make it chordal. A triangulation is ***minimal*** if removing a completion edge creates a hole.

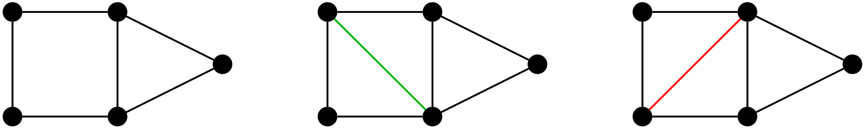
Definition and characterisations

A ***Potential Maximal Clique*** (PMC) is a maximal clique which appears in at least one minimal triangulation.



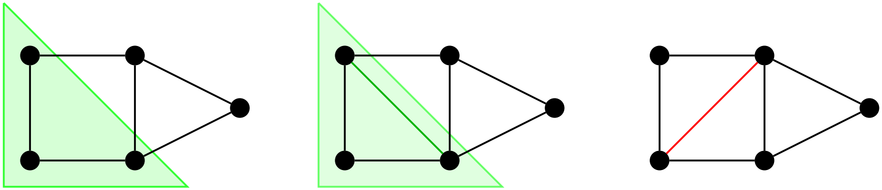
Definition and characterisations

A ***Potential Maximal Clique*** (PMC) is a maximal clique which appears in at least one minimal triangulation.



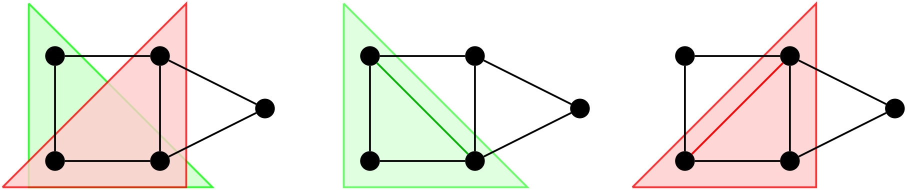
Definition and characterisations

A ***Potential Maximal Clique*** (PMC) is a maximal clique which appears in at least one minimal triangulation.



Definition and characterisations

A ***Potential Maximal Clique*** (PMC) is a maximal clique which appears in at least one minimal triangulation.

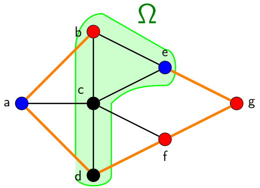


Definition and characterisations

Characterisation : (Bouchitté and Todinca 2002)

A set Ω is a PMC if :

1. Ω does not admit a full component.
2. For any vertices pair $u, v \in \Omega$, either $uv \in E$, or there is a connected component C of $G \setminus \Omega$ such that $N(u) \cap C \neq \emptyset$ and $N(v) \cap C \neq \emptyset$.



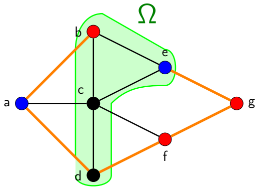
Ω is a valid PMC

Definition and characterisations

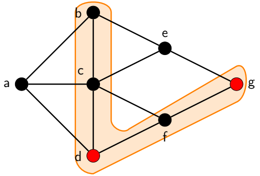
Characterisation : (Bouchitté and Todinca 2002)

A set Ω is a PMC if :

1. Ω does not admit a full component.
2. For any vertices pair $u, v \in \Omega$, either $uv \in E$, or there is a connected component C of $G \setminus \Omega$ such that $N(u) \cap C \neq \emptyset$ and $N(v) \cap C \neq \emptyset$.



Ω is a valid PMC



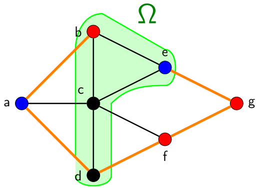
d and g not connected

Definition and characterisations

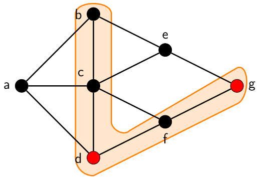
Characterisation : (Bouchitté and Todinca 2002)

A set Ω is a PMC if :

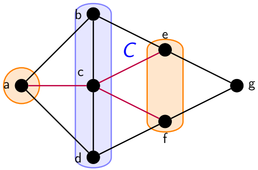
1. Ω does not admit a full component.
2. For any vertices pair $u, v \in \Omega$, either $uv \in E$, or there is a connected component C of $G \setminus \Omega$ such that $N(u) \cap C \neq \emptyset$ and $N(v) \cap C \neq \emptyset$.



Ω is a valid PMC



d and g not connected



C is a full component

Motivations and results

- N : number of PMCs.
- $\mathcal{S}$: number of minimal separators.

Applications to graphs decompositions

Computation of the treewidth.

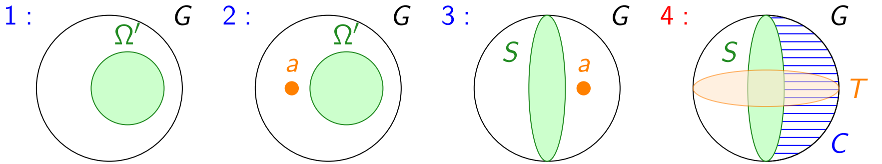
Interesting results from Bouchitté and Todinca in 2002

- An *output polynomial* $O(N^2)$ algorithm to enumerate all the Potential Maximal Cliques.
- If $N = O(n^c)$ then **treewidth calculable in polynomial time**.
- Treewidth is computable in $O(n^2 \times \mathcal{S} \times N)$.

Output quadratic algorithm from Bouchitté and Todinca in 2002

Subgraphs solution

Knowing the PMCs Ω' of G', we can find the PMCs Ω of $G = G' \cup \{a\}$ with 4 cases :



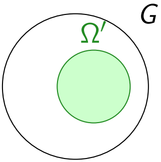
$\Omega = \Omega'$ $\quad$ $\Omega = \Omega' \cup \{a\}$ $\quad$ $\Omega = S \cup \{a\}$ $\quad$ $\Omega = S \cup (C \cap T)$

Output quadratic algorithm from Bouchitté and Todinca in 2002

Subgraphs solution

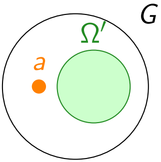
Knowing the PMCs Ω' of G', we can find the PMCs Ω of $G = G' \cup \{a\}$ with 4 cases :

1 :



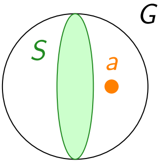
$$\Omega = \Omega'$$

2 :



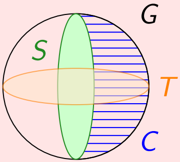
$$\Omega = \Omega' \cup \{a\}$$

3 :



$$\Omega = S \cup \{a\}$$

4 :

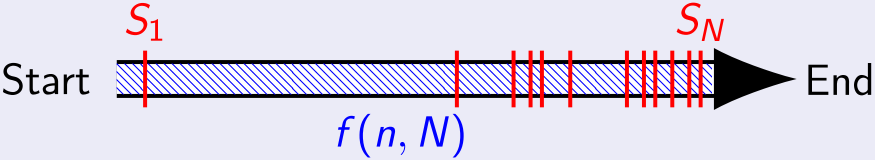


$$\Omega = S \cup (C \cap T)$$

Enumeration complexity classes (Johnson, Yannakakis, Papadimitriou 1988)

Output polynomial algorithm

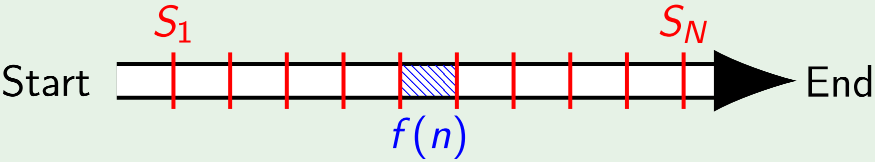
n : number of vertices. | N : number of Solutions. | c : constant.



$O((n+N)^c)$ total

Enumeration complexity classes (Johnson, Yannakakis, Papadimitriou 1988)

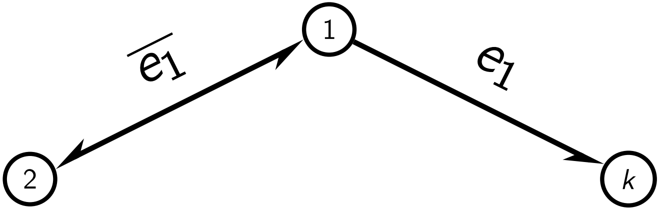
Polynomial delay algorithm



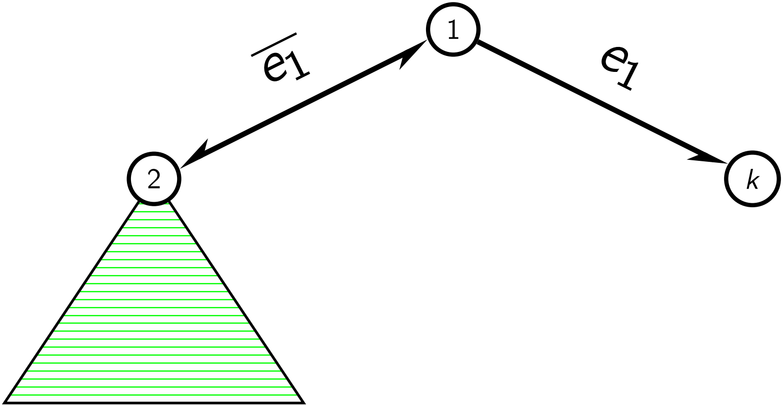
$O(n^c)$ for S_{i+1}

Flashlight Search (Read and Tarjan 1975)

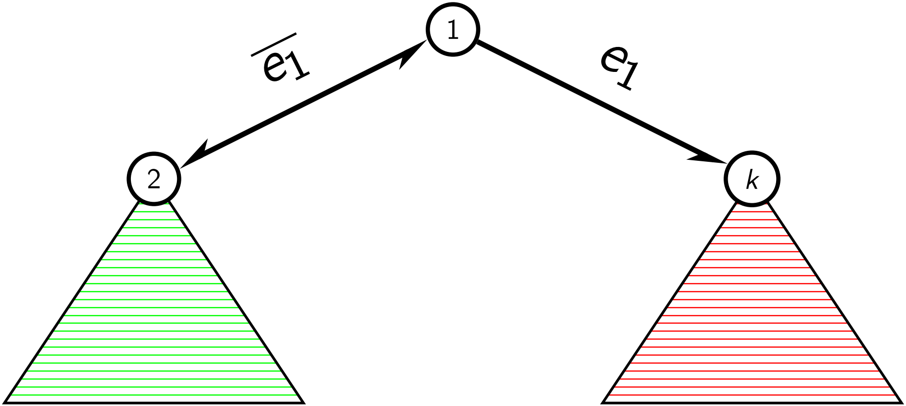
Flashlight Search (Read and Tarjan 1975)



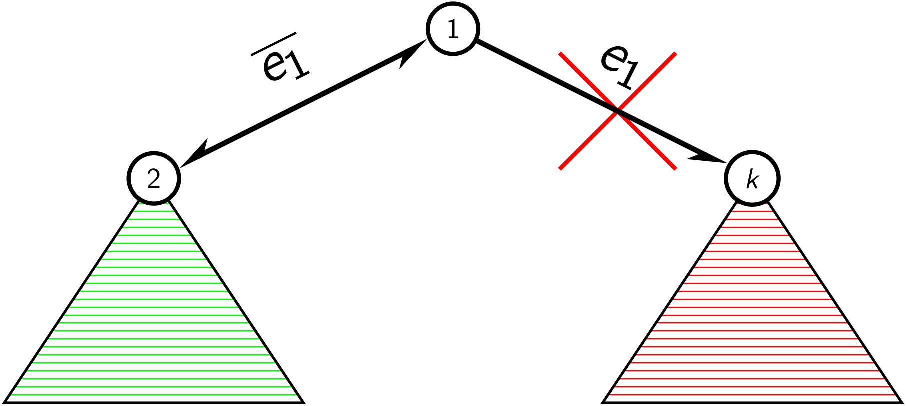
Flashlight Search (Read and Tarjan 1975)



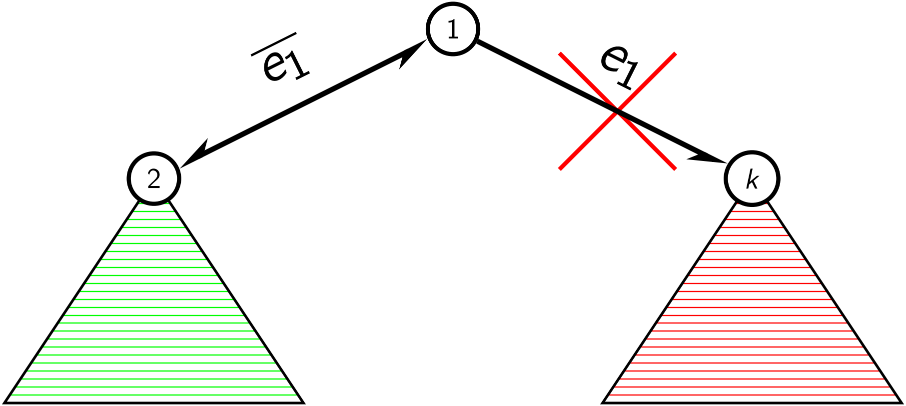
Flashlight Search (Read and Tarjan 1975)



Flashlight Search (Read and Tarjan 1975)



Flashlight Search (Read and Tarjan 1975)



Consequences

Leads to a polynomial delay, polynomial space algorithm.

Flashlight Search (Read and Tarjan 1975)

Principle : extension problem resolution

- Creation and exploration of a tree where the paths from the root to the leaves represent the solutions.
- Resolution of a decision problem called **extension problem** at each step to know if we explore the subtree.

Extension problem scheme

Input: A graph $G = (V, E)$, a subset $M \subset V$ of **mandatory** vertices, a subset $F \subset V$ of **forbidden** vertices such that $M \cap F = \emptyset$ and **P** a property to satisfy.

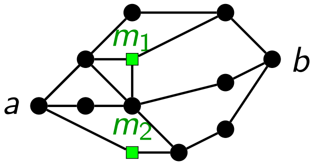
Question: Is there any X, set of vertices with $M \subseteq X$, $X \cap F = \emptyset$ and such that X satisfy **P** ?

Minimal (a, b)-separator extension

Minimal (a, b)-separator extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* vertices such that $M \cap F = \emptyset$ and two vertices a and b from V.
Question: Is there a $X \subseteq V$ a minimal (a, b)-separator with $M \subseteq X$ and $X \cap F = \emptyset$?

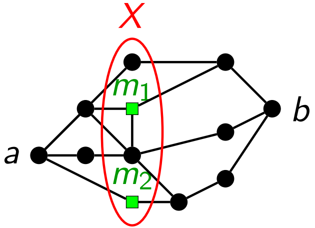
Minimal (a, b)-separator extension

Minimal (a, b)-separator extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* vertices such that $M \cap F = \emptyset$ and two vertices a and b from V.
Question: Is there a $X \subseteq V$ a minimal (a, b)-separator with $M \subseteq X$ and $X \cap F = \emptyset$?



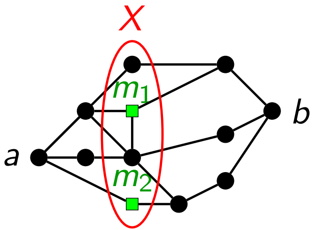
Minimal (a, b)-separator extension

Minimal (a, b)-separator extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* vertices such that $M \cap F = \emptyset$ and two vertices a and b from V.
Question: Is there a $X \subseteq V$ a minimal (a, b)-separator with $M \subseteq X$ and $X \cap F = \emptyset$?



Minimal (a, b)-separator extension

Minimal (a, b)-separator extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* vertices such that $M \cap F = \emptyset$ and two vertices a and b from V.
Question: Is there a $X \subseteq V$ a minimal (a, b)-separator with $M \subseteq X$ and $X \cap F = \emptyset$?



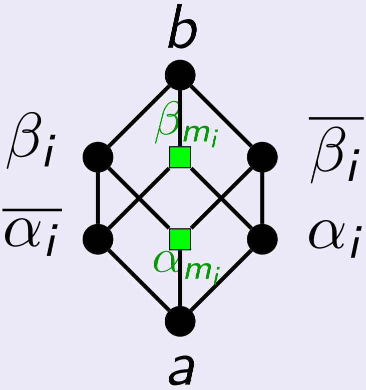
Theorem (Bergé, Limouzy, Schivre 2025)

Minimal (a, b)-separator extension is NP-complete.

Minimal (a, b)-separator extension

Reduction to 3-SAT : Gadget

■ $\in M$

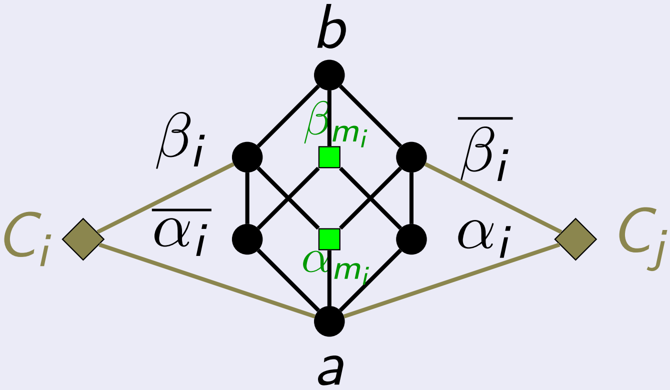


Minimal (a,b)-separator extension

Reduction to 3-SAT : Gadget

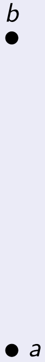
■ $\in M$ ◆ $\in M$

$\Phi = C_i = (? \vee ? \vee \overline{x_i}) \wedge C_j = (x_i \vee ? \vee ?)$



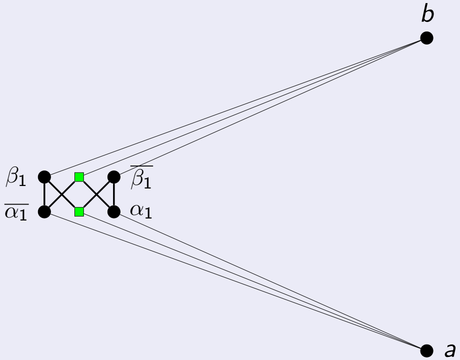
Minimal (a, b)-separator extension

Reduction to 3-SAT : reduction graph for
$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_5) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_5})$



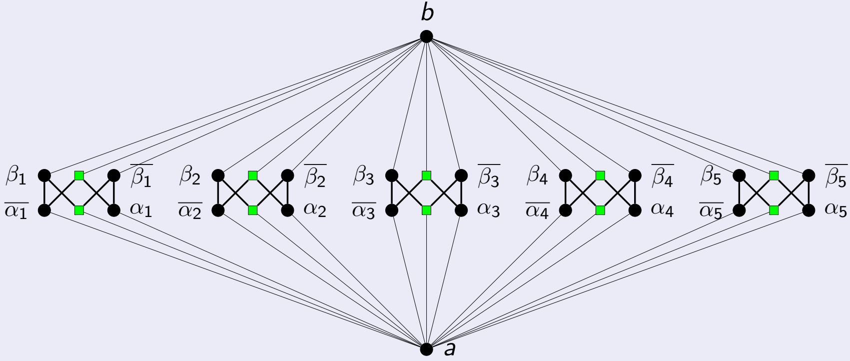
Minimal (a, b)-separator extension

Reduction to 3-SAT : reduction graph for $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_5) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_5})$



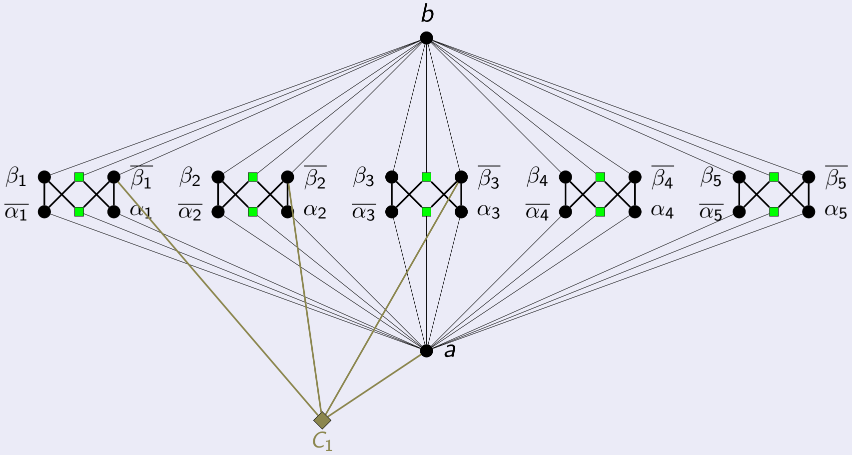
Minimal (a, b)-separator extension

Reduction to 3-SAT : reduction graph for $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_5) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_5})$



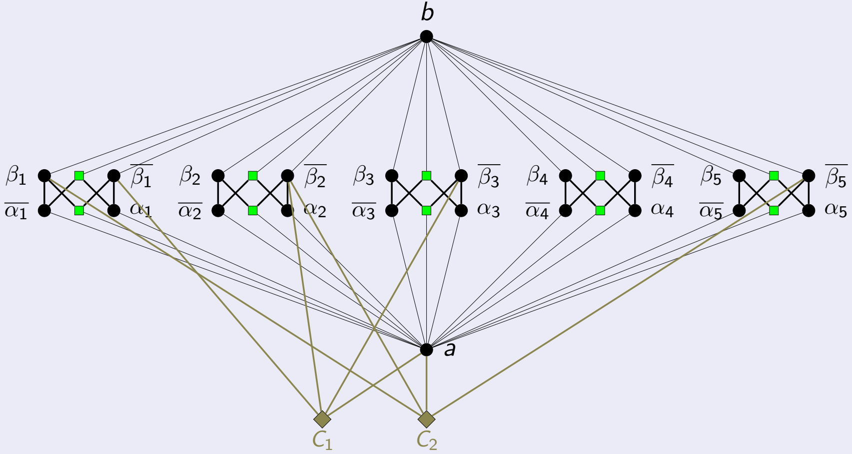
Minimal (a, b)-separator extension

Reduction to 3-SAT : reduction graph for $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_5) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_5})$



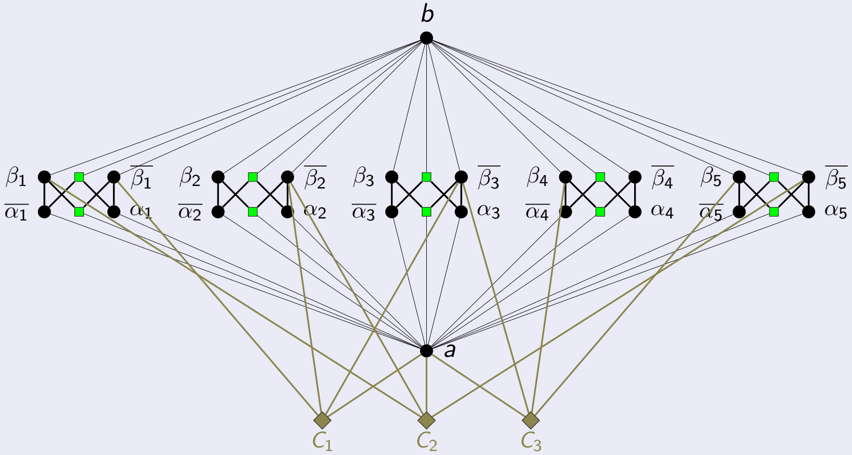
Minimal (a, b)-separator extension

Reduction to 3-SAT : reduction graph for $\Phi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_5) \land (x_3 \lor \overline{x_4} \lor \overline{x_5})$



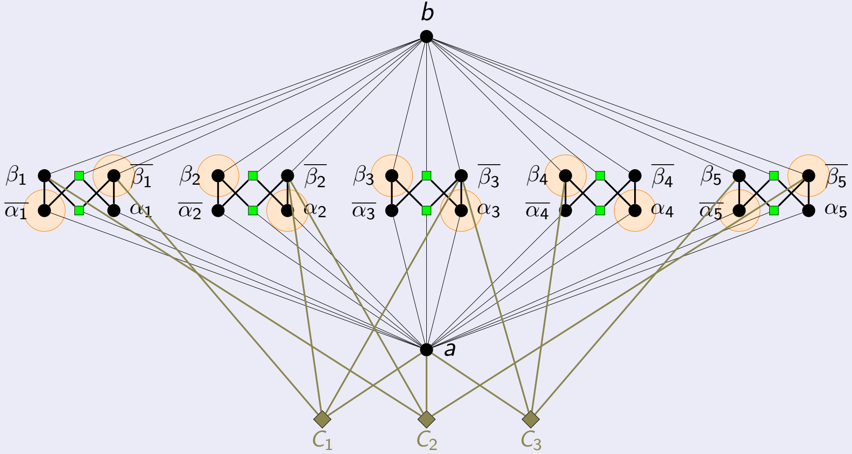
Minimal (a, b)-separator extension

Reduction to 3-SAT : reduction graph for $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_5) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_5})$



Minimal (a, b)-separator extension

Reduction to 3-SAT : reduction graph for $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_5) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_5})$

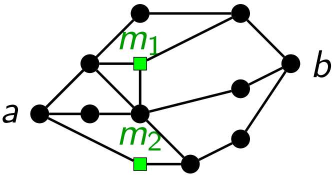


PMC extension

Potential Maximal Clique Extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* such that $M \cap F = \emptyset$.
Question: Is there a X, a potential maximal clique with $M \subseteq X$ and $X \cap F = \emptyset$?

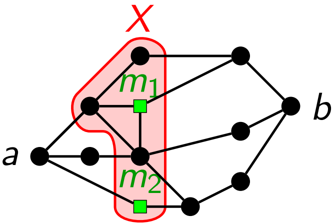
PMC extension

Potential Maximal Clique Extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* such that $M \cap F = \emptyset$.
Question: Is there a X, a potential maximal clique with $M \subseteq X$ and $X \cap F = \emptyset$?



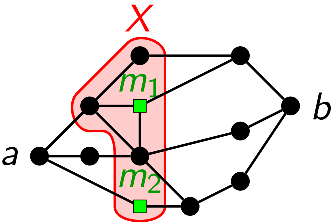
PMC extension

Potential Maximal Clique Extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* such that $M \cap F = \emptyset$.
Question: Is there a X, a potential maximal clique with $M \subseteq X$ and $X \cap F = \emptyset$?



PMC extension

Potential Maximal Clique Extension
Input: A graph $G = (V, E)$, a subset $M \subset V$ of *mandatory* vertices, a subset $F \subset V$ of *forbidden* such that $M \cap F = \emptyset$.
Question: Is there a X, a potential maximal clique with $M \subseteq X$ and $X \cap F = \emptyset$?



Theorem (Bergé, Limouzy, Schivre 2025)

Potential Maximal Clique Extension is NP-complete.

Future work

Amelioration of Bouchitté and Todinca algorithm from 2002 ?

Amelioration of the case **No. 4** could lead to an ***output linear*** complexity.

Polynomial delay algorithm ?

- Conception of an polynomial delay algorithm using other methods such as ***Proximity Search***.
- Algorithm using Flashlight Search using a different extension problem.

Impossibility result ?

Impossibility proof of the constructing of an incremental polynomial or polynomial delay algorithm.

Thank you !

Minimal (a, b)-separator extension

Separator enumeration : known results

- Takata 2010 :
  - Minimal (a, b)-separators enumeration :
    - Polynomial delay ($O(nm)$).
    - Polynomial space ($O(n)$).
  - Minimal separators enumeration :
    - Incremental polynomial ($O(n^3 m)$).
    - Polynomial space ($O(n)$).
- Bergougnoux, M. Kanté and Wasa WEPA 2019 : Improvement of the Takata algorithm for all separators. Use of parallelisation.
  - Polynomial delay ($O(n^3 m)$).
  - Polynomial space ($O(n^3 + m)$).

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

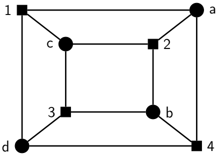
Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



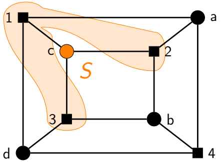
$\mathrm{PMC} = \{1, 2, 3, 4\}$

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



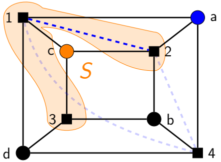
PMC $= \{1, 2, 3, 4\}$

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



PMC $= \{1, 2, 3, 4\}$

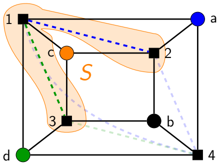
124

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



PMC $= \{1, 2, 3, 4\}$

124

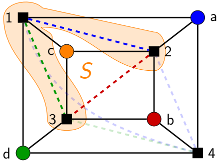
134

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



PMC $= \{1, 2, 3, 4\}$

124

134

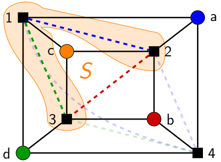
234

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



PMC $= \{1, 2, 3, 4\}$
No active separator

124

134

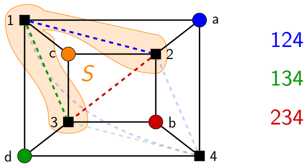
234

S-active separator and S-active PMC

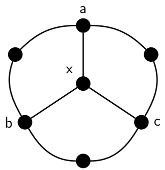
Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



PMC $= \{1, 2, 3, 4\}$
No active separator



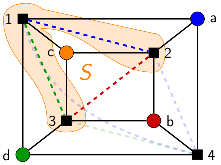
PMC $= \{a, b, c, x\}$

S-active separator and S-active PMC

Correspond to the PMC found by the 4^{th} case of the Bouchitté and Todinca algorithm.

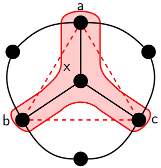
Definition

Ω a PMC of G, Δ_Ω the set of separators of Ω and S a minimal separator.
$\Omega' = \Omega$ to whom we add edges between all the vertices of the separators $X \subseteq \Delta_\Omega \setminus S$ so that any X is a clique.
If Ω' is not a clique, then S is active. Otherwise S is inactive.



124
134
234

PMC $= \{1, 2, 3, 4\}$
No active separator



PMC $= \{a, b, c, x\}$

Separators $\{a, b\}, \{a, c\}, \{b, c\}$ actives

S-active PMC extension

S-active PMC extension
Input: A graph $G = (V, E)$ and a subset $S \subset V$ minimal separator of G.
Question: Is X, a potential maximal clique with $S \subset X$ and S active for X exist ?

S-active PMC extension

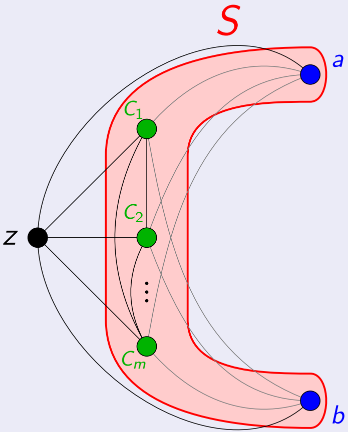
S-active PMC extension
Input: A graph $G = (V, E)$ and a subset $S \subset V$ minimal separator of G.
Question: Is X, a potential maximal clique with $S \subset X$ and S active for X exist ?

Theorem (Bergé, Limouzy, Schivre 2025)

S-active PMC extension is NP-complete.

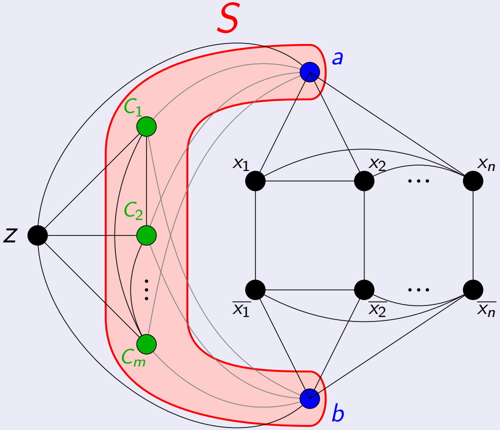
NP-completeness of the S-active PMC extension

Reduction to 3-SAT : reduction graph



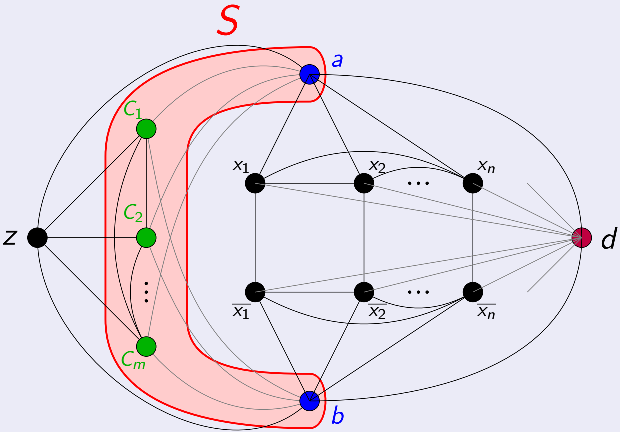
NP-completeness of the S-active PMC extension

Reduction to 3-SAT : reduction graph



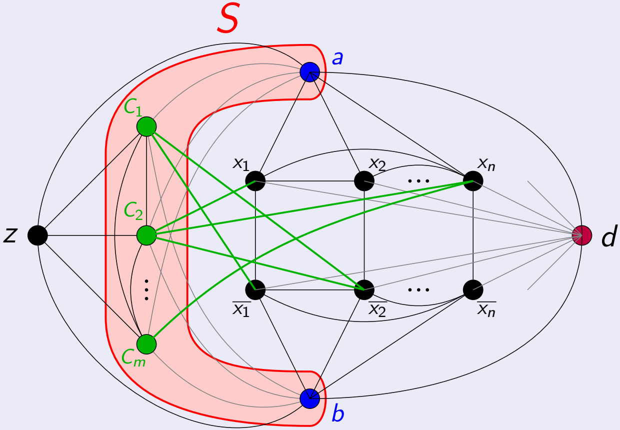
NP-completeness of the S-active PMC extension

Reduction to 3-SAT : reduction graph



NP-completeness of the S-active PMC extension

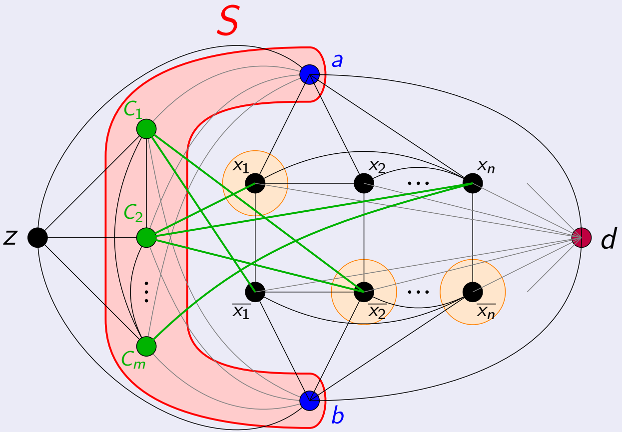
Reduction to 3-SAT : reduction graph



$$\Phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m \text{ with}$$
$$C_1 = x_1 \vee x_2,\ C_2 = \overline{x_1} \vee x_2 \vee \overline{x_n},\ \ldots,\ C_m = \overline{x_n}$$

NP-completeness of the S-active PMC extension

Reduction to 3-SAT : reduction graph



$$\Phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m \text{ with}$$

$$C_1 = x_1 \vee x_2,\ C_2 = \overline{x_1} \vee x_2 \vee \overline{x_n},\ \ldots,\ C_m = \overline{x_n}$$

Proximity Search (Conte and Uno 2019)

Principle

- Establish a function Neighborhood between the solutions.
- Travel with Breadth First Search on the solutions graph.

Difficulties

- The solution graph must be strongly connected.
- Each vertex has a polynomial degree.
- A proximity measure $\widetilde{\cap}$ must exist between solutions.
- For all solutions pair S, T : $\exists S' \in \text{Neighborhood}(S)$, $|S' \widetilde{\cap} T| > |S \widetilde{\cap} T|$.