Infinite Grid Exploration by Disoriented Robots [⋆]

Quentin Bramas[1], Stéphane Devismes[2], and Pascal Lafourcade[3]

[1] University of Strasbourg, ICUBE, CNRS
[2] Université Grenoble Alpes, VERIMAG
[3] University Clermont Auvergne, CNRS UMR 6158, LIMOS

Abstract. We study the *infinite grid exploration* (IGE) problem by a swarm of autonomous mobile robots. Those robots are opaque, have limited visibility capabilities, and run using synchronous Look-Compute-Move cycles. They all agree on a common chirality, but have no global compass. Finally, they may use lights of different colors that can be seen by robots in their surroundings, but except from that, robots have neither persistent memories, nor communication mean. We show that using only three fixed colors, six robots, with a visibility range restricted to one, are necessary and sufficient to solve the non-exclusive IGE problem. We show that using modifiable colors with only five states, five such robots, with a visibility range restricted to one, are necessary and sufficient to solve the (exclusive) IGE problem. Assuming a visibility range of two, we also provide an algorithm that solves the IGE problem using only seven identical robots without any light.

1 Introduction

We deal with a swarm of mobile robots having low computation and communication capabilities. The robots we consider are opaque (*i.e.*, a robot is able to see another robot if and only if no other robot lies in the line segment joining them) and run in synchronous Look-Compute-Move cycles, where they can sense their surroundings within a limited visibility range. All robots agree on a common chirality (*i.e.*, when a robot is located on an axis of symmetry in its surroundings, it is able to distinguish its two sides one from another), but have no global compass (they agree neither on a North-South, nor a East-West direction). However, they may use lights of different colors [17]. These lights can be seen by robots in their surroundings. However, except from those lights, robots have neither persistent memories nor communication capabilities.

We are interested in coordinating such weak robots, endowed with both typically small visibility range (*i.e.*, one or two) and few light colors (only a constant number of them), to solve an infinite task in an infinite discrete environment. As an attempt to tackle this general problem, we consider the exploration of an infinite grid, where nodes represent locations that can be sensed by robots and edges represent the possibility for a robot to move from one location to another. The exploration task requires each node to

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be visited within finite time by at least one robot. In the following, we refer to it as the *Infinite Grid Exploration* (IGE) problem.

Contribution. We give both negative and positive results. We first show that if robots have a common chirality but a bounded visibility range, the IGE problem is unsolvable with:

- two robots, even if those robots agree on common North (the proof of this result is essentially an adaptation to our context of the impossibility proof given in [13]);
- three or four robots equipped with self-inconsistent compasses (*i.e.*, the compasses may change throughout the execution).
- five robots equipped with self-inconsistent compasses if the visibility range is restricted to one, and the lights have fixed (*i.e.*, non-modifiable) colors.

We then propose three algorithms, respectively called $\mathcal{A}_1^{Fixed}$, $\mathcal{A}_1^{Modifiable}$, and $\mathcal{A}_2^{nolight}$, for solving the IGE problem using opaque robots equipped with self-inconsistent compass, yet agreeing on a common chirality. In particular, $\mathcal{A}_1^{Modifiable}$ and $\mathcal{A}_2^{nolight}$ additionally satisfy *exclusiveness* [2], which requires any two robots to never simultaneously occupy the same position nor traverse the same edge. In more detail, Algorithm $\mathcal{A}_1^{Fixed}$ solves the non-exclusive IGE problem using six robots with visibility range restricted to one, and only three fixed (*i.e.*, non-modifiable) colors. In this setting, the algorithm is optimal in terms of number of robots. In Algorithm $\mathcal{A}_1^{Modifiable}$, five robots use modifiable colors with only five states, still with visibility range one. In this setting, the algorithm is optimal in terms of number of robots; moreover it ensures exclusiveness. Algorithm $\mathcal{A}_2^{nolight}$ requires seven identical robots without light (*i.e.*, seven anonymous oblivious[4] robots) and ensures exclusiveness, yet assuming visibility range two. In order to help the reader, animations are available online [6], for each of the three algorithms.

Related Work. The model of robots with lights (also called luminous robots) has been proposed by Peleg in [17]. In [8], the authors use robots with lights and compare the computational power of such robots with respect to the three main execution models: fully-synchronous, semi-synchronous, and asynchronous. Solutions for dedicated problems such as *weak gathering* or *mutual visibility* have been respectively investigated in [15] and [16].

Mobile robot computing in infinite environments has been first studied in the continuous two-dimensional Euclidean space. In this context, studied problems are mostly *terminating* tasks, such as *pattern formation* [11] and *gathering* [14], *i.e.*, problems where robots aim at eventually stopping in a particular configuration specified by their relative positions. A notable exception is the *flocking* problem [18], *i.e.*, the infinite task consisting of forming a desired pattern with the robots and make them moving together while maintaining that formation.

When considering a discrete environment, space is defined as a graph, where the nodes represent the possible locations that a robot can take and the edges the possibility for a robot to move from one location to another. In this setting, researchers have first considered finite graphs and two variants of the exploration problem, respectively called the *terminating* and *perpetual* exploration. The terminating exploration requires every possible location to be eventually visited by at least one robot, with the additional

[4] *Oblivious* means that robots cannot remember the past.

constraint that all robots stop moving after task completion. In contrast, the perpetual exploration requires each location to be visited infinitely often by all or a part of robots. In [9], authors solve terminating exploration of any finite grid using few asynchronous anonymous oblivious robots, yet assuming unbounded visibility range. The exclusive perpetual exploration of a finite grid is considered in the same model in [3].

Various terminating problems have been investigated in infinite grids such as *arbitrary pattern formation* [4], *mutual visibility* [1], and *gathering* [10,12]. The possibly closest related work to our paper is that of Emek *et al.* [13]. They consider the *treasure search problem in an unbounded-size grid* which is closely related to the IGE problem; see [7]. They consider robots that operate in two models: the semi-synchronous and synchronous ones. However, they do not impose the exclusivity at all since their robots can only sense the states of the robots located at the same node (in that sense, the visibility range is zero). The main difference with our settings is that they assume all robots agree on a *global compass*, *i.e.*, they all agree on the same directions North-South and East-West; while we only assume here a *common chirality*. This difference makes the problem somehow easier to solve, indeed they propose two algorithms that respectively need three synchronous and four semi-synchronous robots, while in our settings we show that at least five robots are necessary to solve the IGE problem (even in its non-exclusive variant). Notice that they also exclude solutions for two robots.

In a followup paper [7], Brandt *et al.* extend the impossibility result of Emek *et al.* Indeed, they show the impossibility of *exploring an infinite grid* with three semi-synchronous deterministic robots that agree on a common coordinate system. Although proven using similar techniques, this result is not correlated to ours. Indeed, the lower bound of Brandt *et al.* holds for robots that are weaker in terms of synchrony assumption (semi-synchrony *vs.* fully synchrony in our case), but stronger in terms of coordination capabilities (common coordinate system *vs.* self-inconsistent compass with a common chirality in our case). In other words, our impossibility results do not (even indirectly) follow from those of Brandt *et al.* since in our model difficulties arise from the lack of coordination capabilities and not the level asynchrony. As a matter of facts, based on the results of Emek *et al.* [13], four (asynchronous) robots are actually necessary and sufficient in their settings, while we show that it is five in our context.

Roadmap. In the next section, we define our computational model. In Section 3, we present several lower bounds on the number of robots to solve the IGE problem. In Section 4 and Section 5, we propose algorithms solving the IGE problem under visibility range one and two, respectively. We conclude with some perspectives in Section 6.

Due to the lack of space, some technical results are omitted.

2 Model

We consider a set of $n > 0$ robots located on an *infinite grid* graph with vertex set in $\mathbb{Z} \times \mathbb{Z}$, *i.e.*, there is an edge between two nodes (i, j) and (k, l) if and only if the *Manhattan distance* between those two nodes, *i.e.*, $|i - k| + |j - l|$, is one. Notice that coordinates are used for the analysis only, *i.e.*, robots cannot access them.

We assume time is discrete and at each *round*, the robots synchronously perform a *Look-Compute-Move* cycle. In the *Look* phase, a robot gets a snapshot of the subgraph induced by the nodes within distance $\Phi \in \mathbb{N}^*$ from its position. Φ is called the *visibility range* of the robots. The snapshot is not oriented in any way as the robots do not agree on a common North. However, it is implicitly ego-centered since the robot that performs a Look phase is located at the center of the subgraph in the obtained snapshot. Then, each robot *computes* a destination (either Up, Left, Down, Right or Idle) based only on the snapshot it received. Finally, it *moves* towards its computed destination. We also assume that robots are *opaque* and can obstruct the visibility so that if three robots are aligned, the two extremities cannot see each other.

Robots may have *lights* with different colors that can be seen by robots within distance Φ from them. Let Cl be the set of possible colors. Even when an algorithm does not achieve exclusiveness, we forbid any two robots to occupy the same node simultaneously. A node is *occupied* when a robot is located at this node, otherwise it is *empty*. The *state* of a node is either the light color of the robot located at this node, if it is occupied, or $\perp$ otherwise. In the Look phase, the snapshot includes the state of the nodes (at distance Φ). During the compute phase, and if colors are *modifiable*, a robot may decide to change its color. Otherwise, colors are said to be *fixed*.

Configurations. A *configuration* C is a set of pairs (p, c) where $p \in \mathbb{Z} \times \mathbb{Z}$ is an occupied node and $c \in Cl$ is the light color of the robot located at p. A node p is empty if and only if $\forall c,\ (p, c) \notin C$. We sometimes just write the set of occupied nodes when the colors are clear from the context. Also, for better readability, we sometimes partition the configuration into several subsets $C_1, \ldots, C_k$ and write $C = \{C_1, \ldots, C_k\}$ instead of writing $(C = C_1 \cup \ldots \cup C_k) \wedge (\forall i \neq j, C_i \cap C_j = \emptyset)$.

Views. We denote by G_r the *globally oriented view* centered at the robot r, *i.e.*, the subset of the configuration containing the states of the nodes at distance at most Φ from r, translated so that the coordinates of r is $(0, 0)$. We use this globally oriented view in our analysis to describe the movements of the robots: when we say "the robot moves Up", it is according to the globally oriented view. However, since robots do not agree on a common North, they have no access to the globally oriented view. Instead, when a robot looks at its surroundings, it obtains a snapshot. To model this, we assume that, the *local view* acquired by a robot r in the Look phase is the result of an arbitrary *indistinguishable transformation* on G_r. The set $\mathcal{IT}$ of indistinguishable transformations is closed by composition and depends on the assumptions we make on the robots. The rotations of angle $\pi/2$, and consequently of angle π and $3\pi/2$, centered at r are in $\mathcal{IT}$ if and only if the robots do not agree on a common North direction. A mirroring is in $\mathcal{IT}$ if and only if the robots do not agree on a common *chirality* (they cannot distinguish between clockwise and counterclockwise). Moreover, in the obstructed visibility model, the function that removes the state of a node u if there is another robot between u and r is in $\mathcal{IT}$ and is systematically applied. For a robot r, if the same transformation $f_r \in \mathcal{IT}$ is used for every look phase of r, we say that r is *self-consistent*. Otherwise, the adversary can choose a different transformation for each look phase, and r is said to be *self-inconsistent*.

In the remaining of the paper, all our algorithms assume that all robots agree on a common chirality, *i.e.*, they can distinguish two mirrored views, but we make no assumption on the self-consistency of the coordinate system. On the other hand, we give impossibility results for stronger models when possible.

When a robot r computes a destination d, it is relative to its local view $f(G_r)$, which is its globally oriented view G_r transformed by some $f \in \mathcal{IT}$. It is important to see that the actual movement of the robot in its globally oriented view G_r, and so in the configuration, is $f^{-1}(d)$. Indeed, if $d = Up$ but the robot sees the grid upside-down (f is the π-rotation), then the robot moves $Down = f^{-1}(Up)$. In a configuration C, $V_C(i,j)$ denotes the globally oriented view of a robot located at (i,j).

Algorithm. An algorithm $\mathcal{A}$ is a tuple (Cl, I, T) where Cl is the set of possible colors, I is the initial configuration, and T is the transition function $Views \to \{Idle,\ Up,\ Left,\ Down,\ Right\} \times Cl$, where $Views$ is the set of globally oriented views.

Recall that we assume in our algorithms that the robots are not self-consistent. In this context, we say that an algorithm (Cl, I, T) is *well-defined* if the global destination computed by a robot does not depend on the transformation f chosen by the adversary, *i.e.*, for every globally oriented view V, and every transformation $f \in \mathcal{IT}$, we have $T(V) = f^{-1}(T(f(V)))$. This is usually a property obtained by construction of the algorithm, as we describe the destination d for a given globally oriented view V and then assume that the destination computed from local view $f(V)$ is $f(d)$, for any $f \in \mathcal{IT}$. We can extend the transition function T to the entire configuration. When the robots are in configuration C, the configuration obtained after one round of execution is denoted $T(C)$ and contains the pair $((i,j),c)$ if and only if $\exists c' \in Cl$ for which one of the following conditions holds

- $((i,j),c') \in C$ and $T(V_C(i,j)) = (Idle, c)$,
- $((i-1,j),c') \in C$ and $T(V_C(i-1,j)) = (Right, c)$,
- $((i+1,j),c') \in C$ and $T(V_C(i+1,j)) = (Left, c)$,
- $((i,j-1),c') \in C$ and $T(V_C(i,j-1)) = (Up, c)$,
- $((i,j+1),c') \in C$ and $T(V_C(i,j+1)) = (Down, c)$.

The execution of algorithm $\mathcal{A}$ is the sequence $(C_i)_{i\in\mathbb{N}}$ of configurations such that $C_0 = I$ and $\forall i \geq 0$, $C_{i+1} = T(C_i)$. We sometimes write $\mathcal{A}(C)$ instead of $T(C)$.

Infinite Grid Exploration. An algorithm $\mathcal{A}$ solves the *infinite grid exploration* (IGE) problem if in the execution $(C_i)_{i\in\mathbb{N}}$ of $\mathcal{A}$ and for every node $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ of the grid, there exists $t \in \mathbb{N}$ such that (i,j) is occupied in C_t.

Notations. $\boldsymbol{t}_{(i,j)}(C)$ denotes the translation of the configuration C of vector (i,j).

3 Impossibility Results

The lemma below states the intuitive, yet non trivial, idea that, to explore an infinite grid, the maximum distance between the two farthest robots should tend to infinity. This claim is the cornerstone of our impossibility results.

Lemma 1. *Let $(C_i)_{i\in\mathbb{N}}$ be an execution of an algorithm $\mathcal{A}$. Let d_i be the distance between the two farthest robots in C_i. If $\mathcal{A}$ solves the IGE problem, then* $\lim\limits_{i\to+\infty} d_i = +\infty$.

Proof. We proceed by the contradiction. So we suppose there exists a bound $B > 0$ such that there are infinitely many configurations in the execution where the distance between every pair of robots is less than B. In other words, there is a subsequence of $(C_i)_{i\in\mathbb{N}}$ where the distance between every pair of robots is less than B. Let $(b_i)_{i\in\mathbb{N}}$ be the sequence of indices of this subsequence, *i.e.*, $(b_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of integers such that $d_{b_i} < B$.

When all robots are at distance less than B, then the occupied positions are included in a square sub-grid of size $B \times B$. Since the number of possible configurations included in a sub-grid of size $B \times B$ is finite, there must be two indices k and l such that $C_{b_l} = t(C_{b_k})$ and $k < l$ for a given translation t. The movements done by the robots in configurations C_{b_k} and C_{b_l} are the same because each robot has the same globally oriented view in both configurations, only their positions change. Thus $C_{b_l+1} = t(C_{b_k+1})$ and so on so forth, so that $\forall i$, $C_{b_l+i} = t(C_{b_k+i})$. We obtain that the configurations are periodic (with period $P = b_l - b_k$) and a node u is visited if and only if it is visited before round b_l or if there exists a node v visited between round b_k and b_l such that $u = t^q(v)$ with $q > 0$. So, we claim that there exists a node that is never visited.

To prove this claim, we now exhibit such a node. Let I be the set of integers i such that $(t^{-1})^i(0, 0)$ is visited before round b_l applied i times. I is finite because the number of nodes visited before b_l is finite. Let m be the maximum integer in I (or 0 if I is empty). Let $u = (t^{-1})^{m+1}(0, 0)$. Then, clearly u is not visited before round b_l, otherwise we have a contradiction with the maximality of m. Moreover, u cannot be visited after round b_l, otherwise u would be equals to $t^q(v)$ for a given integer q and a given node v, visited between round b_k and b_l, *i.e.*, $v = (t^{-1})^q(u) = (t^{-1})^{q+m+1}(0, 0)$, which also contradicts the maximality of m. Thus u is never visited. □

Theorem 1. *No algorithm can solve the IGE problems using two robots, even if robots agree on common North and chirality.*

Proof. By Lemma 1, there is a configuration from which the two robots will no more see each other (their distance will remain greater than an arbitrary bound $B \geq \Phi$). For each robot, its next move will only depend on its color. Since the number of color is finite, the movements of each robot are then periodic. So, from that point, each robot r moves by periodically performing the same translation t_r, and thus some nodes are never visited. □

Lemma 2. *Assume the robots are equipped with self-inconsistent compasses, yet agree on a common chirality. Whenever a robot does not see any other one, it either stays idle or the adversary can make it alternatively move between two chosen adjacent nodes.*

Proof. If such a robot does not stay idle, it moves toward a direction $d \in \{Up, Down, Left, Right\}$ but since its orientation is not self-consistent, the adversary can choose, for each activation, a transformation $f \in \mathcal{IT}$ such that the destination $f^{-1}(d)$ in the globally oriented view alternate between two chosen directions (*e.g.*, Up and $Down$). □

Theorem 2. *It is impossible to solve the IGE problem using three robots equipped with self-inconsistent compasses that agree on a common chirality.*

Proof. By Lemma 1, there is a configuration where two robots are always at distance at least B (say $B > 2 \cdot \Phi + 2$), so that it is impossible for any robot to see the all others in the same snapshot. Now, since there are three robots, at least one robot r does not see any other robot. By Lemma 2, if r stays alone, then it remains idle or the adversary can make it alternatively move between two nodes infinitely often. Moreover, the two other robots cannot explore the grid alone, by Theorem 1. Now, they cannot both move towards r because in such a case the distance between the farthest robots would become less than B, a contradiction. Finally, if one of the two other robots moves towards r, at some point all robots are out of the visibility range of each other. In that case, the adversary can make the exploration fail, by Lemma 2. □

Due to the lack of space, the proofs of the next two theorems are only sketched.

Theorem 3. *It is impossible to solve the IGE problem using four robots equipped with self-inconsistent compasses that agree on a common chirality.*

Proof Outline. Assume, by contradiction, that an algorithm $\mathcal{A}$ solves the IGE problem using four robots equipped with self-inconsistent compasses that agree on a common chirality. Then, using Lemma 1, we consider a round where the two farthest robots, called here *extremities*, are always at distance $B \gg \Phi$. Since we know three robots are not enough, no robot stays alone forever. Therefore, infinitely often, there is a moving group of two robots traveling from one extremity to the other. Moreover, whenever traveling an arbitrary long distance, a group of robots necessarily uses periodic movements. We can then show that these periodic movements induce that after some time, the moving group travels infinitely often between two extremities by periodically performing the same translation. This latter claim implies that, after some time, the movements of the robots depend only on configurations of bounded size, which in turn implies that the movements of the two extremities are periodic. Since extremities eventually perform periodic movements, they each one move inside a strip of bounded width that grows in only one direction. Hence, whether they move along collinear vectors or not, the algorithm misses nodes forever in the exploration process. □

Theorem 4. *It is impossible to solve the IGE problem using five robots with self-inconsistent compasses, a common chirality, fixed colors, and visibility range one.*

Proof Outline. One can observe that the main argument of Theorem 3 works with more than one robot at an extremity as soon as they stay idle or move a finite distance when they are not in the range of the other robots, because that makes their movements independent from their distance to the other extremities, and hence makes their movements periodic.

Since there are five robots, there cannot be more than two robots left by the moving group at an extremity, and if there are two robots at an extremity, they cannot move forever because that would mean those two robots never meet the other robots afterwards (all robots move at the same speed). So, in the settings of this theorem, the movements of the extremities are periodic.

We can also deduce that the path taken by a moving group of robots, to travel between extremities, is either a vertical line or an horizontal line. In this case, whether the extremities move along collinear vectors or not, the algorithm misses nodes forever in the exploration process. □

4 Infinite Grid Exploration with $\Phi = 1$

In this section, we present two algorithms assuming visibility range one. The former, Algorithm $\mathcal{A}_1^{Fixed}$, uses six robots with three fixed colors. The latter, Algorithm $\mathcal{A}_1^{Modifiable}$, uses five robots with five modifiable colors and additionally achieves exclusiveness. Recall that animations of these two algorithms are available in our complementary material [6]. The fact that the rules of these algorithms are well-defined has been checked by the script that generated those animations. This has been done by making sure that (1) the view of any rule cannot be transformed into the view of another rule using a combination of $\frac{\pi}{2}$-rotations, and (2) for each rule, the global destination does not depend on the applied local indistinguishable transformation.

4.1 An algorithm using six robots and three fixed colors

Algorithm Overview. First, our robots are divided into two categories: the *beacon robots* — four robots with color B — and the *moving group* — two robots with respective color L and F. The beacons are used to delimit the area which is already explored. The moving group aims at reaching the beacons one by one. Each time a beacon is reached by the moving group, it moves once in the diagonal (two hops) to take the newly explored nodes into account. The moving group then continues toward the next beacon, and so on. Each time the moving group comes back to the first beacon, a so-called *phase* terminates: the border of the area initially delimited by the four beacons is now fully visited, and the area newly delimited by the beacons is bigger; see Fig. 2 to visualize the increasing area that is explored by the moving group (r_L is a particular robot of the moving group, whose role will be explained later).

The moving group successfully performs a phase independently of the distance between the beacons, so that infinitely many growing phases are achieved in sequence. The IGE is then solved as any node of the grid is eventually included in the area delimited by the beacons. Note that we use the same technique for the two other algorithms, yet using areas of different shapes.

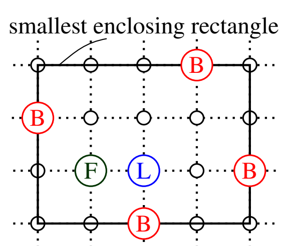


Fig. 1: Initial configuration of Algorithm $\mathcal{A}_1^{Fixed}$.

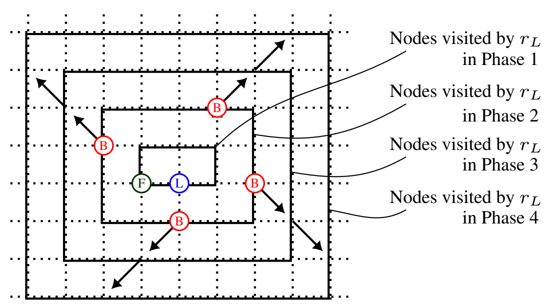


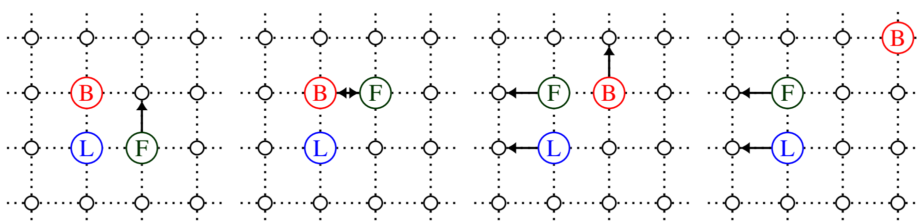
Fig. 2: Visited area after four phases for $\mathcal{A}_1^{fixed}$.

Definition of Algorithm $\mathcal{A}_1^{Fixed}$. We use the set of colors $Cl = \{L, F, B\}$ to partially distinguish robots. The moving group is composed of two robots: one with light color L called the *leader*, and the other with light color F called the *follower*. The four remaining robots, *i.e.*, the beacons, have light color B. The initial configuration I of $\mathcal{A}_1^{Fixed}$ is defined as follows: $I = \{((-1,0),F),((0,0),L),((0,-1),B),((2,0),B),((1,2),B),((-2,1),B)\}$; see Fig. 1.

Recall that $\mathcal{A}_1^{Fixed}$ executes in phases. At the beginning of each phase, we consider the smallest enclosing rectangle, denoted by SER, that encloses the four beacon robots, *e.g.*, in Fig. 1, the SER of the initial configuration I is drawn with solid lines. During a phase, the follower robot r_F explores the borders of the SER, while the leader robot r_L visits the borders of the largest rectangle strictly inside the SER. First, the moving group $\{r_L, r_F\}$ moves straight until the leader robot becomes a neighbor of a beacon robot. Then, the positions of three robots are *adjusted* so that (1) the moving group $\{r_L, r_F\}$ makes a turn, and (2) the beacon robot moves diagonally (two hops) in order to expand the SER. (Notice the execution starts by an adjustment.) Overall, at the end of Phase i (and so at the beginning of Phase $i+1$), both the length and width of SER increases by two.

The rules of $\mathcal{A}_1^{Fixed}$ are defined in Figs. 4, 5, and 6. Some rules aim at moving the group of robots $\{r_L, r_F\}$ straight and the others are used to manage an *adjustment*. In the following, we detail how $\{r_L, r_F\}$ moves straight toward a beacon robot, does a left turn, and how the reached beacon robot moves diagonally. Recall that the rules below also describe the algorithm behavior on equivalent, rotated, local views.

Using Rules of Fig. 6, if we apply $\mathcal{A}_1^{Fixed}$ to $\{((i,j),L),((i+1,j),F)\}$, we obtain $\{((i,j+1),L),((i+1,j+1),F)\}$, *i.e.*, the two robots go through the translation $\boldsymbol{t}_{(0,1)}$. So, the group $\{r_L, r_F\}$ moves on a straight line when isolated. If we rotate the two robots with angle $\pi/2$, π, or $3\pi/2$, then the moving group will move to the left, down, or right, respectively. In fact, the direction of the translation actually depends on the relative positions of r_L and r_F.



(a) $\mathcal{R}_{strF}$ is executed. (b) $\mathcal{R}_{trnB1}$ and $\mathcal{R}_{trnF1}$ are executed. (c) $\mathcal{R}_{strL}$, $\mathcal{R}_{trnB2}$, and $\mathcal{R}_{trnF2}$ are executed. (d) $\mathcal{R}_{strL}$ and $\mathcal{R}_{strF}$ are executed.

Fig. 3: Robots performing a turn.

Before giving the rules for the adjustments and in order to clearly explain how our algorithm works, we show in Fig. 3 the global configurations that occur when the moving group reaches the upper right beacon robot. In the first round, the follower (only) moves straight, as previously, to become neighbor of the beacon. In the second round, the beacon and the follower swap their positions, while the leader stays idle. In the third round, the beacon moves up to finalize its diagonal motion, while the moving group $\{r_L, r_F\}$ starts to move again in a straight line toward the left.

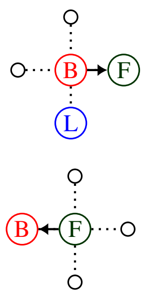


Fig. 4: $\mathcal{R}_{trnB1}$ and $\mathcal{R}_{trnF1}$.

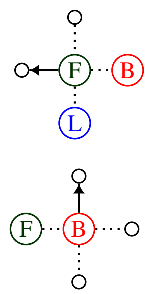


Fig. 5: $\mathcal{R}_{trnB2}$ and $\mathcal{R}_{trnF2}$.

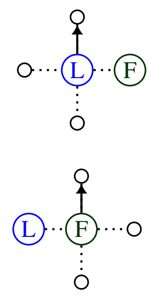


Fig. 6: $\mathcal{R}_{strL}$ and $\mathcal{R}_{strF}$.

In more details, for the first round, there is no rule when r_L sees a beacon robot, thus, when it happens r_L stays idle and r_F continues to move up one more time. For the second round, according to the rules of Fig. 4, when r_F only sees the beacon robot, it moves towards it, and when the beacon sees both r_F and r_L, it moves toward r_F, so that they swap their positions, while r_L stays idle. Finally, the beacon robot makes a last move up, and the moving group moves away from the beacon, according to the two rules of Fig. 5 and the rule of Fig. 6 that makes the leader move straight. With those rules, and with $M = \{((i,j),L),((i+1,j),F)\}$, $X = \{((i,j+1),B)\}$, we can see that by applying $\mathcal{A}_1^{Fixed}$ three times starting from $\{M,X\}$ we obtain $\{((i-1,j),L),((i-1,j+1),F),((i+1,j+2),B)\}$, *i.e.*, $\{\rho(M),\ \boldsymbol{t}_{(1,1)}(X)\}$, where ρ is the rotation centered at $(i-0.5,j-0.5)$ of angle $\pi/2$.

Theorem 5. *Algorithm $\mathcal{A}_1^{Fixed}$ solves the IGE problem using six robots and fixed colors having common chirality and a visibility range of one.*

Proof. We denote by $I = C^0 = \{M^0, C_0^0, C_1^0, C_2^0, C_3^0\}$ the initial configuration given in Fig. 1, where $M^0 = \{((-1,0),F),((0,0),L)\}$, $C_0^0 = \{((0,-1),B)\}$, $C_1^0 = \{((2,0),B)\}$, $C_2^0 = \{((1,2),\ B)\}$, and $C_3^0 = \{((-2,1),B)\}$. We define the configuration $C^i = \{M^i, C_0^i, C_1^i, C_2^i, C_3^i\}$ in Phase i, where $M^i = \boldsymbol{t}_{(-i,-i)}(M^0)$, $C_0^i = \boldsymbol{t}_{(-i,-i)}(C_0^0)$, $C_1^i = \boldsymbol{t}_{(i,-i)}(C_1^0)$, $C_2^i = \boldsymbol{t}_{(i,i)}(C_2^0)$, and $C_3^i = \boldsymbol{t}_{(-i,i)}(C_3^0)$. We now prove that starting with a configuration C^i, the configuration C^{i+1} is eventually reached. Since the initial configuration of our algorithm is C^0, this implies that every configuration C^i, for every $i \geq 0$, is gradually reached. By doing so, the leader robot visits all edges of growing rectangles. Consider the first configuration C^i of Phase i. In C^i, the distance between r_L and the beacon robot on its right is $2i+2$. Indeed, starting from C^i, the robot r_L starts from $(-i,-i)$ and that beacon robot starts from $(i+2,-i)$.

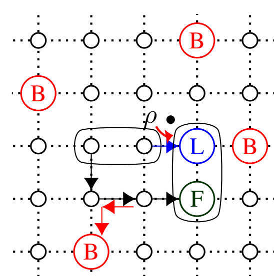


Fig. 7: Configuration after three rounds from C^0.

By executing the algorithm, we remark (see Fig. 7) that after three rounds (1) the configuration is $\{\rho(M^i), C_0^{i+1}, C_1^i, C_2^i, C_3^i\}$ (where ρ is the rotation with center $(0.5, 0.5)$ of angle $\pi/2$) and (2) r_L is at distance $2i+1$ from the bottom down beacon. From that point, the moving group $\{r_L, r_F\}$ starts moving one node to the right at each round (due to the first two rules) until robot r_L sees a beacon robot r in C_1^i; this event occurs at round $3+2i$, *i.e.*, three plus the number of empty nodes between r_L and r. After three more rounds, the moving group performs a left turn again and bottom right beacon robot is translated by a vector $(1, -1)$.

Thus, at round $3+2i+3$, the configuration is $\{\boldsymbol{t}_{(2i,0)}(\rho^2(M^i)), C_0^{i+1}, C_1^{i+1}, C_2^i, C_3^i\}$. After $2i+3$ more rounds, the moving group reaches the top right beacon robot, and performs another left turn. So, at round $3+2(2i+3)$ the configuration is $\{\boldsymbol{t}_{(2i,2i)}(\rho^3(M^i)), C_0^{i+1}, C_1^{i+1}, C_2^{i+1}, C_3^i\}$. Similarly, at round $3+3(2i+3)+1$ the configuration is $\{\boldsymbol{t}_{(-1,2i)}(\rho^4(M^i)), C_0^{i+1}, C_1^{i+1}, C_2^{i+1}, C_3^{i+1}\}$. We observe that the moving group $\{r_L, r_F\}$ required one extra round (as compared to other beacon robots) to reach the beacon robot in C_3^i.

Then, after $2i+1$ more rounds, the group of robots $\{r_L, r_F\}$ moves $2i+1$ nodes down to reach the bottom left beacon robot again, so that, at round $(3+3(2i+3)+1)+2i+1$, the configuration is $\{\boldsymbol{t}_{(-1,-1)}(\rho^4(M^i)), C_0^{i+1}, C_1^{i+1}, C_2^{i+1}, C_3^i\} = C^{i+1}$.

Recursively, if the robots start from configuration C^0, they reach configuration C^i in finite time, for any $i \geq 0$. Also, the nodes V_i visited by r_L between Phase i and $i+1$ contains the edges of the rectangle $\big\{\boldsymbol{t}_{(-i,-i)}(-1,0), \boldsymbol{t}_{(i,-i)}(1,0), \boldsymbol{t}_{(i,i)}(1,1), \boldsymbol{t}_{(-i,i)}(-1,1)\big\}$; see Fig. 2. Since $\bigcup_{i\geq 0} V_i = \mathbb{Z} \times \mathbb{Z}$, our algorithm solves the infinite grid exploration problem. $\square$

4.2 An algorithm using five robots and five modifiable colors

Algorithm $\mathcal{A}_1^{Modifiable}$ we present now solves the exclusive IGE problem using a minimum number of robots. As compared to the previous algorithm, to use one less robot, the moving group of two robots moves along a triangle, delimited by three beacon robots, instead of a rectangle. Except the shape of the growing polygonal, the principles are similar to the previous algorithm. Notice that we require modifiable colors to allow the moving group to follow a diagonal and to make adjustments without violating exclusiveness.

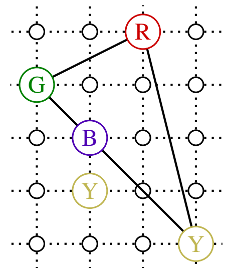


Fig. 8: Initial configuration I of $\mathcal{A}_1^{Modifiable}$.

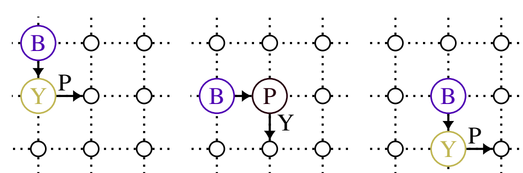


Fig. 9: Sequence of moves for a diagonal motion.

The set of colors is $Cl = \{R, Y, G, B, P\}$. Notice that, to reduce the number of used colors, the meaning of each color changes according to the stage of the exploration,

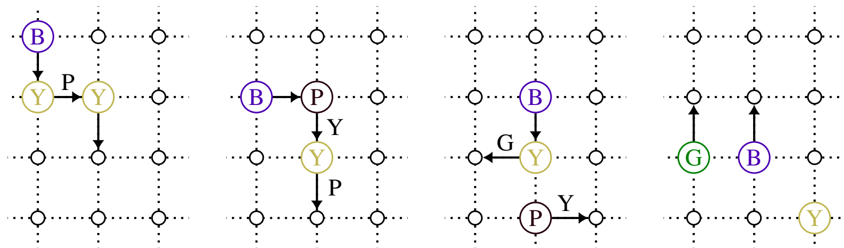


Fig. 10: Sequence of moves for a turn at the bottom beacon robot. A letter is written near each arrow to define the new color of the moving robot in case of change.

i.e., along the exploration they are used for different purposes. The initial configuration I is given in Fig. 8. The three beacon robots at the corners of the growing triangle respectively hold light colors Y, G, and R. The principle of the algorithm is as follows: starting from the initial configuration I and using the diagonal movements described in Fig. 9, the moving group, composed of the two robots initially with lights colored B and Y, goes to the bottom beacon robot Y. During a diagonal move, the color of the light of the robot in the moving group initially colored Y alternates at each move between Y and P, while the light of the robot initially colored B has a fixed color. Robots in the group alternatively move horizontally and vertically (when one moves horizontally, the other moves vertically) according to the colors of the group, either $\{B, Y\}$ or $\{B, P\}$. After the turn at the bottom beacon robot, described in Fig. 10, the lights of the moving group are now colored G and B and the group moves with fixed colors similarly to the previous algorithm, until reaching the third beacon robot. Precisely, they move up towards the top right beacon robot, turns left, and then moves straight to the left towards the third beacon robot, following rules that are identical to the previous algorithm, except that at some point two robots swap their color (and so their role) instead of swapping positions so that the algorithm remains exclusive; precisely a member of the moving group becomes a beacon and conversely. Upon reaching the third beacon robot, the robots perform a turn following the sequence described in Fig. 11. After the turn at the top left beacon robot, the lights of the moving group have again colors B and Y and again moves in diagonal. All rules are given in Fig. 12.

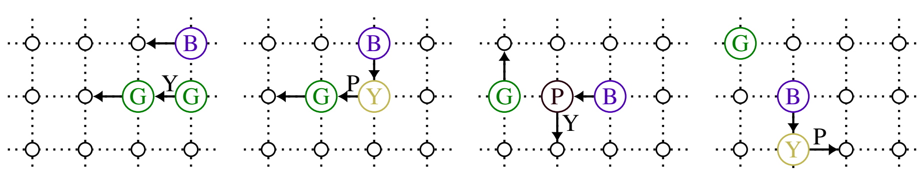
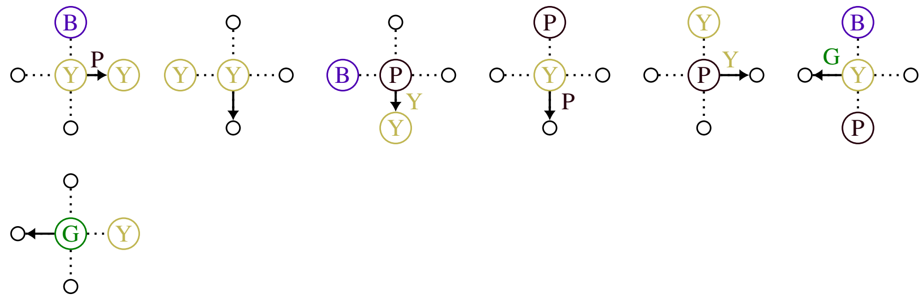


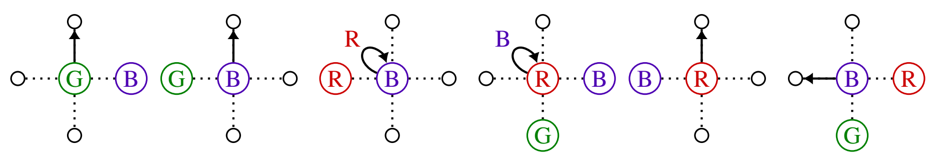
Fig. 11: Sequence of moves of a left turn at the top left beacon robot.

Rules for the diagonal move (the fourth rule is also used during the first and the last turns):

For the first turn:



The rules below allow two robots to move in straight line toward a beacon, turn left, and then move in straight line towards the next beacon. Actually, they are identical to the previous algorithm, except that the two robots swap their colors instead of swapping their positions.



Rules for the last turn:

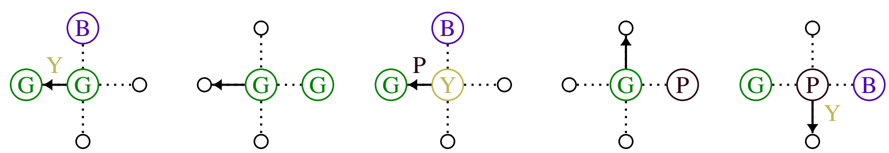


Fig. 12: Rules for Algorithm $\mathcal{A}_1^{Modifiable}$.

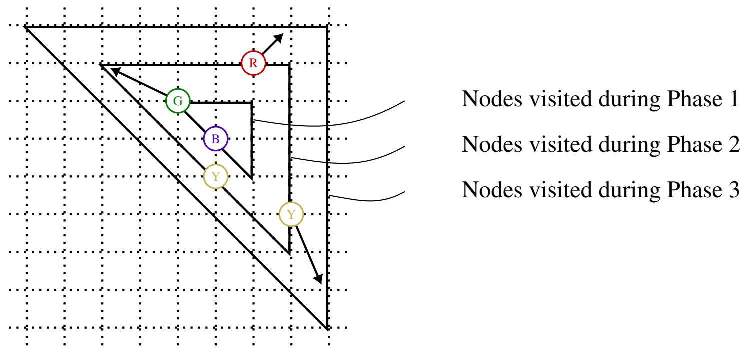


Fig. 13: Visited triangles after three phases for $\mathcal{A}_1^{Modifiable}$.

Due to the lack of space, the proof of the next theorem (which follows the same sketch as the one of Theorem 5) has been omitted.

Theorem 6. *Algorithm $\mathcal{A}_1^{Modifiable}$ solves the exclusive IGE problem using five robots, five modifiable colors, and a visibility range of one.*

5 Infinite Grid Exploration with $\Phi = 2$ and no light

In this section, we describe Algorithm $\mathcal{A}_2^{nolight}$ which solves the exclusive IGE problem assuming visibility range two, yet using no light (or equivalently, using lights with the same fixed color for all robots), *i.e.*, using anonymous oblivious robots. Recall that an animation of this algorithm is available in our complementary material [6]. As previously, the fact that the rules of this algorithm are well-defined and unambiguous has been checked by the script that generated those animations.

First, one can observe that since the visibility range is two, the obstructed visibility can impact the local view of a robot because a robot at distance one can hide a robot behind it at distance two. So, the rules of $\mathcal{A}_2^{nolight}$ should not depend on the states of the nodes that are hidden by a robot. To make it clear, those nodes will be crossed out in the illustrations of our rules, in Figures 15, 16, and 17.

The principle of our algorithm is similar to the first two ones. We still proceed by phases. In Phase i ($i \geq 1$), a moving group, this time of three robots, traverses the edges of a square of length $2i$ (see Fig. 14). The three *moving* robots are always placed in such a way that exactly one of them, the *leader*, has one robot of the group on its horizontal axis and the other on its vertical axis. Again, the two non-leader robots of the group are called the *followers*. Notice however that the leadership changes during a phase. Finally, as previously, the non-members of the moving group are called the *beacon* robots.

The overall idea is that the moving group moves straight according to the relative positions of its members until a follower detects a beacon at distance two. Then, an adjustment is performed in two rounds to push away the beacon and to make the moving group turn left.

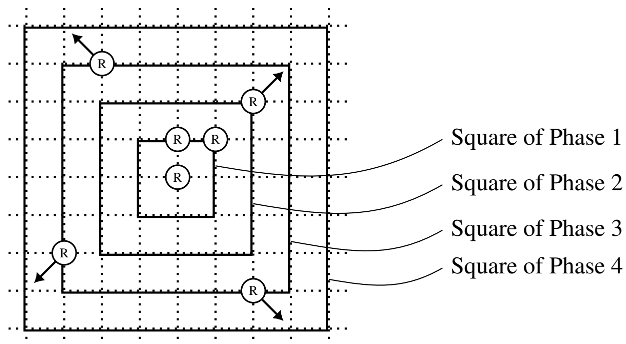


Fig. 14: Initial configuration I of $\mathcal{A}_2^{nolight}$ and visited squares after four phases.

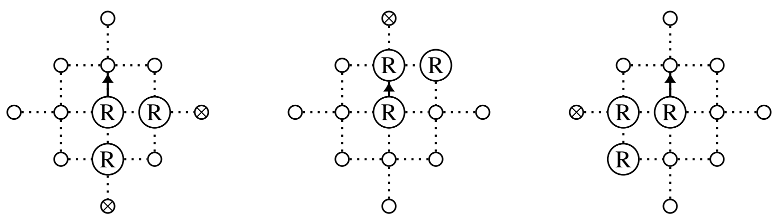


Fig. 15: Moving on a straight line for $\mathcal{A}_2^{nolight}$.

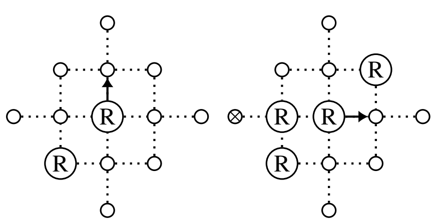


Fig. 16: First round.

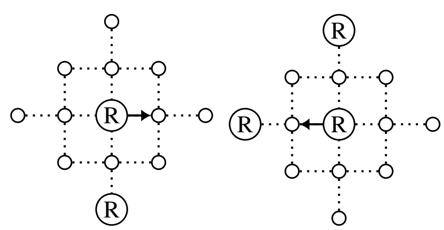


Fig. 17: Second round.

The initial configuration is presented in Fig. 14 and the rules are given in Figs. 15, 16, and 17. During Phase i ($i \geq 1$), the visited square is actually the one of length $2i$ whose center is the initial position of the bottom follower; see Fig. 14. For the movements along a straight line, the moving group forms a right angle. Each of the three moving robots sees the others, can determine its position in the group, and so knows the current direction to follow. Then, when the moving group is close enough from a beacon robot (see the first configuration in Fig. 18), an adjustment is done in two rounds. In the

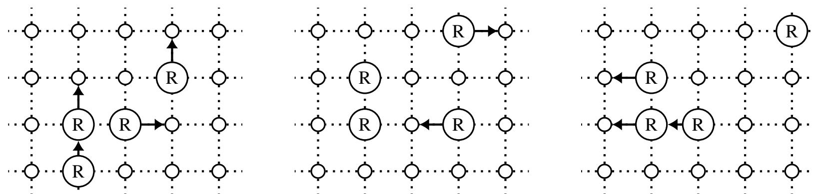


Fig. 18: Sequence moves for a left turn.

first round, a beacon robot sees a follower in diagonal and moves up. Simultaneously, that follower moves towards the node on the right of that beacon robot. The two other members of the moving group move straight, as previously. In the second round, the beacon robot moves away, on the left of the aforementioned follower it sees at distance two (*i.e.*, on the right from a global point of view described in Fig. 18). Simultaneously, that follower, which sees the beacon robot at distance two, catches up with the other robots of the moving group that are on its left and stay idle. Then, the moving group moves again along a straight line, and so on.

Due to the lack of space, the proof of the next theorem has been omitted. Again it follows the same sketch of the proof of Theorem 5.

Theorem 7. *Algorithm $\mathcal{A}_2^{nolight}$ solves the exclusive IGE problem using seven robots without lights and a visibility range of two.*

6 Conclusion and Perspectives

We have considered the problem of exploring an infinite discrete environment, namely an infinite grid-shaped graph, using a small number of mobile synchronous robots with low computation and communication capabilities. In particular, our robots are opaque and only agree on a common chirality. We have shown that using few fixed colors (actually three), six robots, with a visibility range restricted to one, are necessary and sufficient to solve the non-exclusive IGE problem. We have also shown that using modifiable colors with few states (actually five), five such robots, with a visibility range restricted to one, are necessary and sufficient to solve the (exclusive) Infinite Grid Exploration (IGE) problem. We also provide an algorithm that the exclusive IGE problem using seven oblivious anonymous robots, yet assuming visibility range two.

A direct perspective of this work is to study the optimality, in terms of number of robots, when we consider the case of anonymous oblivious robots (*i.e.*, robots without any light). Another line of research would be to study the impact of removing the chirality assumption. As a long-term perspective, we envision to study the IGE problem in fully asynchronous settings.

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