Comparison of Mean Hitting Times for a Degree-Biased Random Walk☆

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Abstract

Consider the random walk on graphs such that, at each step, the next visited vertex is a neighbor of the current vertex, chosen with probability proportional to the inverse of the square root of its degree. On one hand, for every graph with n vertices, the maximal mean hitting time for this degree-biased random walk is asymptotically dominated by n^2. On the other hand, the maximal mean hitting time for the simple random walk is asymptotically dominated by n^3. Yet, in this article, we exhibit for each positive integer n:

- A graph of size n with maximal mean hitting time strictly smaller for the simple random walk than for the degree-biased one.
- A graph of size n with mean hitting time of a so called root vertex strictly smaller for the simple random walk than for the degree-biased one.

Keywords: Markov chains, random walks on graphs, biased random walks
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1. Introduction

This paper deals with mean hitting times of the degree-biased random walk on graphs introduced in [1].

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We assume that every graph is simple, namely all edges are undirected and there are neither self-loops nor multiple edges. Besides, the vertex set, denoted by $\mathcal{V}$, is finite and not reduced to a singleton. Furthermore, every graph is connected.

The degree-biased random walk, denoted by $(X_n)_{n\geqslant 0}$, has the following transition kernel, parameterized by a real number λ. At each step n, the next visited vertex X_{n+1} is a random neighbor of the current vertex X_n, chosen with probability inversely proportional to its degree to the power λ. In particular, if $\lambda = 0$, then $(X_n)_{n\geqslant 0}$ is a simple random walk: the next visited vertex is uniform among the neighbors of the current vertex.

More specifically, for every vertex x, let $\deg(x)$ denote the degree of x, *i.e.*, its number of neighbors. The degree-biased random walk $(X_n)_{n\geqslant 0}$ follows the same law as the random walk on the weighted graph constructed from G by endowing every edge $\{x, y\}$ with the weight $\deg(x)^{-\lambda} \deg(y)^{-\lambda}$. Thereby, we remark that the random walk $(X_n)_{n\geqslant 0}$ is reversible.

For each vertex y, the hitting time T_y is the random number of steps needed by the walker to reach y: $T_y = \inf\{n \geqslant 0 : X_n = y\}$. We emphasize the dependence of T_y on λ by writing $T_y(\lambda)$. Furthermore, we write E for expectation and the subscript x in E_x indicates that the walk starts from the vertex x: $X_0 = x$. In [1], Ikeda *et al* give an upper bound of $\max_{x,y\in\mathcal{V}} E_x T_y(\lambda)$, for every real number λ and every graph of n vertices. In particular:

- $\max_{x,y\in\mathcal{V}} E_x T_y(0) = O(n^3)$, as firstly stated in [2]; and
- $\max_{x,y\in\mathcal{V}} E_x T_y(1/2) = O(n^2)$.

Our first result, Proposition 1.3, implies that for every positive integer n, there exists a graph with n vertices such that $\max_{x,y\in\mathcal{V}} E_x T_y(0) < \max_{x,y\in\mathcal{V}} E_x T_y(1/2)$.

A graph is rooted if a particular vertex, called *root*, is distinguished. Throughout this paper, the root is denoted by o. Our second result, Proposition 1.5, yields for every positive integer n a rooted graph with n vertices such that $E_x T_o(0) < E_x T_o(1/2)$, for every vertex x distinct from the root. This result is motivated by the routing problem on wireless sensor networks [3, 4], as described after the statement of Proposition 1.5.

Both propositions are of independent interest. Moreover, they do not seem to be directly related. Indeed, each proposition exhibits an infinite set of rooted graphs and the two sets are neither disjoint nor included in each other.

From now on, we restrict ourselves to spherically symmetric graphs, defined below. The distance between any two vertices of a graph is the minimum number of edges among all paths joining them. The level of a vertex in a rooted graph is its distance to the root. The height of a graph is its maximum level. Since we have assumed that the vertex set of each graph is not reduced to a singleton, the height is always positive.

Definition 1.1. *A spherically symmetric graph is a rooted graph such that for every positive integer ℓ, all vertices at level ℓ have the same number of neighbors at level $\ell + 1$, the same number of neighbors at level ℓ, and the same number of neighbors at level $\ell - 1$.*

Proposition 1.3 deals with spherically trees, *i.e.*, spherically symmetric graphs without any cycle as subgraph. A spherically symmetric tree is displayed in Figure 1. Our proof uses the following lemma. Consider a spherically symmetric tree with height h. We express, for every real number λ, the maximal mean hitting time $\max_{x,y\in\mathcal{V}} E_xT_y(\lambda)$. For every integer ℓ in $\{0, \dots, h\}$, d_ℓ is the degree of any vertex at level ℓ.

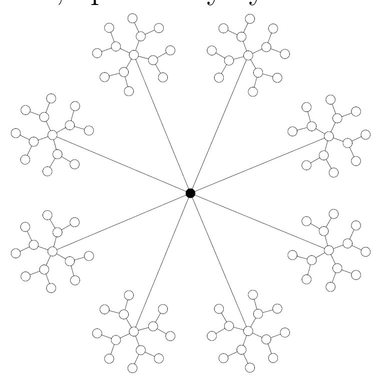

Figure 1: A spherically symmetric tree with $h = 3$, $d_0 = 8$, $d_1 = 5$, $d_2 = 3$, and $d_3 = 1$. The root is colored black.

Lemma 1.2. *Consider a spherically symmetric tree with vertex set $\mathcal{V}$ and positive height h. For every real number λ, the maximal mean hitting time* $\max_{x,y\in\mathcal{V}} E_xT_y(\lambda)$ *is equal to*

$$2d_0 \sum_{\ell=1}^{h} \left(\prod_{k=1}^{\ell-1} (d_k - 1) \right) \sum_{k=1}^{h} \left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k} \right)^{-\lambda}$$

Proposition 1.3 states that if the two sequences of degrees $(d_{2\ell})_{\ell=0}^{\lfloor h/2 \rfloor}$ and $(d_{2\ell+1})_{\ell=0}^{\lfloor h/2 \rfloor}$ do not increase, then $\max_{x,y\in\mathcal{V}} E_xT_y(\lambda)$ increases in λ.

Below, we call *path of length two* every graph composed of three vertices v_1, v_2, v_3, rooted at v_1, and such that v_2 is the only neighbor of v_1 and v_3.

Proposition 1.3. *Consider a spherically symmetric tree with vertex set $\mathcal{V}$ and height h greater or equal to 2, which is not a path of length two. Assume that $d_{\ell+1} \leqslant d_{\ell-1}$ for every integer ℓ in $\{1, \dots, h - 1\}$. Then, the maximal mean hitting time* $\max_{x,y\in\mathcal{V}} E_xT_y(\lambda)$ *increases in λ.*

Now, we enunciate our second result, Proposition 1.5. Our proof needs an explicit expression of the mean hitting time E_xT_o of the root, starting from any vertex x. Consider a spherically symmetric graph with height h. We introduce the following notations. For each integer ℓ in $\{0,\dots,h\}$ and any vertex at level ℓ, let $d_{\ell,\ell+1}$ denote the number of its neighbors at level $\ell+1$, $d_{\ell,\ell}$ the number of its neighbors at level ℓ and $d_{\ell,\ell-1}$ the number of its neighbors at level $\ell-1$. By definition, $d_\ell = d_{\ell,\ell+1} + d_{\ell,\ell} + d_{\ell,\ell-1}$. We remark that $d_{0,0} = 0$, $d_{0,-1} = 0$, $d_{1,0} = 1$ and $d_{h,h+1} = 0$. Besides, although the symbol d_{h+1} does not have any mathematical meaning, since there is no vertex at level $h+1$, we keep the notations easier by setting $d_{h,h+1}d_{h+1}^{-\lambda} = 0$. Figure 2 shows a spherically symmetric graph and associated integers h, d_ℓ, $d_{\ell,\ell+1}$, $d_{\ell,\ell}$, and $d_{\ell,\ell-1}$, for ℓ running through $\{0,\dots,h\}$.

For every integer ℓ in $\{0,\dots,h\}$, let

$$p_\ell = \frac{d_{\ell,\ell+1}d_{\ell+1}^{-\lambda}}{d_{\ell,\ell+1}d_{\ell+1}^{-\lambda} + d_{\ell,\ell}d_\ell^{-\lambda} + d_{\ell,\ell-1}d_{\ell-1}^{-\lambda}}$$

and

$$q_\ell = \frac{d_{\ell,\ell-1}d_{\ell-1}^{-\lambda}}{d_{\ell,\ell+1}d_{\ell+1}^{-\lambda} + d_{\ell,\ell}d_\ell^{-\lambda} + d_{\ell,\ell-1}d_{\ell-1}^{-\lambda}}$$

In particular, $p_0 = 1$, $q_0 = 0$, and $p_h = 0$. Consider a walker that performs a degree-biased random walk with parameter λ. Assume that the walker is currently at level ℓ. The real number p_ℓ is the probability that the walker reaches level $\ell+1$ at its next step and the real number q_ℓ is the probability that the walker reaches level $\ell-1$ at its next step.

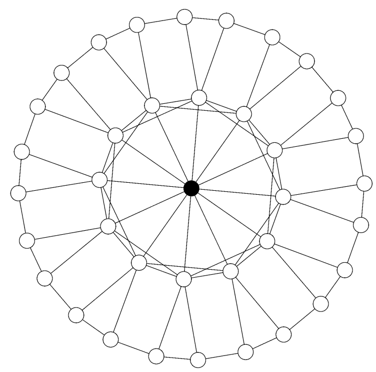

Figure 2: A spherically symmetric graph that is not a tree, with $h = 2$, $d_0 = d_{0,1} = 12$, $d_1 = 7$, $d_{1,2} = 2$, $d_{1,1} = 4$, $d_{1,0} = 1$, $d_2 = 3$, $d_{2,2} = 2$, and $d_{2,1} = 1$. The root is colored black.

Lemma 1.4. *Consider a spherically symmetric graph with height h. For every real number λ, every integer ℓ in $\{0,\dots,h\}$, and every vertex x at level ℓ:*

$$E_xT_o = \sum_{i=0}^{\ell-1} \frac{1}{p_i} \sum_{k=i+1}^{h} \prod_{j=i}^{k-1} \frac{p_j}{q_{j+1}}$$

Proposition 1.5 states that if the sequence of degrees $(d_\ell)_{\ell=0}^h$ is nonincreasing, then $E_xT_o(\lambda)$ increases in λ, for every vertex $x \neq o$.

Proposition 1.5. *Consider a spherically symmetric graph with height h greater or equal to 2. Assume that $d_{\ell+1} \leqslant d_\ell$ for every integer ℓ in $\{0, \dots, h-1\}$. Then, for every vertex x distinct from the root, the mean hitting time $E_x T_o(\lambda)$ increases in λ.*

This result applies to routing in wireless sensor networks. Such networks are formed by a large number of sensors together with an information collector, the sink. The sensors collect information on their environment and data are routed to the sink, usually *via* radio channel; see [5]. A wireless sensor network may be modeled by a rooted graph, the root standing for the sink. Each sensor is battery powered and experimental results show that the sensors closest to the sink tend to deplete their energy faster than other sensors, see for example [6]. Hence, in order to maximize lifetime, it is relevant to design a network such that the closer to the sink a node is, the higher its degree is. The degree-biased random walk with $\lambda = 1/2$ defines a probabilistic routing protocol well suited to wireless sensor networks, as described in [7]. The mean hitting time of the root represents the mean number of steps needed by a data packet to reach the sink. Therefore, comparing the mean hitting times of the root may be relevant to compare efficiency of probabilistic routing protocols. Yet, for a network represented by a spherically symmetric graph with height h greater or equal to 2 and nonincreasing sequence of degrees $(d_i)_{i=0}^h$, we have shown that for every vertex x distinct from the root, $E_x T_o(1/2) > E_x T_o(0)$. Thus, the simple random walk is a probabilistic routing protocol more efficient than the degree-biased random walk with $\lambda = 1/2$.

The remainder of the paper is organized as follows: Lemma 1.2 is proved in Section 2, Proposition 1.3 in Section 3, Lemma 1.4 in Section 4, and Proposition 1.5 in Section 5. (An alternative proof of Proposition 1.5 is also proposed at the end of this latter section.)

2. Proof of Lemma 1.2

Let G be a spherically symmetric tree. Let u and v be any two vertices at level h such that the root belongs to all paths from u to v. Consequently, $\max_{x,y\in\mathcal{V}} E_x T_y = E_u T_v$ and $E_u T_v = E_u T_o + E_o T_v$. By symmetry, $E_o T_v = E_o T_u$. Hence, $\max_{x,y\in\mathcal{V}} E_x T_y$ is the mean commute time $E_u T_o + E_o T_u$.

Yet, the random walk $(X_n)_{n\geqslant 0}$ is reversible. Indeed, the degree-biased random walk $(X_n)_{n\geqslant 0}$ follows the same law as the random walk on the weighted graph constructed from G by endowing every edge $\{x, y\}$ with the weight $\deg(x)^{-\lambda}\deg(y)^{-\lambda}$, where for every vertex z, $\deg(z)$ denotes the degree of z. Hence, according to Corollary 11, Chapter 5 of [8], $E_u T_o + E_o T_u$ is equal to $2mR$, where R is the effective resistance between o and u, and m is the sum of all weights:

$$m = \sum_{\{x,y\}\in\mathcal{E}} \deg(x)^{-\lambda}\deg(y)^{-\lambda}$$

with $\mathcal{E}$ the edge set of G.

For every integer ℓ in $\{1, \ldots, h\}$, we denote by n_ℓ the number of vertices at level ℓ: $n_\ell = d_0 \prod_{k=1}^{\ell-1}(d_k - 1)$. Thus, $m = \sum_{\ell=1}^{h} n_\ell (d_{\ell-1}d_\ell)^{-\lambda}$. By analogy between electrical networks and weighted graphs, the effective resistance between o and u is the sum of the inverse of each weight $R = \sum_{k=1}^{h}(d_{k-1}d_k)^{\lambda}$. Thereby, $\max_{x,y\in\mathcal{V}} E_x T_y = 2f(\lambda)$, where

$$f(\lambda) = \sum_{\ell=1}^{h}\sum_{k=1}^{h} n_\ell \left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right)^{-\lambda}$$

□

3. Proof of Proposition 1.3

We keep the notations used in the proof of Lemma 1.2. It suffices to show that the function f increases on the set of real numbers. Since f is differentiable and

$$f'(\lambda) = -\sum_{\ell=1}^{h}\sum_{k=1}^{h} n_\ell \log\left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right) \times \left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right)^{-\lambda}$$

we get

$$f'(\lambda) = \sum_{\ell=1}^{h-1}\sum_{k=\ell+1}^{h} n_\ell \log\left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right) \times \left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right)^{-\lambda}\left(\frac{n_k}{n_\ell}\left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right)^{2\lambda} - 1\right)$$

Proposition 1.3 assumes that the degrees at even levels and the degrees at odd levels do not increase with the level, *i.e.*, $d_{\ell+1} \leqslant d_{\ell-1}$ for every integer ℓ in $\{1, \ldots, h-1\}$. Hence, for any two integers ℓ and k such that $1 \leqslant \ell < k \leqslant h$, we infer that $d_{k-1}d_k \leqslant d_{\ell-1}d_\ell$. Besides, $n_k \geqslant n_\ell$. Therefore,

$$\frac{n_k}{n_\ell}\left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right)^{2\lambda} \geqslant 1$$

and

$$n_\ell \log\left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right) \times \left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right)^{-\lambda} \left(\frac{n_k}{n_\ell}\left(\frac{d_{\ell-1}d_\ell}{d_{k-1}d_k}\right)^{2\lambda} - 1\right) \geqslant 0$$

Consequently, $f'(\lambda) \geqslant 0$. Besides, $f'(\lambda) = 0$ if and only if:

- The number of vertices at each level is constant: for every integer ℓ in $\{1, \ldots, h\}$, $n_\ell = n_1$.
- The degree of vertices at each even level is constant: for every integer k in $\{1, \ldots, \lfloor h/2 \rfloor\}$, $d_{2k} = d_0$.
- The degree of vertices at each odd level is constant: for every integer k in $\{1, \ldots, \lfloor (h-1)/2 \rfloor\}$, $d_{2k+1} = d_1$.

Consequently, if $f'(\lambda) = 0$ then $d_h = d_{h-2}$. By definition, the vertices at level h are the leaves of the tree: $d_h = 1$. Hence, $h = 2$ and $d_0 = 1$. Moreover, $n_1 = n_2$. Since $n_1 = d_0$, it follows that there is only one vertex at level one and only one leaf. In other words, the tree is a path of length two. Consequently, $f'(\lambda)$ is positive. □

4. Proof of Lemma 1.4

The degree-biased random walk can be lumped by aggregating vertices at the same level, as follows. Let π denote the function from $\mathcal{V}$ to $\{0, \ldots, h\}$ that maps each vertex to its level. According to Theorem 6.4.1 in [9, p. 133], the stochastic process $(\pi(X_n))_{n \geqslant 0}$ is a birth-and-death Markov chain on the set $\{0, \ldots, h\}$ with transition kernel Q defined for every two integers i and j in $\{0, \ldots, h\}$ by

$$Q(j, \{k\}) = \begin{cases} p_j & \text{if } k = j+1\,, \\ 1 - p_j - q_j & \text{if } k = j\,, \\ q_j & \text{if } k = j-1\,, \\ 0 & \text{otherwise.} \end{cases}$$

For each integer ℓ in $\{0, \ldots, h\}$, let H_ℓ denote the hitting time of ℓ for the lumped Markov chain $(\pi(X_n))_{n \geqslant 0}$, *i.e.*, $H_\ell = \inf\{n \geqslant 0 : \pi(X_n) = \ell\}$. By construction, for every integer ℓ in $\{1, \ldots, h\}$ and every vertex x at level ℓ, the mean hitting times $E_x T_o$ and $E_\ell H_0$ are equal.

Let ℓ be an integer in $\{1, \ldots, h\}$. Since $E_\ell H_0 = \sum_{i=0}^{\ell-1} E_{i+1} H_i$, it suffices to express $E_{i+1}H_i$ for every integer i in $\{0, \ldots, h-1\}$, using methods of Chapter 5 in [8]. Fix an integer i in $\{0, \ldots, \ell-1\}$. Let $(\widetilde{X}_n)_{n\geqslant 0}$ denote the Markov chain with states space $\{i, \ldots, h\}$ and transition kernel $\widetilde{Q}$ defined by

$$\widetilde{Q}(j, \{k\}) = \begin{cases} Q(j, \{k\}) & \text{if } j \neq i\,, \\ p_i/(1-q_i) & \text{if } (j,k) = (i, i+1)\,, \\ (1-p_i-q_i)/(1-q_i) & \text{if } (j,k) = (i,i)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Starting from every vertex j greater or equal to $i+1$, the Markov chains $(\widetilde{X}_n)_{n\geqslant 0}$ and $(\pi(X_n))_{n\geqslant 0}$ have the same behavior until time H_i. Hence, H_i and the hitting time $\widetilde{H}_i$ of the state i for the Markov chain $(\widetilde{X}_n)_{n\geqslant 0}$ follow the same law, conditioned by the event $X_0 = \widetilde{X}_0$. Let $\widetilde{H}_i^+$ denote the first return time to i for the Markov chain $(\widetilde{X}_n)_{n\geqslant 0}$: $\widetilde{H}_i^+ = \inf\{n \geqslant 1 \,:\, \widetilde{X}_n = i\}$. On one hand, if the walker starts at i, then at its first step, one of the two following disjoint events occurs: either the walker stays at state i, with probability $\widetilde{Q}(i, \{i\})$, or he hits the vertex $i+1$, with probability $\widetilde{Q}(i, \{i+1\})$. Thereby,

$$E_i \widetilde{H}_i^+ = 1 + \frac{p_i}{1-q_i} E_{i+1} \widetilde{H}_i$$

Since $E_{i+1}H_i = E_{i+1}\widetilde{H}_i$, it follows

$$E_{i+1}H_i = \frac{1-q_i}{p_i}\left(E_i \widetilde{H}_i^+ - 1\right)$$

On the other hand, $E_i\widetilde{H}_i^+ = 1/\eta(\{i\})$, where η is the unique invariant probability of $(\widetilde{X}_n)_{n\geqslant 0}$. Classical computations, see for example the book [10, p. 106], yield the equality

$$\frac{1}{\eta(\{i\})} = 1 + \frac{1}{1-q_i} \sum_{k=i+1}^{h} \prod_{j=i}^{k-1} \frac{p_j}{q_{j+1}}$$

Consequently,

$$E_{i+1}H_i = \frac{1}{p_i} \sum_{k=i+1}^{h} \prod_{j=i}^{k-1} \frac{p_j}{q_{j+1}}$$

Now, $E_\ell H_0 = \sum_{i=0}^{\ell-1} E_{i+1} H_i$, therefore

$$E_\ell H_0 = \sum_{i=0}^{\ell-1} \frac{1}{p_i} \sum_{k=i+1}^{h} \prod_{j=i}^{k-1} \frac{p_j}{q_{j+1}}$$

□

5. Proof of Proposition 1.5

Assume that for every integer ℓ in $\{0, \ldots, h-1\}$, $d_{\ell+1} \leqslant d_\ell$. According to Lemma 1.4, it is enough to show that for every integer i in $\{0, \ldots, h\}$, the quantity

$$\frac{1}{p_i} \sum_{k=i+1}^{h} \prod_{j=i}^{k-1} \frac{p_j}{q_{j+1}}$$

increases in λ. Yet,

$$\frac{1}{p_i} \sum_{k=i+1}^{h} \prod_{j=i}^{k-1} \frac{p_j}{q_{j+1}} = \sum_{k=i+1}^{h} \frac{1}{q_k} \prod_{j=i+1}^{k-1} \frac{p_j}{q_j}$$

For every integer ℓ in $\{0, \ldots, h\}$, the function $\lambda \mapsto p_\ell$ increases and the function $\lambda \mapsto q_\ell$ is nonincreasing. Moreover, if $d_{\ell+1} < d_\ell$, then $\lambda \mapsto p_\ell$ increases and $\lambda \mapsto q_\ell$ decreases. By definition, the vertices at level h are the leaves of the tree: $d_h = 1$. Proposition 1.5 assumes that h is greater or equal to 2. Hence, d_{h-1} is greater or equal to 2. Since $d_h < d_{h-1}$, the result follows. □

We remark that Proposition 1.5 can be proved by stochastic comparison, without using Lemma 1.4. Indeed, consider two real numbers μ and λ such that $\mu < \lambda$. An alternative proof is obtained from a direct application of Corollary 1 of [11], with the following parameters:

- The states space E_1 is the vertex set of T.
- The closed partial ordering $\leqq$ is the distance ℓ to the root in T.
- $(X_n)_{n\geqslant 0}$ is the μ-degree-biased random walk $(X_n(\mu))_{n\geqslant 0}$ and $(Y_n)_{n\geqslant 0}$ is the λ-degree-biased random walk $(X_n(\lambda))_{n\geqslant 0}$, with μ and λ two real numbers such that $\mu < \lambda$.

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